CONJUGATE TANGENT VECTORS AND ASYMPTOTIC DIRECTIONS FOR SURFACES AT A CONSTANT DISTANCE FROM EDGE OF REGRESSION ON A SURFACE IN $E_3^1$

DERYA SAĞLAM$^1$ AND ÖZGÜR BOYACİOĞLU KALKAN$^2$

Abstract. In this paper we give conjugate tangent vectors and asymptotic directions for surfaces at a constant distance from edge of regression on a surface in $E_3^1$.

1. Introduction

Conjugate tangent vectors and asymptotic directions in Euclidean space $E^3$ can be found in [9]. In 1984, A. Kılıç and H. H. Hacisalihoğlu found the Euler theorem and Dupin indicatrix for parallel hypersurfaces in $E^n$ [13]. Also the Euler theorem and Dupin indicatrix are obtained for the parallel hypersurfaces in pseudo-Euclidean spaces $E_1^{n+1}$ and $E_0^{n+1}$ in the papers ([5], [7], [8]).

In 2005 H. H. Hacisalihoğlu and Ö. Tarakçı introduced surfaces at a constant distance from edge of regression on a surface. These surfaces are a generalization of parallel surfaces in $E_3^3$. Because the authors took any vector instead of normal vector [17]. Euler theorem and Dupin indicatrix for these surfaces are given in [2]. Conjugate tangent vectors and asymptotic directions are given in [1]. In 2010 we obtained the surfaces at a constant distance from edge of regression on a surface in $E_3^1$ [15]. We obtained the Euler theorem and Dupin indicatrix for these surfaces in $E_3^1$ [16].

In this paper we give conjugate tangent vectors and asymptotic directions for surfaces at a constant distance from edge of regression on a surface in $E_3^1$.

2. Preliminaries

Let $E_3^1$ be the Minkowski 3-space is the real vector space $R^3$ endowed with the standard flat Lorentzian metric given by

$$\langle , \rangle = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2$$

*Date:* January 1, 2013 and, in revised form, February 2, 2013.

2000 *Mathematics Subject Classification.* 51B20, 53B30.

*Key words and phrases.* conjugate tangent vectors, asymptotic direction, edge of regression.
where \((x_1, x_2, x_3)\) is a rectangular coordinate system of \(E^3_1\). An arbitrary vector \(x \in E^3_1\) is called spacelike if \(\langle x, x \rangle > 0\) or \(x = 0\), timelike if \(\langle x, x \rangle < 0\) and lightlike (null) if \(\langle x, x \rangle = 0\) and \(x \neq 0\).

The timelike-cone of \(E^3_1\) is defined as the set of all timelike vectors of \(E^3_1\), that is
\[
\mathcal{T} = \{(x, y, z) \in E^3_1: x^2 + y^2 - z^2 < 0\}.
\]
The set of lightlike vectors is defined by \(\mathcal{C}\) and it is the following set:
\[
\mathcal{C} = \{(x, y, z) \in E^3_1: x^2 + y^2 - z^2 = 0\} - \{(0, 0, 0)\}.
\]
The cross product \(x \times y\) of vectors \(x = (x_1, x_2, x_3)\) and \(y = (y_1, y_2, y_3)\) in \(E^3_1\) is defined as
\[
\langle x \times y, z \rangle = \det(x, y, z) \quad \text{for all} \ z = (z_1, z_2, z_3) \in E^3_1.
\]
More explicitly, if \(x, y\) belong to \(E^3_1\), then
\[
\begin{align*}
\langle x, y \rangle &= -x_1y_1 + x_2y_2 + x_3y_3 \\
x \times y &= \begin{pmatrix} x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1 \end{pmatrix} \\
(a \times b, x \times y) &= -\frac{1}{2} \begin{vmatrix} \langle a, x \rangle & \langle b, x \rangle \\ \langle a, y \rangle & \langle b, y \rangle \end{vmatrix}
\end{align*}
\]
where \(a = (a_1, a_2, a_3)\) and \(b = (b_1, b_2, b_3)\) in \(E^3_1\) (Lagrange identity in \(E^3_1\)).

Let \(e_1, e_2 \in E^3_1\) be such that \(< e_1, e_1 > = \pm 1\) and \(\langle e_1, e_2 \rangle = 0\) and \(e_3 = e_1 \times e_2\). Then these three vectors form an orthonormal frame. If \(\langle e_1, e_1 \rangle = \varepsilon_1\) and \(\langle e_2, e_2 \rangle = \varepsilon_2\) where \(\varepsilon_1, \varepsilon_2 = \pm 1\), it follows from the Lagrange identity that \(\langle e_3, e_3 \rangle = -\varepsilon_1\varepsilon_2\).

