Konuralp Journal of Mathematics
Volume 2 No. 1 pp. 24–35 (2014) ©KJM

CONJUGATE TANGENT VECTORS AND ASYMPTOTIC DIRECTIONS FOR SURFACES AT A CONSTANT DISTANCE FROM EDGE OF REGRESSION ON A SURFACE IN E_1^3

DERYA SAĞLAM[1] AND ÖZGÜR BOYACIOĞLU KALKAN[2]

Abstract. In this paper we give conjugate tangent vectors and asymptotic directions for surfaces at a constant distance from edge of regression on a surface in E_1^3.

1. Introduction

Conjugate tangent vectors and asymptotic directions in Euclidean space E^3 can be found in [9]. In 1984, A. Kılıç and H. H. Hacısalihoğlu found the Euler theorem and Dupin indicatrix for parallel hypersurfaces in E^n [13]. Also the Euler theorem and Dupin indicatrix are obtained for the parallel hypersurfaces in pseudo-Euclidean spaces E_1^{n+1} and E_ν^{n+1} in the papers ([5], [7], [8]).

In 2005 H. H. Hacısalihoğlu and Ö. Tarakçı introduced surfaces at a constant distance from edge of regression on a surface. These surfaces are a generalization of parallel surfaces in E^3. Because the authors took any vector instead of normal vector [17]. Euler theorem and Dupin indicatrix for these surfaces are given in [2]. Conjugate tangent vectors and asymptotic directions are given in [1]. In 2010 we obtained the surfaces at a constant distance from edge of regression on a surface in E_1^3 [15]. We obtained the Euler theorem and Dupin indicatrix for these surfaces in E_1^3 [16].

In this paper we give conjugate tangent vectors and asymptotic directions for surfaces at a constant distance from edge of regression on a surface in E_1^3.

2. Preliminaries

Let E_1^3 be the Minkowski 3-space is the real vector space R^3 endowed with the standard flat Lorentzian metric given by

$$\langle , \rangle = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2$$

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* 51B20, 53B30.
Key words and phrases. conjugate tangent vectors, asymptotic direction, edge of regression.

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $x \in E_1^3$ is called spacelike if $\langle x, x\rangle > 0$ or $x = 0$, timelike if $\langle x, x\rangle < 0$ and lightlike (null) if $\langle x, x\rangle = 0$ and $x \neq 0$.

The timelike-cone of E_1^3 is defined as the set of all timelike vectors of E_1^3, that is

$$\mathcal{T} = \{(x, y, z) \in E_1^3;\ x^2 + y^2 - z^2 < 0\}.$$

The set of lightlike vectors is defined by $\mathcal{C}$ and it is the following set:

$$\mathcal{C} = \{(x, y, z) \in E_1^3;\ x^2 + y^2 - z^2 = 0\} - \{0, 0, 0\}.$$

The cross product $x \times y$ of vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in E_1^3 is defined as

$$\langle x \times y, z\rangle = \det(x, y, z) \quad \text{for all } z = (z_1, z_2, z_3) \in E_1^3.$$

More explicitly, if x, y belong to E_1^3, then

$$\begin{aligned} \langle x, y\rangle &= -x_1y_1 + x_2y_2 + x_3y_3 \\ x \times y &= (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \\ \langle a \times b, x \times y\rangle &= -\begin{vmatrix} \langle a, x\rangle & \langle b, x\rangle \\ \langle a, y\rangle & \langle b, y\rangle \end{vmatrix} \end{aligned}$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in E_1^3 (Lagrange identity in E_1^3).

Let $e_1, e_2 \in E_1^3$ be such that $< e_i, e_i >= \pm 1$ and $\langle e_1, e_2\rangle = 0$ and $e_3 = e_1 \times e_2$. Then these three vectors form an orthonormal frame. If $\langle e_1, e_1\rangle = \varepsilon_1$ and $\langle e_2, e_2\rangle = \varepsilon_2$ where $\varepsilon_1, \varepsilon_2 = \pm 1$, it follows from the Lagrange identity that $\langle e_3, e_3\rangle = -\varepsilon_1\varepsilon_2$. Each vector $x \in E_1^3$ can be written uniquely in terms of e_1, e_2, e_3 by

$$x = \varepsilon_1 \langle x, e_1\rangle e_1 + \varepsilon_2 \langle x, e_2\rangle e_2 - \varepsilon_1\varepsilon_2 \langle x, e_3\rangle e_3.$$

The angle between two vectors in Minkowski 3-space is defined by ([3], [10], [11], [12]):

Definition 2.1. i. Hyperbolic angle: Let x and y be timelike vectors in the same timecone of Minkowski space. Then there is a unique real number $\theta \geq 0$, called the hyperbolic angle between x and y, such that

$$< x, y >= -\|x\| \|y\| \cosh\theta.$$

ii. Central angle: Let x and y be spacelike vectors in Minkowski space that span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$, called the central angle between x and y, such that

$$|< x, y >| = \|x\| \|y\| \cosh\theta.$$

iii. Spacelike angle: Let x and y be spacelike vectors in Minkowski space that span a spacelike vector subspace. Then there is a unique real number θ between 0 and π called the spacelike angle between x and y, such that

$$< x, y >= \|x\| \|y\| \cos\theta.$$

iv. Lorentzian timelike angle: Let x be a spacelike vector and y be a timelike vector in Minkowski space. Then there is a unique real number $\theta \geq 0$, called the Lorentzian timelike angle between x and y, such that

