



## SOME GRÜSS TYPE INEQUALITIES FOR THE RIEMANN–STIELTJES INTEGRAL WITH LIPSCHITZIAN INTEGRATORS

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ABSTRACT. In this paper several new inequalities of Grüss' type for the Riemann–Stieltjes integral with Lipschitzian integrators are proved.

### 1. INTRODUCTION

The Čebyšev functional

$$(1.1) \quad \mathcal{T}(f, g) = \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt,$$

has interesting applications in the approximation of weighted integrals as one can see from the literature below.

Bounding Čebyšev functional has a long history, starting with Grüss inequality [14] in 1935, where Grüss had proved that for two integrable mappings  $f, g$  such that  $\phi \leq f(x) \leq \Phi$  and  $\gamma \leq g(x) \leq \Gamma$ , the inequality

$$(1.2) \quad |\mathcal{T}(f, g)| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma)$$

holds, and the constant  $\frac{1}{4}$  is the best possible.

After that many authors have studied the functional (1.1) and several bounds under various assumptions for the functions involved have been obtained. For new results and generalizations the reader may refer to [2]–[15].

A generalization of (1.1) for Riemann–Stieltjes integral was considered by Dragomir in [10]. Namely, the author has introduced the following Čebyšev functional for the

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Riemann–Stieltjes integral:

$$(1.3) \quad \mathcal{T}(f, g; u) := \frac{1}{u(b) - u(a)} \int_a^b f(t) g(t) du(t) \\ - \frac{1}{u(b) - u(a)} \int_a^b f(t) du(t) \cdot \frac{1}{u(b) - u(a)} \int_a^b g(t) du(t)$$

under the assumptions that,  $f, g$  are continuous on  $[a, b]$  and  $u$  is of bounded variation on  $[a, b]$  with  $u(b) \neq u(a)$ .

By simple computations with Riemann–Stieltjes integral, Dragomir [10] has introduced the identity,

$$(1.4) \quad \mathcal{T}(f, g; u) := \frac{1}{u(b) - u(a)} \int_a^b \left[ f(t) - \frac{f(a) + f(b)}{2} \right] \\ \times \left[ g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t),$$

to obtain several sharp bounds of the Čebyšev functional for the Riemann–Stieltjes integral (1.3).

In this work, several sharp inequalities of Grüss' type for the Riemann–Stieltjes integral with Lipschitzian integrators are proved.

## 2. THE RESULTS

We recall that a function  $f : [a, b] \rightarrow \mathbb{C}$  is  $p$ - $H_f$ -Holder continuous on  $[a, b]$ , if

$$|f(t) - f(s)| \leq H_f |t - s|^p$$

for all  $t, s \in [a, b]$ , where  $p \in (0, 1]$  and  $H_f > 0$  are given. If  $p = 1$  we call  $f$   $H_f$ -Lipschitzian.

We are ready to state our first result as follows:

**Theorem 2.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a  $p$ - $H_f$ -Hölder continuous on  $[a, b]$ , where  $p \in (0, 1]$  and  $H_f > 0$ ; are given. Let  $g, u : [a, b] \rightarrow \mathbb{R}$  be such that  $g$  is Lebesgue integrable on  $[a, b]$  and there exists the real numbers  $m, M$  such that  $m \leq g(x) \leq M$  for all  $x \in [a, b]$ , and  $u$  is  $L_u$ -Lipschitzian on  $[a, b]$  then*

$$(2.1) \quad |\mathcal{T}(f, g; u)| \leq \frac{L_u H_f}{(p+1)} \cdot \frac{(M-m)}{|u(b) - u(a)|} \cdot (b-a)^{p+1}.$$

*Proof.* Taking the modulus in (1.4) and utilizing the triangle inequality, we get

$$\begin{aligned}
|\mathcal{T}(f, g; u)| &= \left| \frac{1}{u(b) - u(a)} \int_a^b \left[ f(t) - \frac{f(a) + f(b)}{2} \right] \right. \\
&\quad \left. \times \left[ g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t) \right| \\
&\leq \frac{L_u}{|u(b) - u(a)|} \cdot \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| \\
&\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\
&\leq \frac{L_u}{|u(b) - u(a)|} \cdot \sup_{t \in [a, b]} \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| \\
&\quad \times \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| dt \\
&\leq \frac{L_u}{|u(b) - u(a)|} \cdot \frac{L_u(M - m)}{|u(b) - u(a)|} (b - a) \cdot \frac{H_f}{2} \int_a^b [(t - a)^p + (b - t)^p] dt \\
&= \frac{L_u^2 H_f}{p + 1} \cdot \frac{(M - m)}{(u(b) - u(a))^2} (b - a)^{p+2},
\end{aligned}$$