Each vector \(x \in E^3_1\) can be written uniquely in terms of \(e_1, e_2, e_3\) by
\[
x = \varepsilon_1 \langle x, e_1 \rangle e_1 + \varepsilon_2 \langle x, e_2 \rangle e_2 - \varepsilon_1\varepsilon_2 \langle x, e_3 \rangle e_3.
\]

The angle between two vectors in Minkowski 3-space is defined by ([3], [10], [11], [12]):

**Definition 2.1.**

i. **Hyperbolic angle:** Let \(x\) and \(y\) be timelike vectors in the same timecone of Minkowski space. Then there is a unique real number \(\theta \geq 0\), called the hyperbolic angle between \(x\) and \(y\), such that
\[
< x, y > = -\|x\| \|y\| \cosh \theta.
\]

ii. **Central angle:** Let \(x\) and \(y\) be spacelike vectors in Minkowski space that span a timelike vector subspace. Then there is a unique real number \(\theta \geq 0\), called the central angle between \(x\) and \(y\), such that
\[
< x, y > = \|x\| \|y\| \cos \theta.
\]

iii. **Spacelike angle:** Let \(x\) and \(y\) be spacelike vectors in Minkowski space that span a spacelike vector subspace. Then there is a unique real number \(\theta \geq 0\) and \(\pi\) called the spacelike angle between \(x\) and \(y\), such that
\[
< x, y > = \|x\| \|y\| \cos \theta.
\]

iv. **Lorentzian timelike angle:** Let \(x\) be a spacelike vector and \(y\) be a timelike vector in Minkowski space. Then there is a unique real number \(\theta \geq 0\), called the Lorentzian timelike angle between \(x\) and \(y\), such that
\[
< x, y > = \|x\| \|y\| \sinh \theta.
\]
Definition 2.2. Let $M$ and $M^f$ be two surfaces in $E^3_1$ and $N_p$ be a unit normal vector of $M$ at the point $P \in M$. Let $T_pM$ be tangent space at $P \in M$ and $\{X_p, Y_p\}$ be an orthonormal bases of $T_pM$. Let $Z_p = d_1X_p + d_2Y_p + d_3N_p$ be a unit vector, where $d_1, d_2, d_3 \in \mathbb{R}$ are constant numbers and $\epsilon_1d_1^2 + \epsilon_2d_2^2 - \epsilon_1\epsilon_2d_3^2 = \pm 1$. If a function $f$ exists and satisfies the condition $f : M \to M^f$, $f(P) = P + rZ_p$, $r$ constant, $M^f$ is called as the surface at a constant distance from the edge of regression on $M$ and $M^f$ denoted by the pair $(M, M^f)$.

If $d_1 = d_2 = 0$, then we have $Z_p = N_p$ and $f(P) = P + rN_p$. In this case $M$ and $M^f$ are parallel surfaces [15].

Theorem 2.1. Let the pair $(M, M^f)$ be given in $E_1^3$. For any $W \in \chi(M)$, we have $f_*(W) = \overline{W} + rD_{\overline{W}}Z$, where $W = \sum_{i=1}^{3} w_i \frac{\partial}{\partial x_i}$, $\overline{W} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}$ and $\forall P \in M$, $w_i(P) = \pi_i(f(p))$, $1 \leq i \leq 3$ [15].

Let $(\phi, U)$ be a parametrization of $M$, so we can write that

$$\phi : U_{(u,v)} \subset E_1^3 \to M_{P=\phi(u,v)}.$$ 

In this case $\{\phi_u|_P, \phi_v|_P\}$ is a basis of $T_pM$. Let $N_p$ is a unit normal vector at $P \in M$ and $d_1, d_2, d_3 \in \mathbb{R}$ be constant numbers then we can write that $Z_p = d_1\phi_u|_P + d_2\phi_v|_P + d_3N_p$. Since $M^f = \{f(P) \mid f(P) = P + rZ_p\}$, a parametric representation of $M^f$ is $\psi(u, v) = \phi(u, v) + rZ(u, v)$. Thus we can write

$$M^f = \{\psi(u, v) \mid \psi(u, v) = \phi(u, v) + r(d_1\phi_u(u, v) + d_2\phi_v(u, v) + d_3N(u, v)),
\quad d_1, d_2, d_3, r \text{ are constant, } \epsilon_1d_1^2 + \epsilon_2d_2^2 - \epsilon_1\epsilon_2d_3^2 = \pm 1\}.$$ 

If we take $rd_1 = \lambda_1$, $rd_2 = \lambda_2$, $rd_3 = \lambda_3$ then we have

$$M^f = \{\psi(u, v) \mid \psi(u, v) = \phi(u, v) + \lambda_1\phi_u(u, v) + \lambda_2\phi_v(u, v) + \lambda_3N(u, v),
\quad \lambda_1, \lambda_2, \lambda_3 \text{ are constant}\}.$$ 

Let $\{\phi_u, \phi_v\}$ is basis of $\chi(M^f)$. If we take $(\phi_u, \phi_u) = \epsilon_1$, $(\phi_v, \phi_v) = \epsilon_2$ and $(N, N) = -\epsilon_1\epsilon_2$, then

$$\psi_u = (1 + \lambda_3k_1)\phi_u + \epsilon_2\lambda_1k_1N,$$

$$\psi_v = (1 + \lambda_3k_2)\phi_v + \epsilon_1\lambda_2k_2N$$

is a basis of $\chi(M^f)$, where $N$ is the unit normal vector field on $M$ and $k_1, k_2$ are principal curvatures of $M$ [15].