$$|< x, y >| = \|x\| \|y\| \sinh\theta.$$

Definition 2.2. Let M and M^f be two surfaces in E_1^3 and N_p be a unit normal vector of M at the point $P \in M$. Let T_pM be tangent space at $P \in M$ and $\{X_p, Y_p\}$ be an orthonormal bases of T_pM. Let $Z_p = d_1X_p + d_2Y_p + d_3N_p$ be a unit vector, where $d_1, d_2, d_3 \in R$ are constant numbers and $\varepsilon_1 d_1^2 + \varepsilon_2 d_2^2 - \varepsilon_1\varepsilon_2 d_3^2 = \pm 1$. If a function f exists and satisfies the condition $f : M \to M^f$, $f(P) = P + rZ_p$, r constant, M^f is called as the surface at a constant distance from the edge of regression on M and M^f denoted by the pair (M, M^f).

If $d_1 = d_2 = 0$, then we have $Z_p = N_p$ and $f(P) = P + rN_p$. In this case M and M^f are parallel surfaces [15].

Theorem 2.1. *Let the pair (M, M^f) be given in E_1^3. For any $W \in \chi(M)$, we have $f_*(W) = \overline{W} + r\overline{D_W Z}$, where $W = \sum_{i=1}^{3} w_i \dfrac{\partial}{\partial x_i}$,$\overline{W} = \sum_{i=1}^{3} \overline{w_i} \dfrac{\partial}{\partial x_i}$ and $\forall P \in M$, $w_i(P) = \overline{w_i}(f(p))$, $1 \le i \le 3$ [15].*

Let (ϕ, U) be a parametrization of M, so we can write that

$$\phi : \underset{(u,v)}{U} \subset E_1^3 \to \underset{P=\phi(u,v)}{M}.$$

In this case $\{\phi_u|_p, \phi_v|_p\}$ is a basis of T_pM. Let N_p is a unit normal vector at $P \in M$ and $d_1, d_2, d_3 \in R$ be constant numbers then we can write that $Z_p = d_1\phi_u|_p + d_2\phi_v|_p + d_3N_p$. Since $M^f = \{f(P) \mid f(P) = P + rZ_p\}$, a parametric representation of M^f is $\psi(u,v) = \phi(u,v) + rZ(u,v)$. Thus we can write

$$\begin{aligned} M^f \;\; = \;\; & \{\psi(u,v) \mid \psi(u,v) = \phi(u,v) + r(d_1\phi_u(u,v) + d_2\phi_v(u,v) + d_3N(u,v)), \\ & \quad d_1,\ d_2,\ d_3, r \text{ are constant}, \quad \varepsilon_1 d_1^2 + \varepsilon_2 d_2^2 - \varepsilon_1\varepsilon_2 d_3^2 = \pm 1\big\}. \end{aligned}$$

If we take $rd_1 = \lambda_1,\ rd_2 = \lambda_2,\ rd_3 = \lambda_3$ then we have

$$M^f = \{\psi(u,v)|\psi(u,v) = \phi(u,v)+\lambda_1\phi_u(u,v)+\lambda_2\phi_v(u,v)+\lambda_3 N(u,v), \quad \lambda_1, \lambda_2, \lambda_3 \text{ are constant}\}.$$

Let $\{\phi_u, \phi_v\}$ is basis of $\chi(M^f)$. If we take $\langle\phi_u, \phi_u\rangle = \varepsilon_1$, $\langle\phi_v, \phi_v\rangle = \varepsilon_2$ and $\langle N, N\rangle = -\varepsilon_1\varepsilon_2$, then

$$\begin{aligned} \psi_u &= (1 + \lambda_3 k_1)\phi_u + \varepsilon_2\lambda_1 k_1 N, \\ \psi_v &= (1 + \lambda_3 k_2)\phi_v + \varepsilon_1\lambda_2 k_2 N \end{aligned}$$

is a basis of $\chi(M^f)$, where N is the unit normal vector field on M and k_1, k_2 are principal curvatures of M [15].

Theorem 2.2. *Let the pair (M, M^f) be given in E_1^3. Let $\{\phi_u, \phi_v\}$ (orthonormal and principal vector fields on M) be basis of $\chi(M)$ and k_1, k_2 be principal curvatures of M. The matrix of the shape operator of M^f with respect to the basis $\{\psi_u = (1 + \lambda_3 k_1)\phi_u + \varepsilon_2\lambda_1 k_1 N,\ \psi_v = (1 + \lambda_3 k_2)\phi_v + \varepsilon_1\lambda_2 k_2 N\}$ of $\chi(M^f)$ is*

$$S^f = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix}$$

where

$$\begin{aligned}
\mu_1 &= \frac{(1+\lambda_3 k_2)}{A^3}\left\{\varepsilon\lambda_1\frac{\partial k_1}{\partial u}(\lambda_2^2 k_2^2-\varepsilon_1(1+\lambda_3 k_2)^2)+k_1A^2\right\},\\
\mu_2 &= \frac{\varepsilon\lambda_1^2\lambda_2 k_1 k_2(1+\lambda_3 k_2)}{A^3}\frac{\partial k_1}{\partial u},\\
\mu_3 &= \frac{-\varepsilon\lambda_1\lambda_2^2 k_1 k_2(1+\lambda_3 k_1)}{A^3}\frac{\partial k_2}{\partial v},\\
\mu_4 &= \frac{(1+\lambda_3 k_1)}{A^3}\left\{-\varepsilon\lambda_2\frac{\partial k_2}{\partial v}(\lambda_1^2 k_1^2-\varepsilon_2(1+\lambda_3 k_1)^2)+k_2A^2\right\}
\end{aligned}$$

and $A=\sqrt{\varepsilon\left(\varepsilon_1\lambda_1^2k_1^2(1+\lambda_3k_2)^2+\varepsilon_2\lambda_2^2k_2^2(1+\lambda_3k_1)^2-\varepsilon_1\varepsilon_2(1+\lambda_3k_1)^2(1+\lambda_3k_2)^2\right)}$ [15].