since  $m \leq g(x) \leq M$ , for all  $x \in [a, b]$ , then

$$\begin{aligned}
\left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| &\leq \left| \frac{\int_a^b [g(t) - g(s)] du(s)}{u(b) - u(a)} \right| \\
&\leq \frac{L_u}{|u(b) - u(a)|} \int_a^b |g(t) - g(s)| ds \\
(2.2) \qquad \qquad \qquad &\leq \frac{L_u(M - m)}{|u(b) - u(a)|} (b - a),
\end{aligned}$$

which completes the proof.  $\square$

**Corollary 2.1.** *Let  $g, u$  be as in Theorem 2.1. If  $f : [a, b] \rightarrow \mathbb{R}$  is  $L_f$ -Lipschitzian on  $[a, b]$ , then*

$$(2.3) \qquad |\mathcal{T}(f, g; u)| \leq \frac{L_u^2 L_f (M - m)}{2(u(b) - u(a))^2} (b - a)^3$$

*Remark 2.1.* Under the assumptions of Theorem 2.1, we have

$$(2.4) \qquad |\mathcal{T}(f, g)| \leq \frac{H_f}{(p + 1)} (M - m) \cdot (b - a)^p.$$

In particular, if  $f$  is  $L_f$ -Lipschitzian, then

$$(2.5) \qquad |\mathcal{T}(f, g)| \leq \frac{1}{2} L_f (b - a) (M - m).$$

**Theorem 2.2.** *Let  $g, u$  be as in Theorem 2.1. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation on  $[a, b]$ , then we have*

$$(2.6) \quad |\mathcal{T}(f, g; u)| \leq \frac{1}{2} \frac{L_u(M-m)}{(u(b)-u(a))^2} (b-a) \cdot \bigvee_a^b(f).$$

*Proof.* Since  $u$  is  $L_u$ -Lipschitzian on  $[a, b]$ , as in Theorem 2.1, we have

$$\begin{aligned} |\mathcal{T}(f, g; u)| &\leq \frac{L_u}{|u(b)-u(a)|} \cdot \sup_{t \in [a, b]} \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right| \\ &\quad \times \int_a^b \left| f(t) - \frac{f(a)+f(b)}{2} \right| dt \end{aligned}$$

Since  $m \leq g \leq M$ , by (2.2) we have

$$(2.7) \quad \begin{aligned} \frac{1}{|u(b)-u(a)|} \int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right| du(t) \\ \leq \frac{L_u(M-m)}{(u(b)-u(a))^2} (b-a). \end{aligned}$$

Now as  $f$  is of bounded variation on  $[a, b]$ , we have

$$(2.8) \quad \begin{aligned} \sup_{t \in [a, b]} \left| f(t) - \frac{f(a)+f(b)}{2} \right| &= \sup_{t \in [a, b]} \left| \frac{f(t)-f(a)+f(t)-f(b)}{2} \right| \\ &\leq \frac{1}{2} \sup_{t \in [a, b]} [|f(t)-f(a)| + |f(t)-f(b)|] \leq \frac{1}{2} \bigvee_a^b(f), \end{aligned}$$

for all  $t \in [a, b]$ . Finally, combining the inequalities (2.7)–(2.8), we obtain the required result (2.6).  $\square$

**Theorem 2.3.** *Let  $g, u : [a, b] \rightarrow \mathbb{R}$  be such that  $g$  is of bounded variation on  $[a, b]$ , and  $u$  be  $L_u$ -Lipschitzian on  $[a, b]$ , then we have*

$$(2.9) \quad |\mathcal{T}(f, g; u)| \leq \begin{cases} \frac{H_f L_u^2 (b-a)^{p+2}}{(p+1)(u(b)-u(a))^2} \cdot \bigvee_a^b(g), & \text{if } f \text{ is } H_f\text{-}p\text{-Hölder} \\ \frac{L_u^2 (b-a)^2}{2(u(b)-u(a))^2} \bigvee_a^b(g) \cdot \bigvee_a^b(f), & \text{if } f \text{ is of bounded variation} \end{cases}$$

where,  $L_u, H_f > 0$  and  $p \in (0, 1]$  are given.