Theorem 2.2. Let the pair $(M, M^f)$ be given in $E_1^3$. Let $\{\phi_u, \phi_v\}$ (orthonormal and principal vector fields on $M$) be basis of $\chi(M)$ and $k_1, k_2$ be principal curvatures of $M$. The matrix of the shape operator of $M^f$ with respect to the basis $\{\psi_u = (1 + \lambda_3k_1)\phi_u + \epsilon_2\lambda_1k_1N, \psi_v = (1 + \lambda_3k_2)\phi_v + \epsilon_1\lambda_2k_2N\}$ of $\chi(M^f)$ is

$$S^f = \begin{bmatrix}
\mu_1 & \mu_2 \\
\mu_3 & \mu_4
\end{bmatrix}$$
Let \( M \) be a surface in \( \mathbb{E}^3 \) and \( S \) be shape operator of \( M \). For any \( X_p, Y_p \in T_pM \), if
\[
\langle S(X_p), Y_p \rangle = 0
\]
then \( X_p \) and \( Y_p \) are called conjugate tangent vectors of \( M \) at \( p \) [9].

**Definition 2.4.** Let \( M \) be an Euclidean surface in \( \mathbb{E}^3 \) and \( S \) be shape operator of \( M \). For any \( X_p \in T_pM \), if
\[
\langle S(X_p), X_p \rangle = 0
\]
then \( X_p \) is called an asymptotic direction of \( M \) at \( p \) [9].

We can get the definitions of conjugate tangent vectors and asymptotic direction in Minkowski 3-space similar to Definition 2.3 and 2.4 as below:

**Definition 2.5.** Let \( M \) be a surface in \( \mathbb{E}_1^3 \) and \( S \) be shape operator of \( M \). For any \( X_p, Y_p \in T_pM \), if
\[
\langle S(X_p), Y_p \rangle = 0
\]
then \( X_p \) and \( Y_p \) are called conjugate tangent vectors of \( M \) at \( p \).

**Definition 2.6.** Let \( M \) be a surface in \( \mathbb{E}_1^3 \) and \( S \) be shape operator of \( M \). For any \( X_p \in T_pM \), if
\[
\langle S(X_p), X_p \rangle = 0
\]
then \( X_p \) is called an asymptotic direction of \( M \) at \( p \).

3. Conjugate tangent vectors for surfaces at a constant distance from edge of regression on a surface in \( \mathbb{E}_1^3 \)

**Theorem 3.1.** Let \( M' \) be a surface at a constant distance from edge of regression on a \( M \) in \( \mathbb{E}_1^3 \). Let \( k_1 \) and \( k_2 \) denote principal curvature function of \( M \) and let \( \{ \phi_u, \phi_v \} \) be orthonormal basis such that \( \phi_u \) and \( \phi_v \) are principal directions on \( M \). For \( X_p, Y_p \in T_pM \), \( f_*(X_p) \) and \( f_*(Y_p) \) are conjugate tangent vectors if and only if
\[
\varepsilon_1 \mu_1^* x_1 y_1 + \varepsilon_2 \mu_2^* x_1 y_2 + \varepsilon_2 \mu_2^* x_2 y_1 + \varepsilon_2 \mu_2^* x_2 y_2 = 0
\]
Let
\[
\begin{align*}
x_1 &= (X_p, \phi_u), & x_2 &= (X_p, \phi_v), \\
y_1 &= (Y_p, \phi_u), & y_2 &= (Y_p, \phi_v),
\end{align*}
\]  
(3.2)

\(\mu_1^* = \mu_1(1 + \lambda_3k_1)^2 - \lambda_1k_1(e_2\mu_1\lambda_1k_1 + e_1\mu_2\lambda_2k_2),\)
\(\mu_2^* = \mu_2(1 + \lambda_3k_2)^2 - \lambda_2k_2(e_2\mu_1\lambda_1k_1 + e_1\mu_2\lambda_2k_2),\)
(3.5)
\(\mu_3^* = \mu_3(1 + \lambda_3k_1)^2 - \lambda_1k_1(e_2\mu_3\lambda_1k_1 + e_1\mu_4\lambda_2k_2),\)
\(\mu_4^* = \mu_4(1 + \lambda_3k_2)^2 - \lambda_2k_2(e_2\mu_3\lambda_1k_1 + e_1\mu_4\lambda_2k_2).\)

Thus using equations (3.4) and (3.6) in equation (2.3) we obtain (3.1).

Proof. Let \(f_*(X_p) \in T_{f(p)}M^f.\) Then let us calculate \(f_*(X_p)\) and \(S^f(f_*(X_p)).\) Since \(\phi_u\) and \(\phi_v\) are orthonormal we have
\[
X_p = \varepsilon_1 (X_p, \phi_u) \phi_u + \varepsilon_2 (X_p, \phi_v) \phi_v
= \varepsilon_1 x_1 \phi_u + \varepsilon_2 x_2 \phi_v.
\]

Further without lost of generality, we suppose that \(X_p\) is a unit vector. Then
\[
f_*(X_p) = \varepsilon_1 x_1 f_*(\phi_u) + \varepsilon_2 x_2 f_*(\phi_v)
= \varepsilon_1 x_1 \psi_u + \varepsilon_2 x_2 \psi_v.
\]

On the other hand we find that
\[
S^f(f_*(X_p)) = \varepsilon_1 x_1 S^f(\psi_u) + \varepsilon_2 x_2 S^f(\psi_v)
= \varepsilon_1 x_1 (\mu_1(1 + \lambda_3k_1)\phi_u + \mu_2(1 + \lambda_3k_2)\phi_v + (\mu_1\varepsilon_2\lambda_1k_1 + \mu_2\varepsilon_1\lambda_2k_2)N)
+ \varepsilon_2 x_2 (\mu_3(1 + \lambda_3k_1)\phi_u + \mu_4(1 + \lambda_3k_2)\phi_v + (\mu_3\varepsilon_2\lambda_1k_1 + \mu_4\varepsilon_1\lambda_2k_2)N)
\]
and for \(Y_p \in T_p M\) we have
\[
Y_p = \varepsilon_1 (Y_p, \phi_u) \phi_u + \varepsilon_2 (Y_p, \phi_v) \phi_v
= \varepsilon_1 y_1 \phi_u + \varepsilon_2 y_2 \phi_v.
\]