Definition 2.3. Let M be an Euclidean surface in E^3 and S be shape operator of M. For any $X_p, Y_p \in T_pM$, if

$$\langle S(X_p), Y_p\rangle = 0 \tag{2.1}$$

then X_p and Y_p are called conjugate tangent vectors of M at p [9].

Definition 2.4. Let M be an Euclidean surface in E^3 and S be shape operator of M. For any $X_p \in T_pM$, if

$$\langle S(X_p), X_p\rangle = 0 \tag{2.2}$$

then X_p is called an asymptotic direction of M at p [9].

We can get the definitions of conjugate tangent vectors and asymptotic direction in Minkowski 3-space similar to Definition 2.3 and 2.4 as below:

Definition 2.5. Let M be a surface in E_1^3 and S be shape operator of M. For any $X_p, Y_p \in T_pM$, if

$$\langle S(X_p), Y_p\rangle = 0 \tag{2.3}$$

then X_p and Y_p are called conjugate tangent vectors of M at p.

Definition 2.6. M be a surface in E_1^3 and S be shape operator of M. For any $X_p \in T_pM$, if

$$\langle S(X_p), X_p\rangle = 0 \tag{2.4}$$

then X_p is called an asymptotic direction of M at p.

3. Conjugate tangent vectors for surfaces at a constant distance from edge of regression on a surface in E_1^3

Theorem 3.1. *Let M^f be a surface at a constant distance from edge of regression on a M in E_1^3. Let k_1 and k_2 denote principal curvature function of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. For $X_p, Y_p \in T_pM$, $f_*(X_p)$ and $f_*(Y_p)$ are conjugate tangent vectors if and only if*

$$\varepsilon_1\mu_1^*x_1y_1+\varepsilon_1\mu_2^*x_1y_2+\varepsilon_2\mu_3^*x_2y_1+\varepsilon_2\mu_4^*x_2y_2=0 \tag{3.1}$$

where

$$\begin{aligned}
x_1 &= \langle X_p, \phi_u \rangle, \qquad x_2 = \langle X_p, \phi_v \rangle, \\
y_1 &= \langle Y_p, \phi_u \rangle, \qquad y_2 = \langle Y_p, \phi_v \rangle, \\
\mu_1^* &= \mu_1(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_2 \mu_1 \lambda_1 k_1 + \varepsilon_1 \mu_2 \lambda_2 k_2), \\
\mu_2^* &= \mu_2(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\varepsilon_2 \mu_1 \lambda_1 k_1 + \varepsilon_1 \mu_2 \lambda_2 k_2), \\
\mu_3^* &= \mu_3(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_2 \mu_3 \lambda_1 k_1 + \varepsilon_1 \mu_4 \lambda_2 k_2), \\
\mu_4^* &= \mu_4(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\varepsilon_2 \mu_3 \lambda_1 k_1 + \varepsilon_1 \mu_4 \lambda_2 k_2).
\end{aligned} \tag{3.2}$$

Proof. Let $f_*(X_p) \in T_{f(p)}M^f$. Then let us calculate $f_*(X_p)$ and $S^f(f_*(X_p))$. Since ϕ_u and ϕ_v are orthonormal we have

$$\begin{aligned}
X_p &= \varepsilon_1 \langle X_p, \phi_u \rangle \phi_u + \varepsilon_2 \langle X_p, \phi_v \rangle \phi_v \\
&= \varepsilon_1 x_1 \phi_u + \varepsilon_2 x_2 \phi_v.
\end{aligned}$$

Further without lost of generality, we suppose that X_p is a unit vector. Then

$$\begin{aligned}
f_*(X_p) &= \varepsilon_1 x_1 f_*(\phi_u) + \varepsilon_2 x_2 f_*(\phi_v) \\
&= \varepsilon_1 x_1 \psi_u + \varepsilon_2 x_2 \psi_v.
\end{aligned} \tag{3.3}$$

On the other hand we find that

(3.4)

$$\begin{aligned}
S^f(f_*(X_p)) &= \varepsilon_1 x_1 S^f(\psi_u) + \varepsilon_2 x_2 S^f(\psi_v) \\
&= \varepsilon_1 x_1 \left(\mu_1(1+\lambda_3 k_1)\phi_u + \mu_2(1+\lambda_3 k_2)\phi_v + (\mu_1 \varepsilon_2 \lambda_1 k_1 + \mu_2 \varepsilon_1 \lambda_2 k_2)N\right) \\
&\quad + \varepsilon_2 x_2 \left(\mu_3(1+\lambda_3 k_1)\phi_u + \mu_4(1+\lambda_3 k_2)\phi_v + (\mu_3 \varepsilon_2 \lambda_1 k_1 + \mu_4 \varepsilon_1 \lambda_2 k_2)N\right)
\end{aligned}$$

and for $Y_p \in T_pM$ we have

$$\begin{aligned}
Y_p &= \varepsilon_1 \langle Y_p, \phi_u \rangle \phi_u + \varepsilon_2 \langle Y_p, \phi_v \rangle \phi_v \\
&= \varepsilon_1 y_1 \phi_u + \varepsilon_2 y_2 \phi_v.
\end{aligned} \tag{3.5}$$