Then
\[
f_*(Y_p) = \varepsilon_1 y_1 f_*(\phi_u) + \varepsilon_2 y_2 f_*(\phi_v)
= \varepsilon_1 y_1 \psi_u + \varepsilon_2 y_2 \psi_v.
\]

Thus using equations (3.4) and (3.6) in equation (2.3) we obtain (3.1).

\[\square\]

Theorem 3.2. Let \(M^f\) be a surface at a constant distance from edge of regression on \(M\) in \(E^3.\) Let \(k_1\) and \(k_2\) denote principal curvatures of \(M\) and let \(\{\phi_u, \phi_v\}\) be orthonormal basis such that \(\phi_u\) and \(\phi_v\) are principal directions on \(M.\)

Let us denote the angle between \(X_p \in T_p M\) and \(\phi_u, \phi_v\) by \(\theta_1, \theta_2\) respectively and the angle between \(Y_p \in T_p M\) and \(\phi_u, \phi_v\) by \(\theta'_1, \theta'_2\) respectively. \(f_*(X_p)\) and \(f_*(Y_p)\) are conjugate tangent vectors if and only if
\(\theta'_1 = \theta_1, \theta'_2 = \theta_2\)

(a) Let \(N_p\) be a timelike vector then
\(\mu_1^* \cos \theta_1 \cos \theta'_1 + \mu_2^* \cos \theta_1 \cos \theta'_2 + \mu_3^* \cos \theta_2 \cos \theta'_1 + \mu_4^* \cos \theta_2 \cos \theta'_2 = 0.\)

(b) Let \(\phi_u\) be a timelike vector.

(b.1) If \(X_p\) and \(Y_p\) are spacelike vectors then
\[0 = -\delta_1^2 \mu_1^* \sinh \theta_1 \sinh \theta'_1 - \delta_1^2 \mu_3^* \sinh \theta_1 \cosh \theta'_2
+ \delta_1^2 \mu_2^* \cosh \theta_2 \sinh \theta'_1 + \delta_2^2 \mu_4^* \cosh \theta_2 \cosh \theta'_2.\]
(b.2) If \(X_p, Y_p\) and \(\phi_u\) are timelike vectors in the same timecone then
\[
0 = \mu_1^t \cosh \theta_1 \cosh \theta_1' + \delta \mu_2^t \cosh \theta_1 \sinh \theta_2' \\
-\delta \mu_2^t \sinh \theta_2 \cosh \theta_1' + \delta \mu_2^t \sinh \theta_2 \cosh \theta_2',
\]
(b.3) If \(X_p, \phi_u\) are timelike vectors in the same timecone and \(Y_p\) is spacelike vector then
\[
0 = \delta \mu_1^t \cosh \theta_1 \sinh \theta_1' + \delta \mu_2^t \cosh \theta_2 \sinh \theta_1' \\
+ \delta \mu_2^t \sinh \theta_2' \cosh \theta_1' + \delta \mu_2^t \sinh \theta_2' \cosh \theta_2',
\]
(b.4) If \(Y_p\) and \(\phi_u\) are timelike vectors in the same timecone and \(X_p\) is spacelike vector then
\[
0 = \delta \mu_1^t \sinh \theta_1 \cosh \theta_1' - \delta \mu_2^t \sinh \theta_2 \cosh \theta_1' \\
-\delta \mu_2^t \cosh \theta_2 \cosh \theta_1' + \delta \mu_2^t \cosh \theta_2 \cosh \theta_2'.
\]
(c.1) If \(X_p\) and \(Y_p\) are spacelike vectors then
\[
0 = \delta \mu_1^t \sinh \theta_1 \cosh \theta_1' - \delta \mu_2^t \sinh \theta_2 \cosh \theta_1' \\
-\delta \mu_2^t \cosh \theta_2 \cosh \theta_1' - \delta \mu_2^t \cosh \theta_2 \cosh \theta_2'.
\]
(c.2) If \(X_p, Y_p\) and \(\phi_u\) are timelike vectors in the same timecone then
\[
0 = \delta \mu_1^t \sinh \theta_1 \cosh \theta_1' - \delta \mu_2^t \sinh \theta_2 \cosh \theta_1' \\
+ \delta \mu_2^t \cosh \theta_2 \cosh \theta_1' + \delta \mu_2^t \cosh \theta_2 \cosh \theta_2'.
\]
(c.3) If \(X_p\) and \(\phi_u\) are timelike vectors in the same timecone and \(Y_p\) is spacelike vector then
\[
0 = \delta \mu_1^t \cosh \theta_1 \sinh \theta_1' - \delta \mu_2^t \cosh \theta_2 \sinh \theta_1' \\
+ \delta \mu_2^t \cosh \theta_2 \cosh \theta_1' + \delta \mu_2^t \cosh \theta_2 \cosh \theta_2'.
\]
(c.4) If \(Y_p\) and \(\phi_u\) are timelike vectors in the same timecone and \(X_p\) is spacelike vector then
\[
0 = \delta \mu_1^t \cosh \theta_1 \sinh \theta_1' - \delta \mu_2^t \cosh \theta_2 \sinh \theta_1' \\
-\delta \mu_2^t \sinh \theta_2' \cosh \theta_1' + \delta \mu_2^t \sinh \theta_2' \cosh \theta_2'.
\]
Above mentioned \(\mu_1^t, \mu_2^t, \mu_3^t\) and \(\mu_4^t\) are given in (3.2),
\[
\delta_i = \begin{cases} 
1, & x_i \text { is positive} \\
-1, & x_i \text { is negative}
\end{cases} \quad i = (1, 2)
\]
and
\[
\delta_i' = \begin{cases} 
1, & y_i \text { is positive} \\
-1, & y_i \text { is negative}
\end{cases} \quad i = (1, 2).
\]
Proof. (a) Let \(N_p\) be a timelike vector. In this case \(\theta_1, \theta_2, \theta_1', \theta_2'\) are spacelike angles then
\[
x_1 = \langle X_p, \phi_u \rangle = \cos \theta_1 \\
x_2 = \langle X_p, \phi_u \rangle = \cos \theta_2.
\]
and
\[
y_1 = \langle Y_p, \phi_u \rangle = \cos \theta_1' \\
y_2 = \langle Y_p, \phi_u \rangle = \cos \theta_2'.
\]
Substituting these equations in (3.1) the proof is obvious.
(b) Let $\phi_v$ be a timelike vector.

(b.1) If $X_p$ and $Y_p$ are spacelike vectors and $\phi_v$ is timelike vector then there are Lorentzian timelike angles $\theta_1$, $\theta'_1$ and central angles $\theta_2$, $\theta'_2$. Thus

\[
\begin{align*}
  x_1 &= \delta_1 \sinh \theta_1 \text{ and } x_2 = \delta_2 \cosh \theta_2 \\
  y_1 &= \delta'_1 \sinh \theta'_1 \text{ and } y_2 = \delta'_2 \cosh \theta'_2.
\end{align*}
\]

(b.2) If $X_p$, $Y_p$ and $\phi_v$ are timelike vectors in the same timecone then there are hyperbolic angles $\theta_1$, $\theta'_1$ and Lorentzian timelike angles $\theta_2$, $\theta'_2$. Thus

\[
\begin{align*}
  x_1 &= -\cosh \theta_1 \text{ and } x_2 = \delta_2 \sinh \theta_2 \\
  y_1 &= -\cosh \theta'_1 \text{ and } y_2 = \delta'_2 \sinh \theta'_2.
\end{align*}
\]

(b.3) If $X_p$ and $\phi_v$ are timelike vectors in the same timecone and $Y_p$ is spacelike vector then there is a hyperbolic angle $\theta_1$, a central angle $\theta'_1$ and there are Lorentzian timelike angles $\theta_2$, $\theta'_2$. Thus

\[
\begin{align*}
  x_1 &= -\cosh \theta_1 \text{ and } x_2 = \delta_2 \sinh \theta_2 \\
  y_1 &= -\cosh \theta'_1 \text{ and } y_2 = \delta'_2 \sinh \theta'_2.
\end{align*}
\]

(c) Let $\phi_v$ be a timelike vector.

(c.1) If $X_p$ and $Y_p$ are spacelike vectors and $\phi_v$ is timelike vector then there are central angles $\theta_1$, $\theta'_1$ and Lorentzian timelike angles $\theta_2$, $\theta'_2$. Thus

\[
\begin{align*}
  x_1 &= \delta_1 \cosh \theta_1 \text{ and } x_2 = \delta_2 \sinh \theta_2 \\
  y_1 &= \delta'_1 \cosh \theta'_1 \text{ and } y_2 = \delta'_2 \sinh \theta'_2.
\end{align*}
\]

(c.2) If $X_p$, $Y_p$ and $\phi_v$ are timelike vectors in the same timecone then there are Lorentzian timelike angles $\theta_1$, $\theta'_1$ and hyperbolic angles $\theta_2$, $\theta'_2$. Thus

\[
\begin{align*}
  x_1 &= \delta_1 \sinh \theta_1 \text{ and } x_2 = -\cosh \theta_2 \\
  y_1 &= \delta'_1 \sinh \theta'_1 \text{ and } y_2 = -\cosh \theta'_2.
\end{align*}
\]

(c.3) If $X_p$ and $\phi_v$ are timelike vectors in the same timecone and $Y_p$ is spacelike vector then there is a hyperbolic angle $\theta_1$, a central angle $\theta'_1$ and there are Lorentzian timelike vectors $\theta_2$, $\theta'_2$. Thus

\[
\begin{align*}
  x_1 &= \delta_1 \sinh \theta_1 \text{ and } x_2 = -\cosh \theta_2 \\
  y_1 &= \delta'_1 \cosh \theta'_1 \text{ and } y_2 = \delta'_2 \sinh \theta'_2.
\end{align*}
\]

(c.4) If $Y_p$ and $\phi_v$ are timelike vectors in the same timecone and $X_p$ is spacelike vector then then there is a central angle $\theta_1$, a hyperbolic angle $\theta'_1$ and there are Lorentzian timelike angles $\theta'_2$, $\theta_2$. Thus

\[
\begin{align*}
  x_1 &= \delta_1 \cosh \theta_1 \text{ and } x_2 = \delta_2 \sinh \theta_2 \\
  y_1 &= \delta'_1 \sinh \theta'_1 \text{ and } y_2 = -\cosh \theta'_2.
\end{align*}
\]
As a special case if we take $\lambda_1 = \lambda_2 = 0, \lambda_3 = r = \text{constant}$, then we obtain that $M$ and $M'$ are parallel surfaces. Hence we give the following corollaries.