Then

$$\begin{aligned}
f_*(Y_p) &= \varepsilon_1 y_1 f_*(\phi_u) + \varepsilon_2 y_2 f_*(\phi_v) \\
&= \varepsilon_1 y_1 \psi_u + \varepsilon_2 y_2 \psi_v.
\end{aligned} \tag{3.6}$$

Thus using equations (3.4) and (3.6) in equation (2.3) we obtain (3.1). □

Theorem 3.2. *Let M^f be a surface at a constant distance from edge of regression on M in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively and the angle between $Y_p \in T_pM$ and ϕ_u, ϕ_v by θ_1', θ_2' respectively. $f_*(X_p)$ and $f_*(Y_p)$ are conjugate tangent vectors if and only if*

(a)Let N_p be a timelike vector then

$$\mu_1^* \cos\theta_1 \cos\theta_1' + \mu_2^* \cos\theta_1 \cos\theta_2' + \mu_3^* \cos\theta_2 \cos\theta_1' + \mu_4^* \cos\theta_2 \cos\theta_2' = 0.$$

(b) Let ϕ_u be a timelike vector.

(b.1) If X_p and Y_p are spacelike vectors then

$$\begin{aligned}
0 = &-\delta_1 \delta_1' \mu_1^* \sinh\theta_1 \sinh\theta_1' - \delta_1 \delta_2' \mu_2^* \sinh\theta_1 \cosh\theta_2' \\
&+ \delta_1' \delta_2 \mu_3^* \cosh\theta_2 \sinh\theta_1' + \delta_2 \delta_2' \mu_4^* \cosh\theta_2 \cosh\theta_2'.
\end{aligned}$$

(b.2) If X_p, Y_p and ϕ_u are timelike vectors in the same timecone then

$$\begin{aligned} 0 &= \mu_1^* \cosh\theta_1 \cosh\theta_1' + \delta_2' \mu_2^* \cosh\theta_1 \sinh\theta_2' \\ &\quad -\delta_2 \mu_3^* \sinh\theta_2 \cosh\theta_1' + \delta_2 \delta_2' \mu_4^* \sinh\theta_2 \sinh\theta_2'. \end{aligned}$$

(b.3) If X_p, ϕ_u are timelike vectors in the same timecone and Y_p is spacelike vector then

$$\begin{aligned} 0 &= \delta_1' \mu_1^* \cosh\theta_1 \sinh\theta_1' + \delta_2' \mu_2^* \cosh\theta_1 \cosh\theta_2' \\ &\quad +\delta_1' \delta_2 \mu_3^* \sinh\theta_2 \sinh\theta_1' + \delta_2 \delta_2' \mu_4^* \sinh\theta_2 \cosh\theta_2'. \end{aligned}$$

(b.4) If Y_p and ϕ_u are timelike vectors in the same timecone and X_p is spacelike vector then

$$\begin{aligned} 0 &= \delta_1 \mu_1^* \sinh\theta_1 \cosh\theta_1' - \delta_1 \delta_2' \mu_2^* \sinh\theta_1 \sinh\theta_2' \\ &\quad -\delta_2 \mu_3^* \cosh\theta_2 \cosh\theta_1' + \delta_2 \delta_2' \mu_4^* \cosh\theta_2 \sinh\theta_2'. \end{aligned}$$

(c) Let ϕ_v be a timelike vector.
(c.1) If X_p and Y_p are spacelike vectors then

$$\begin{aligned} 0 &= \delta_1 \delta_1' \mu_1^* \cosh\theta_1 \cosh\theta_1' + \delta_1 \delta_2' \mu_2^* \cosh\theta_1 \sinh\theta_2' \\ &\quad -\delta_1' \delta_2 \mu_3^* \sinh\theta_2 \cosh\theta_1' - \delta_2 \delta_2' \mu_4^* \sinh\theta_2 \sinh\theta_2'. \end{aligned}$$

(c.2) If X_p, Y_p and ϕ_v are timelike vectors in the same timecone then

$$\begin{aligned} 0 &= \delta_1 \delta_1' \mu_1^* \sinh\theta_1 \sinh\theta_1' - \delta_1 \mu_2^* \sinh\theta_1 \cosh\theta_2' \\ &\quad -\delta_1' \mu_3^* \cosh\theta_2 \sinh\theta_1' - \mu_4^* \cosh\theta_2 \cosh\theta_2'. \end{aligned}$$

(c.3) If X_p and ϕ_v are timelike vectors in the same timecone and Y_p is spacelike vector then

$$\begin{aligned} 0 &= \delta_1 \delta_1' \mu_1^* \sinh\theta_1 \cosh\theta_1' + \delta_1 \delta_2' \mu_2^* \sinh\theta_1 \sinh\theta_2' \\ &\quad +\delta_1' \mu_3^* \cosh\theta_2 \cosh\theta_1' + \delta_2' \mu_4^* \cosh\theta_2 \sinh\theta_2'. \end{aligned}$$