**Corollary 3.1.** Let $M$ and $M_r$ be parallel surfaces in $E^3_1$. Let $k_1$ and $k_2$ denote principal curvature functions of $M$ and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that $\phi_u$ and $\phi_v$ are principal directions on $M$. Let us denote the angle between $X_p \in T_p M$ and $\phi_u$, $\phi_v$ by $\theta_1, \theta_2$ respectively and the angle between $Y_p \in T_p M$ and $\phi_u$, $\phi_v$ by $\theta'_1, \theta'_2$ respectively. $f_*(X_p)$ and $f_*(Y_p)$ are conjugate tangent vectors if and only if

$$
(3.7) \quad \varepsilon_1 k_1 (1 + r k_1) x_1 y_1 + \varepsilon_2 k_2 (1 + r k_2) x_2 y_2 = 0.
$$

**Proof.** Since

$$
\mu'_1 = k_1 (1 + r k_1), \quad \mu'_2 = 0, \quad \mu'_3 = 0, \quad \mu'_4 = k_2 (1 + r k_2)
$$

from (3.1) we find (3.7). \hfill \Box

**Corollary 3.2.** Let $M$ and $M_r$ be parallel surfaces in $E^3_1$. Let $k_1$ and $k_2$ denote principal curvature functions of $M$ and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that $\phi_u$ and $\phi_v$ are principal directions on $M$. Let us denote the angle between $X_p \in T_p M$ and $\phi_u$, $\phi_v$ by $\theta_1, \theta_2$ respectively and the angle between $Y_p \in T_p M$ and $\phi_u$, $\phi_v$ by $\theta'_1, \theta'_2$ respectively. $f_*(Y_p)$ are conjugate tangent vectors if and only if

(a) Let $N_p$ be a timelike vector then

$$
k_1 (1 + r k_1) \cos \theta_1 \cos \theta'_1 + k_2 (1 + r k_2) \cos \theta_2 \cos \theta'_2 = 0.
$$

(b) Let $\phi_u$ be a timelike vector.

(b.1) If $X_p$ and $Y_p$ are spacelike vectors then

$$
-\delta_1 \delta'_1 k_1 (1 + r k_1) \sinh \theta_1 \sinh \theta'_1 + \delta_2 \delta'_2 k_2 (1 + r k_2) \cosh \theta_2 \cosh \theta'_2 = 0.
$$

(b.2) If $X_p$, $Y_p$ and $\phi_u$ are timelike vectors in the same timecone then

$$
-k_1 (1 + r k_1) \cosh \theta_1 \cosh \theta'_1 + k_2 (1 + r k_2) \sinh \theta_2 \sinh \theta'_2 = 0.
$$

(b.3) If $X_p$ and $\phi_u$ are timelike vectors in the same timecone and $Y_p$ is spacelike vector then

$$
\delta'_1 k_1 (1 + r k_1) \cosh \theta_1 \sinh \theta'_1 + \delta_2 \delta'_2 k_2 (1 + r k_2) \sinh \theta_2 \cosh \theta'_2 = 0.
$$

(b.4) If $Y_p$ and $\phi_u$ are timelike vectors in the same timecone and $X_p$ is spacelike vector then

$$
\delta_1 k_1 (1 + r k_1) \cosh \theta_1 \sinh \theta'_1 + \delta_2 \delta'_2 k_2 (1 + r k_2) \cosh \theta_2 \sinh \theta'_2 = 0.
$$

(c) Let $\phi_v$ be a timelike vector.

(c.1) If $X_p$ and $Y_p$ are spacelike vectors then

$$
\delta_1 \delta'_1 k_1 (1 + r k_1) \cosh \theta_1 \cosh \theta'_1 - \delta_2 \delta'_2 k_2 (1 + r k_2) \sinh \theta_2 \sinh \theta'_2 = 0.
$$

(c.2) If $X_p$, $Y_p$ and $\phi_v$ are timelike vectors in the same timecone then

$$
\delta_1 \delta'_1 k_1 (1 + r k_1) \sinh \theta_1 \sinh \theta'_1 - k_2 (1 + r k_2) \cosh \theta_2 \cosh \theta'_2 = 0.
$$

(c.3) If $X_p$ and $\phi_v$ are timelike vectors in the same timecone and $Y_p$ is spacelike vector then

$$
\delta_1 \delta'_1 k_1 (1 + r k_1) \sinh \theta_1 \cosh \theta'_1 + \delta_2 \delta'_2 k_2 (1 + r k_2) \cosh \theta_2 \sinh \theta'_2 = 0.
$$
(c.4) If $Y_p$ and $\phi_v$ are timelike vectors in the same timecone and $X_p$ is spacelike vector then
\[ \delta_1 \delta'_1 k_1 (1 + r k_1) \cosh \theta_1 \sinh \theta'_1 + \delta_2 k_2 (1 + r k_2) \sinh \theta_2 \cosh \theta'_2 = 0. \]
For the above equations
\[ \delta_i = \begin{cases} 1, & x_i \text{ is positive} \\ -1, & x_i \text{ is negative} \end{cases}, \quad i = (1,2) \]
and
\[ \delta'_i = \begin{cases} 1, & y_i \text{ is positive} \\ -1, & y_i \text{ is negative} \end{cases}, \quad i = (1,2). \]