(c.4) If Y_p and ϕ_v are timelike vectors in the same timecone and X_p is spacelike vector then

$$\begin{aligned} 0 &= \delta_1 \delta_1' \mu_1^* \cosh\theta_1 \sinh\theta_1' - \delta_1 \mu_2^* \cosh\theta_1 \cosh\theta_2' \\ &\quad -\delta_2 \delta_1' \mu_3^* \sinh\theta_2 \sinh\theta_1' + \delta_2 \mu_4^* \sinh\theta_2 \cosh\theta_2'. \end{aligned}$$

Abovementioned $\mu_1^*, \mu_2^*, \mu_3^*$ and μ_4^* are given in (3.2),

$$\delta_i = \begin{cases} 1, & x_i \text{ is positive} \\ -1, & x_i \text{ is negative} \end{cases}, \quad i = (1,2)$$

and

$$\delta_i' = \begin{cases} 1, & y_i \text{ is positive} \\ -1, & y_i \text{ is negative} \end{cases}, \quad i = (1,2).$$

Proof. (a) Let N_p be a timelike vector. In this case θ_1, θ_2, θ_1', θ_2' are spacelike angles then

$$\begin{aligned} x_1 &= \langle X_p, \phi_u \rangle = \cos\theta_1 \\ x_2 &= \langle X_p, \phi_v \rangle = \cos\theta_2. \end{aligned}$$

and

$$\begin{aligned} y_1 &= \langle Y_p, \phi_u \rangle = \cos\theta_1' \\ y_2 &= \langle Y_p, \phi_v \rangle = \cos\theta_2'. \end{aligned}$$

Substituting these equations in (3.1) the proof is obvious.

(b) Let ϕ_u be a timelike vector.
(b.1) If X_p and Y_p are spacelike vectors and ϕ_u is timelike vector then there are Lorentzian timelike angles θ_1, θ_1' and central angles θ_2, θ_2'. Thus

$$\begin{aligned} x_1 &= \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \cosh\theta_2 \\ y_1 &= \delta_1' \sinh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \cosh\theta_2'. \end{aligned}$$

(b.2) If X_p, Y_p and ϕ_u are timelike vectors in the same timecone then there are hyperbolic angles θ_1, θ_1' and Lorentzian timelike angles θ_2, θ_2'. Thus

$$\begin{aligned} x_1 &= -\cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2 \\ y_1 &= -\cosh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \sinh\theta_2'. \end{aligned}$$

(b.3) If X_p and ϕ_u are timelike vectors in the same timecone and Y_p is spacelike vector then there is a hyperbolic angle θ_1, a central angle θ_2' and there are Lorentzian timelike angles θ_2, θ_1'. Thus

$$\begin{aligned} x_1 &= -\cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2 \\ y_1 &= \delta_1' \sinh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \cosh\theta_2'. \end{aligned}$$

(b.4) If Y_p and ϕ_u are timelike vectors in the same timecone and X_p is spacelike vector then there is a central angle θ_2, a hyperbolic angle θ_1' and there are Lorentzian timelike angles θ_1, θ_2'. Thus

$$\begin{aligned} x_1 &= \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \cosh\theta_2 \\ y_1 &= -\cosh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \sinh\theta_2'. \end{aligned}$$

(c) Let ϕ_v be a timelike vector.
(c.1) If X_p and Y_p are spacelike vectors and ϕ_v is timelike vector then there are central angles θ_1, θ_1' and Lorentzian timelike angles θ_2, θ_2'.Thus

$$\begin{aligned} x_1 &= \delta_1 \cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2 \\ y_1 &= \delta_1' \cosh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \sinh\theta_2'. \end{aligned}$$

(c.2) If X_p, Y_p and ϕ_v are timelike vectors in the same timecone then there are Lorentzian timelike angles θ_1, θ_1' and hyperbolic angles θ_2, θ_2'. Thus

$$\begin{aligned} x_1 &= \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = -\cosh\theta_2 \\ y_1 &= \delta_1' \sinh\theta_1' \quad \text{and} \quad y_2 = -\cosh\theta_2'. \end{aligned}$$

(c.3) If X_p and ϕ_v are timelike vectors in the same timecone and Y_p is spacelike vector then there is a hyperbolic angle θ_2, a central angle θ_1' and there are Lorentzian timelike vectors θ_1, θ_2'. Thus

$$\begin{aligned} x_1 &= \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = -\cosh\theta_2 \\ y_1 &= \delta_1' \cosh\theta_1' \quad \text{and} \quad y_2 = \delta_2' \sinh\theta_2'. \end{aligned}$$

(c.4) If Y_p and ϕ_v are timelike vectors in the same timecone and X_p is spacelike vector then then there is a central angle θ_1, a hyperbolic angle θ_2' and there are Lorentzian timelike angles θ_1', θ_2. Thus

$$\begin{aligned} x_1 &= \delta_1 \cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2 \\ y_1 &= \delta_1' \sinh\theta_1' \quad \text{and} \quad y_2 = -\cosh\theta_2'. \end{aligned}$$

□

As a special case if we take $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = r = constant$, then we obtain that M and M^f are parallel surfaces. Hence we give the following corollaries.