4. Asymptotic directions for surfaces at a constant distance from edge of regression on a surface in $E^3_1$

**Theorem 4.1.** Let $M^f$ be a surface at a constant distance from edge of regression on a $M$ in $E^3_1$. Let $k_1$ and $k_2$ denote principal curvature functions of $M$ and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that $\phi_u$ and $\phi_v$ are principal directions on $M$. $f_s(X_p) \in T_{f(p)}(M^f)$ is an asymptotic direction if and only if
\[ \mu_1^* x_1^2 + \varepsilon_1 \varepsilon_2 \mu_2^2 x_1 x_2 + \mu_3^* x_2^2 = 0 \]

where
\[ \mu_1^* = \varepsilon_1 \varepsilon_2 (1 + \lambda_3 k_1)^2 - \lambda_1 k_1 (\varepsilon_1 \varepsilon_2 \mu_1 \lambda_1 k_1 + \mu_2 \lambda_2 k_2), \]
\[ \mu_2^* = \varepsilon_2 \mu_2 (1 + \lambda_3 k_2)^2 - \lambda_2 k_2 (\mu_1 \lambda_1 k_1 + \varepsilon_1 \varepsilon_2 \mu_2 \lambda_2 k_2) + \varepsilon_1 \varepsilon_2 \lambda_1 k_1 (\varepsilon_2 \mu_3 \lambda_3 k_1 + \mu_4 \lambda_2 k_2), \]
\[ \mu_3^* = \varepsilon_2 \mu_4 (1 + \lambda_3 k_2)^2 - \lambda_2 k_2 (\mu_3 \lambda_1 k_1 + \varepsilon_1 \varepsilon_2 \mu_4 \lambda_2 k_2). \]

**Proof.** Let $f_s(X_p) \in T_{f(p)}(M^f)$. Then let us calculate $f_s(X_p)$ and $S^f(f_s(X_p))$. Since $\phi_u$ and $\phi_v$ are orthonormal we have
\[ X_p = \varepsilon_1 (X_p, \phi_u) \phi_u + \varepsilon_2 (X_p, \phi_v) \phi_v \]
\[ = \varepsilon_1 x_1 \phi_u + \varepsilon_2 x_2 \phi_v \]
Further without lost of generality, we suppose that $X_p$ is a unit vector. Then
\[ f_s(X_p) = \varepsilon_1 x_1 f_s(\phi_u) + \varepsilon_2 x_2 f_s(\phi_v) \]
\[ = \varepsilon_1 x_1 \psi_u + \varepsilon_2 x_2 \psi_v. \]

On the other hand we find that
\[ S^f(f_s(X_p)) = \varepsilon_1 x_1 S^f(\psi_u) + \varepsilon_2 x_2 S^f(\psi_v) \]
\[ = \varepsilon_1 x_1 (\mu_1 (1 + \lambda_3 k_1) \phi_u + \mu_2 (1 + \lambda_3 k_2) \phi_v + (\mu_1 \varepsilon_2 \lambda_1 k_1 + \mu_2 \varepsilon_1 \lambda_2 k_2) N) \]
\[ + \varepsilon_2 x_2 (\mu_3 (1 + \lambda_3 k_1) \phi_u + \mu_4 (1 + \lambda_3 k_2) \phi_v + (\mu_3 \varepsilon_2 \lambda_1 k_1 + \mu_4 \varepsilon_1 \lambda_2 k_2) N) \]
Thus using equations (4.3) and (4.4) in equation (2.4) we obtain (4.1). \[ \square \]

**Corollary 4.1.** Let $M^f$ be a surface at a constant distance from edge of regression on $M$ in $E^3_1$. Let $k_1$ and $k_2$ denote principal curvature functions of $M$ and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that $\phi_u$ and $\phi_v$ are principal directions on $M$. Let us denote the angle between $X_p \in T_p M$ and $\phi_u, \phi_v$ by $\theta_1, \theta_2$ respectively. $f_s(X_p) \in T_{f(p)}(M^f)$ is an asymptotic direction if and only if
\[ \mu_1^* x_1^2 + \varepsilon_1 \varepsilon_2 \mu_2^2 x_1 x_2 + \mu_3^* x_2^2 = 0 \]
(a) Let $N_p$ be a timelike vector then
\[ \mu_1^* \cos^2 \theta_1 + \mu_2^* \cos \theta_1 \cos \theta_2 + \mu_3^* \cos^2 \theta_2 = 0. \]

(b) Let $N_p$ be a spacelike vector.