Corollary 3.1. *Let M and M_r be parallel surfaces in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively and the angle between $Y_p \in T_pM$ and ϕ_u, ϕ_v by θ_1', θ_2' respectively. $f_*(X_p)$ and $f_*(Y_p)$ are conjugate tangent vectors if and only if*

$$\varepsilon_1 k_1(1 + rk_1)x_1y_1 + \varepsilon_2 k_2(1 + rk_2)x_2y_2 = 0. \tag{3.7}$$

Proof. Since

$$\begin{aligned} \mu_1^* &= k_1(1 + rk_1), \\ \mu_2^* &= 0, \qquad \mu_3^* = 0, \\ \mu_4^* &= k_2(1 + rk_2) \end{aligned}$$

from (3.1) we find (3.7). □

Corollary 3.2. *Let M and M_r be parallel surfaces in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively and the angle between $Y_p \in T_pM$ and ϕ_u, ϕ_v by θ_1', θ_2' respectively. $f_*(Y_p)$ are conjugate tangent vectors if and only if*
(a)Let N_p be a timelike vector then

$$k_1(1 + rk_1)\cos\theta_1\cos\theta_1' + k_2(1 + rk_2)\cos\theta_2\cos\theta_2' = 0.$$

(b) Let ϕ_u be a timelike vector.
(b.1) If X_p and Y_p are spacelike vectors then

$$-\delta_1\delta_1' k_1(1 + rk_1)\sinh\theta_1\sinh\theta_1' + \delta_2\delta_2' k_2(1 + rk_2)\cosh\theta_2\cosh\theta_2' = 0.$$

(b.2) If X_p, Y_p and ϕ_u are timelike vectors in the same timecone then

$$-k_1(1 + rk_1)\cosh\theta_1\cosh\theta_1' + k_2(1 + rk_2)\sinh\theta_2\sinh\theta_2' = 0.$$

(b.3) If X_p and ϕ_u are timelike vectors in the same timecone and Y_p is spacelike vector then

$$\delta_1' k_1(1 + rk_1)\cosh\theta_1\sinh\theta_1' + \delta_2\delta_2' k_2(1 + rk_2)\sinh\theta_2\cosh\theta_2' = 0.$$

(b.4) If Y_p and ϕ_u are timelike vectors in the same timecone and X_p is spacelike vector then

$$\delta_1 k_1(1 + rk_1)\sinh\theta_1\cosh\theta_1' + \delta_2\delta_2' k_2(1 + rk_2)\cosh\theta_2\sinh\theta_2' = 0.$$

(c) Let ϕ_v be a timelike vector.
(c.1) If X_p and Y_p are spacelike vectors then

$$\delta_1\delta_1' k_1(1 + rk_1)\cosh\theta_1\cosh\theta_1' - \delta_2\delta_2' k_2(1 + rk_2)\sinh\theta_2\sinh\theta_2' = 0.$$

(c.2) If X_p, Y_p and ϕ_v are timelike vectors in the same timecone then

$$\delta_1\delta_1' k_1(1 + rk_1)\sinh\theta_1\sinh\theta_1' - k_2(1 + rk_2)\cosh\theta_2\cosh\theta_2' = 0.$$

(c.3) If X_p and ϕ_v are timelike vectors in the same timecone and Y_p is spacelike vector then

$$\delta_1\delta_1' k_1(1 + rk_1)\sinh\theta_1\cosh\theta_1' + \delta_2' k_2(1 + rk_2)\cosh\theta_2\sinh\theta_2' = 0.$$

(c.4) If Y_p and ϕ_v are timelike vectors in the same timecone and X_p is spacelike vector then

$$\delta_1\delta_1' k_1(1+rk_1)\cosh\theta_1 \sinh\theta_1' + \delta_2 k_2(1+rk_2)\sinh\theta_2\cosh\theta_2' = 0.$$

For the above equations

$$\delta_i = \begin{cases} 1, & x_i \text{ is positive} \\ -1, & x_i \text{ is negative} \end{cases}, \quad i=(1,2)$$

and

$$\delta_i' = \begin{cases} 1, & y_i \text{ is positive} \\ -1, & y_i \text{ is negative} \end{cases}, \quad i=(1,2).$$

4. Asymptotic directions for surfaces at a constant distance from edge of regression on a surface in E_1^3

Theorem 4.1. *Let M^f be a surface at a constant distance from edge of regression on a M in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. $f_*(X_p) \in T_{f(p)}(M^f)$ is an asymptotic direction if and only if*

$$\mu_1^* x_1^2 + \varepsilon_1\varepsilon_2\mu_2^* x_1 x_2 + \mu_3^* x_2^2 = 0 \tag{4.1}$$

where

$$\begin{aligned} x_1 &= \langle X_p, \phi_u\rangle, \qquad x_2 = \langle X_p, \phi_v\rangle, \\ \mu_1^* &= \varepsilon_1\mu_1(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_1\varepsilon_2\mu_1\lambda_1 k_1 + \mu_2\lambda_2 k_2), \\ \mu_2^* &= \varepsilon_2\mu_2(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\mu_1\lambda_1 k_1 + \varepsilon_1\varepsilon_2\mu_2\lambda_2 k_2) \\ &\quad +\varepsilon_1\mu_3(1+\lambda_3 k_1)^2 - \lambda_1 k_1(\varepsilon_1\varepsilon_2\mu_3\lambda_1 k_1 + \mu_4\lambda_2 k_2), \\ \mu_3^* &= \varepsilon_2\mu_4(1+\lambda_3 k_2)^2 - \lambda_2 k_2(\mu_3\lambda_1 k_1 + \varepsilon_1\varepsilon_2\mu_4\lambda_2 k_2). \end{aligned} \tag{4.2}$$

Proof. Let $f_*(X_p) \in T_{f(p)}(M^f)$. Then let us calculate $f_*(X_p)$ and $S^f(f_*(X_p))$. Since ϕ_u and ϕ_v are orthonormal we have

$$\begin{aligned} X_p &= \varepsilon_1 \langle X_p, \phi_u\rangle \phi_u + \varepsilon_2 \langle X_p, \phi_v\rangle \phi_v \\ &= \varepsilon_1 x_1\phi_u + \varepsilon_2 x_2 \phi_v \end{aligned}$$