(b.1) If $X_p$ and $\phi_u$ are timelike vectors in the same timecone then
\[ \mu_1^* \cosh^2 \theta_1 + \delta_2 \mu_2^* \cosh \theta_1 \sinh \theta_2 + \mu_3^* \sinh^2 \theta_2 = 0. \]

(b.2) If $X_p$ and $\phi_v$ are timelike vectors in the same timecone then
\[ \mu_1^* \sinh^2 \theta_1 + \delta_1 \mu_2^* \sinh \theta_1 \cosh \theta_2 + \mu_3^* \cosh^2 \theta_2 = 0. \]

(b.3) If $X_p$ is a spacelike vector and $\phi_u$ is timelike vector then
\[ \mu_1^* \sinh^2 \theta_1 - \delta_1 \delta_2 \mu_2^* \sinh \theta_1 \cosh \theta_2 + \mu_3^* \cosh^2 \theta_2 = 0. \]

(b.4) If $X_p$ is a spacelike vector and $\phi_v$ is timelike vector then
\[ \mu_1^* \cosh^2 \theta_1 - \delta_1 \delta_2 \mu_2^* \cosh \theta_1 \sinh \theta_2 + \mu_3^* \sinh^2 \theta_2 = 0. \]

Above mentioned $\mu_1^*, \mu_2^*$ and $\mu_3^*$ are given in (4.2) and
\[ \delta_i = \begin{cases} 
1, & x_i \text{ is positive} \\
-1, & x_i \text{ is negative} 
\end{cases}, \quad i = (1, 2). \]

Proof. (a) Let $N_p$ be a timelike vector. In this case $\theta_1$ and $\theta_2$ are spacelike angles then
\[ x_1 = \langle X_p, \phi_u \rangle = \cos \theta_1 \]
\[ x_2 = \langle X_p, \phi_v \rangle = \cos \theta_2. \]

Substituting these equations in (4.1) the proof is obvious.

(b) Let $N_p$ be a spacelike vector.

(b.1) If $X_p$ and $\phi_u$ are timelike vectors in the same timecone then there is a hyperbolic angle $\theta_1$ and a Lorentzian timelike angle $\theta_2$. Since
\[ x_1 = -\cosh \theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh \theta_2 \]
the proof is obvious.

(b.2) If $X_p$ and $\phi_v$ are timelike vectors in the same timecone then there is a Lorentzian timelike angle $\theta_1$ and a hyperbolic angle $\theta_2$. Thus
\[ x_1 = \delta_1 \sinh \theta_1 \quad \text{and} \quad x_2 = -\cosh \theta_2. \]

(b.3) If $X_p$ is a spacelike vector and $\phi_u$ is timelike vector then there is a Lorentzian timelike angle $\theta_1$ and a central angle $\theta_2$. Thus
\[ x_1 = \delta_1 \sinh \theta_1 \quad \text{and} \quad x_2 = \delta_2 \cosh \theta_2. \]

(b.4) If $X_p$ is a spacelike vector and $\phi_v$ is timelike vector then there is a central angle $\theta_1$ and a Lorentzian timelike angle $\theta_2$. Thus
\[ x_1 = \delta_1 \cosh \theta_1 \quad \text{and} \quad x_2 = \delta_2 \sinh \theta_2. \]

\[ \Box \]

As a special case if $M$ and $M_r$ be parallel surfaces from (4.1) and (4.2) we obtain that $f_*(X_p) \in T_f(p)M_r$ is an asymptotic direction if and only if
\[ \varepsilon_1 k_1 (1 + rk_1)x_1^2 + \varepsilon_2 k_2 (1 + rk_2)x_2^2 = 0. \]
Corollary 4.2. Let $M$ and $M_r$ be parallel surfaces in $E^{3}_{1}$. Let $k_1$ and $k_2$ denote principal curvature function of $M$ and let $\{\phi_u, \phi_v\}$ be orthonormal basis such that $\phi_u$ and $\phi_v$ are principal directions on $M$. Let us denote the angle between $X_p \in T_pM$ and $\phi_u$, $\phi_v$ by $\theta_1$, $\theta_2$ respectively. $f_*(X_p) \in T_{f(p)}M_r$ is an asymptotic direction if and only if

(a) Let $N_p$ be a timelike vector then

$$k_1(1 + rk_1) \cos^2 \theta_1 + k_2(1 + rk_2) \cos^2 \theta_2 = 0.$$  

(b) Let $N_p$ be a spacelike vector.

(b.1) If $X_p$ and $\phi_u$ are timelike vectors in the same timecone then

$$-k_1(1 + rk_1) \cosh^2 \theta_1 + k_2(1 + rk_2) \sinh^2 \theta_2 = 0.$$  

(b.2) If $X_p$ and $\phi_v$ are timelike vectors in the same timecone then

$$k_1(1 + rk_1) \sinh^2 \theta_1 - k_2(1 + rk_2) \cosh^2 \theta_2 = 0.$$  

(b.3) If $X_p$ is a spacelike vector and $\phi_u$ is timelike vector then

$$-k_1(1 + rk_1) \sinh^2 \theta_1 + k_2(1 + rk_2) \cosh^2 \theta_2 = 0.$$  

(b.4) If $X_p$ is a spacelike vector and $\phi_v$ is timelike vector then

$$k_1(1 + rk_1) \cosh^2 \theta_1 - k_2(1 + rk_2) \sinh^2 \theta_2 = 0.$$  

References


CONJUGATE TANGENT VECTORS AND ASYMPTOTIC DIRECTIONS FOR SURFACES AT A CONSTANT DISTANCE FROM EDGE OF REGRESSION ON A SURFACE IN $E_3^{1}$


1Gazi University, Polatlı Science and Art Faculty, Department of Mathematics, Polatlı-TURKEY
E-mail address: deryasaglam@gazi.edu.tr

2Afyon Vocational School, Afyon Kocatepe University, Afyon - Turkey
E-mail address: bozgur@aku.edu.tr