Further without lost of generality, we suppose that X_p is a unit vector. Then

$$\begin{aligned} f_*(X_p) &= \varepsilon_1 x_1 f_*(\phi_u) + \varepsilon_2 x_2 f_*(\phi_v) \\ &= \varepsilon_1 x_1 \psi_u + \varepsilon_2 x_2 \psi_v. \end{aligned} \tag{4.3}$$

On the other hand we find that

(4.4)

$$\begin{aligned} S^f(f_*(X_p)) &= \varepsilon_1 x_1 S^f(\psi_u) + \varepsilon_2 x_2 S^f(\psi_v) \\ &= \varepsilon_1 x_1 \left(\mu_1(1+\lambda_3 k_1)\phi_u + \mu_2(1+\lambda_3 k_2)\phi_v + (\mu_1\varepsilon_2\lambda_1 k_1 + \mu_2\varepsilon_1\lambda_2 k_2)N\right) \\ &\quad +\varepsilon_2 x_2 \left(\mu_3(1+\lambda_3 k_1)\phi_u + \mu_4(1+\lambda_3 k_2)\phi_v + (\mu_3\varepsilon_2\lambda_1 k_1 + \mu_4\varepsilon_1\lambda_2 k_2)N\right) \end{aligned}$$

Thus using equations (4.3) and (4.4) in equation (2.4) we obtain (4.1). □

Corollary 4.1. *Let M^f be a surface at a constant distance from edge of regression on M in E_1^3. Let k_1 and k_2 denote principal curvature functions of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively. $f_*(X_p) \in T_{f(p)}M^f$ is an asymptotic direction if and only if* *(a)Let N_p be a timelike vector then*

$$\mu_1^* \cos^2\theta_1 + \mu_2^* \cos\theta_1 \cos\theta_2 + \mu_3^* \cos^2\theta_2 = 0.$$

(b) Let N_p be a spacelike vector.
(b.1) If X_p and ϕ_u are timelike vectors in the same timecone then

$$\mu_1^* \cosh^2\theta_1 + \delta_2\mu_2^* \cosh\theta_1 \sinh\theta_2 + \mu_3^* \sinh^2\theta_2 = 0.$$

(b.2) If X_p and ϕ_v are timelike vectors in the same timecone then

$$\mu_1^* \sinh^2\theta_1 + \delta_1\mu_2^* \sinh\theta_1 \cosh\theta_2 + \mu_3^* \cosh^2\theta_2 = 0.$$

(b.3) If X_p is a spacelike vector and ϕ_u is timelike vector then

$$\mu_1^* \sinh^2\theta_1 - \delta_1\delta_2\mu_2^* \sinh\theta_1 \cosh\theta_2 + \mu_3^* \cosh^2\theta_2 = 0.$$

(b.4) If X_p is a spacelike vector and ϕ_v is timelike vector then

$$\mu_1^* \cosh^2\theta_1 - \delta_1\delta_2\mu_2^* \cosh\theta_1 \sinh\theta_2 + \mu_3^* \sinh^2\theta_2 = 0.$$

Abovementioned μ_1^*, μ_2^* and μ_3^* are given in (4.2) and

$$\delta_i = \begin{cases} 1, & x_i \text{ is positive} \\ -1, & x_i \text{ is negative} \end{cases}, \quad i = (1,2).$$

Proof. (a) Let N_p be a timelike vector. In this case θ_1 and θ_2 are spacelike angles then

$$\begin{aligned} x_1 &= \langle X_p, \phi_u\rangle = \cos\theta_1 \\ x_2 &= \langle X_p, \phi_v\rangle = \cos\theta_2. \end{aligned}$$

Substituting these equations in (4.1) the proof is obvious.
(b) Let N_p be a spacelike vector.
(b.1) If X_p and ϕ_u are timelike vectors in the same timecone then there is a hyperbolic angle θ_1 and a Lorentzian timelike angle θ_2. Since

$$x_1 = -\cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2$$

the proof is obvious.
(b.2) If X_p and ϕ_v are timelike vectors in the same timecone then there is a Lorentzian timelike angle θ_1 and a hyperbolic angle θ_2. Thus

$$x_1 = \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = -\cosh\theta_2.$$

(b.3) If X_p is a spacelike vector and ϕ_u is timelike vector then there is a Lorentzian timelike angle θ_1 and a central angle θ_2. Thus

$$x_1 = \delta_1 \sinh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \cosh\theta_2.$$

(b.4) If X_p is a spacelike vector and ϕ_v is timelike vector then there is a central angle θ_1 and a Lorentzian timelike angle θ_2. Thus

$$x_1 = \delta_1 \cosh\theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh\theta_2.$$

□

As a special case if M and M_r be parallel surfaces from (4.1) and (4.2) we obtain that $f_*(X_p) \in T_{f(p)}M_r$ is an asymptotic direction if and only if

$$\varepsilon_1 k_1(1 + rk_1)x_1^2 + \varepsilon_2 k_2(1 + rk_2)x_2^2 = 0.$$

Corollary 4.2. *Let M and M_r be parallel surfaces in E_1^3. Let k_1 and k_2 denote principal curvature function of M and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principal directions on M. Let us denote the angle between $X_p \in T_pM$ and ϕ_u, ϕ_v by θ_1, θ_2 respectively. $f_*(X_p) \in T_{f(p)}M_r$ is an asymptotic direction if and only if*
(a)Let N_p be a timelike vector then

$$k_1(1 + rk_1)\cos^2\theta_1 + k_2(1 + rk_2)\cos^2\theta_2 = 0.$$

(b) Let N_p be a spacelike vector.
(b.1) If X_p and ϕ_u are timelike vectors in the same timecone then

$$-k_1(1 + rk_1)\cosh^2\theta_1 + k_2(1 + rk_2)\sinh^2\theta_2 = 0.$$

(b.2) If X_p and ϕ_v are timelike vectors in the same timecone then

$$k_1(1 + rk_1)\sinh^2\theta_1 - k_2(1 + rk_2)\cosh^2\theta_2 = 0.$$

(b.3) If X_p is a spacelike vector and ϕ_u is timelike vector then

$$-k_1(1 + rk_1)\sinh^2\theta_1 + k_2(1 + rk_2)\cosh^2\theta_2 = 0.$$

(b.4) If X_p is a spacelike vector and ϕ_v is timelike vector then

$$k_1(1 + rk_1)\cosh^2\theta_1 - k_2(1 + rk_2)\sinh^2\theta_2 = 0.$$

References

[1] Aktan N., Görgülü A., Özüsağlam E. and Ekici C., Conjugate Tangent Vectors and Asymptotic Directions for Surfaces at a Constant Distance From Edge of Regression on a Surface, IJPAM, 33, No. 1 (2006), 127-133.

[2] Aktan N., Özüsağlam E. and Görgülü A., The Euler Theorem and Dupin Indicatrix for Surfaces at a Constant Distance from Edge of Regression on a Surface, International Journal of Applied Mathematics &Statistics, 14, No.S09 (2009), 37-43.

[3] Bilici, M. and Çalışkan, M., On the involutes of the spacelike curve with a timelike binormal in Minkowski 3-space, International Mathematical Forum, 4, no.31, 1497-1509, (2009).

[4] Çöken A. C., Dupin Indicatrix for Pseudo-Euclidean Hypersurfaces in Pseudo-Euclidean Space R_v^{n+1}, Bull. Cal. Math. Soc., 89 (1997), 343-348.

[5] Çöken A. C., The Euler Theorem and Dupin indicatrix for Parallel Pseudo-Euclidean Hypersurfaces in Pseudo-Euclidean Space in Semi-Euclidean Space E_ν^{n+1}, Hadronic Journal Supplement, 16, (2001), 151-162.

[6] Duggal K. L., Bejancu A., Lightlike submanifolds of semi-Riemannian manifolds and it's applications, Kluwer Dortrecth, 1996.

[7] Görgülü A., Çöken A. C., The Euler Theorem for Parallel Pseudo-Euclidean Hypersurfaces in Pseudo- Euclidean Space E_1^{n+1}, Jour Inst.Math. & Comp. Sci. (Math. Ser.), 6, No.2 (1993), 161-165.

[8] Görgülü A., Çöken A. C., The Dupin indicatrix for Parallel Pseudo-Euclidean Hypersurfaces in Pseudo-Euclidean Space in Semi-Euclidean Space E_1^{n+1}, Journ. Inst. Math. and Comp. Sci. (Math Series), **7**, No.3 (1994), 221-225.

[9] Hacısalihoğlu H. H., Diferensiyel Geometri, İnönü Üniversitesi Fen Edeb. Fak. Yayınları, Mat. No.2 895s., 1983.

[10] Kazaz, M., Uğurlu, H. H., Onder, M.and Kahraman M., Mannheim partner D-curves in Minkowski 3-space E_1^3, arXiv: 1003.2043v3 [math.DG].

[11] Kazaz M., Önder M. , Mannheim offsets of timelike ruled surfaces in Minkowski 3-space arXiv:0906.2077v5 [math.DG].

[12] Kazaz M., Uğurlu H. H. ,Önder M., Mannheim offsets of spacelike ruled surfaces in Minkowski 3-space, arXiv:0906.4660v3 [math.DG].

[13] Kılıç A. and Hacısalihoğlu H. H., Euler's Theorem and the Dupin Representation for Parallel Hypersurfaces, Journal of Sci. and Arts of Gazi Univ. Ankara, 1, No.1 (1984), 21-26.

[14] O'Neill B., Semi-Riemannian Geometry With Applications To Relativity, Academic Press, New York, London,1983.
[15] Sağlam D., Boyacıoğlu Kalkan Ö , Surfaces at a constant distance from edge of regression on a surface in E_1^3, Differential Geometry-Dynamical Systems, 12, (2010), 187-200.
[16] Sağlam D., Kalkan Boyacıoğlu Ö., The Euler Theorem and Dupin Indicatrix for Surfaces at a Constant Distance from Edge of Regression on a Surface in E_1^3, Matematicki Vesnik, 65, No.2 (2013), 242–249.
[17] Tarakci Ö., Hacısalihoğlu H. H. , Surfaces at a constant distance from edge of regression on a surface, Applied Mathematics and Computation, 155, (2004), 81-93.

[1] Gazi University, Polatli Science and Art Faculty, Department of Mathematics, Polatli-TURKEY
E-mail address: deryasaglam@gazi.edu.tr

[2] Afyon Vocational School, Afyon Kocatepe University, Afyon - Turkey
E-mail address: bozgur@aku.edu.tr