*Proof.* Using (1.4) we may write

$$\begin{aligned}
(2.10) \quad & |\mathcal{T}(f, g; u)| \\
& \leq \frac{L_u}{|u(b) - u(a)|} \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\
& = \frac{L_u}{(u(b) - u(a))^2} \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| \left| \int_a^b [g(t) - g(s)] du(s) \right| dt \\
& \leq \frac{L_u^2}{(u(b) - u(a))^2} \int_a^b \left[ \left| f(t) - \frac{f(a) + f(b)}{2} \right| \cdot \int_a^b |g(t) - g(s)| ds \right] dt,
\end{aligned}$$

but since  $g$  is of bounded variation then we have,

$$(2.11) \quad \int_a^b |g(t) - g(s)| ds \leq \sup_{s \in [a, b]} |g(t) - g(s)| \cdot \int_a^b ds \leq (b - a) \bigvee_a^b(g).$$

Therefore, if  $f$  is of  $p$ -Hölder type, then we have

$$\begin{aligned}
& |\mathcal{T}(f, g; u)| \\
& \leq \frac{1}{2} \cdot \frac{L_u^2 (b - a)}{(u(b) - u(a))^2} \cdot \bigvee_a^b(g) \cdot \int_a^b [|f(t) - f(a)| + |f(t) - f(b)|] dt \\
& \leq \frac{H_f}{2} \cdot \frac{L_u^2 (b - a)}{(u(b) - u(a))^2} \cdot \bigvee_a^b(g) \cdot \int_a^b [|t - a|^p + |t - b|^p] dt \\
& = \frac{H_f}{(p + 1)} \cdot \frac{L_u^2 (b - a)^{p+2}}{(u(b) - u(a))^2} \cdot \bigvee_a^b(g),
\end{aligned}$$

which prove the first part of inequality (2.9).

To prove the second part of (2.9), assume that  $f$  is of bounded variation, then by (2.10) we have

$$\begin{aligned}
& |\mathcal{T}(f, g; u)| \\
& \leq \frac{L_u^2}{(u(b) - u(a))^2} \int_a^b \left[ \left| f(t) - \frac{f(a) + f(b)}{2} \right| \cdot \int_a^b |g(t) - g(s)| ds \right] dt \\
& \leq \frac{L_u^2 (b - a)^2}{2(u(b) - u(a))^2} \bigvee_a^b(g) \cdot \bigvee_a^b(f),
\end{aligned}$$

and thus the theorem is proved.  $\square$

*Remark 2.2.* Under the assumptions of Theorem 2.3, we have

$$(2.12) \quad |\mathcal{T}(f, g)| \leq \begin{cases} \frac{H_f}{p+1} (b - a)^p \cdot \bigvee_a^b(g), & \text{if } f \text{ is } H_f - p\text{-Hölder} \\ \frac{1}{2} \bigvee_a^b(g) \cdot \bigvee_a^b(f), & \text{if } f \text{ is of bounded variation} \end{cases}$$

where,  $H_f, > 0$  and  $p \in (0, 1]$  are given.

An improvement for the first inequality in (2.9) may be stated as follows:

**Corollary 2.2.** *Let  $g, u$  be as in Theorem 2.3 and  $f : [a, b] \rightarrow \mathbb{R}$  be of  $p$ - $H_f$ -Holder type on  $[a, b]$ , then*

$$(2.13) \quad |\mathcal{T}(f, g; u)| \leq \frac{L_u^2 H_f (b-a)^{p+2}}{2^p (u(b)-u(a))^2} \bigvee_a^b(g).$$

*Proof.* By Theorem 2.3 we have

$$\begin{aligned} & |\mathcal{T}(f, g; u)| \\ & \leq \frac{L_u^2}{(u(b)-u(a))^2} \int_a^b \left[ \left| f(t) - \frac{f(a)+f(b)}{2} \right| \cdot \int_a^b |g(t)-g(s)| ds \right] dt \\ & \leq \frac{L_u^2 (b-a)}{(u(b)-u(a))^2} \bigvee_a^b(g) \int_a^b \left| f(t) - \frac{f(a)+f(b)}{2} \right| dt \\ & \leq \frac{L_u^2 (b-a)^2}{(u(b)-u(a))^2} \bigvee_a^b(g) \cdot \sup_{t \in [a,b]} \left| f(t) - \frac{f(a)+f(b)}{2} \right| \\ & \leq \frac{L_u^2 (b-a)^2}{(u(b)-u(a))^2} \bigvee_a^b(g) \cdot H_f \left( \frac{b-a}{2} \right)^p, \end{aligned}$$

and since  $f$  is of  $p$ - $H_f$ -Holder type on  $[a, b]$ , we have

$$\begin{aligned} \left| f(t) - \frac{f(a)+f(b)}{2} \right| &= \left| \frac{f(t)-f(a)+f(t)-f(b)}{2} \right| \\ &\leq \frac{1}{2} |f(t)-f(a)| + \frac{1}{2} |f(t)-f(b)| \\ &\leq \frac{H_f}{2} [(t-a)^p + (b-t)^p], \end{aligned}$$

it follows that

$$(2.14) \quad \sup_{t \in [a,b]} \left| f(t) - \frac{f(a)+f(b)}{2} \right| \leq H_f \left( \frac{b-a}{2} \right)^p.$$

which completes the proof.  $\square$

*Remark 2.3.* Under the assumptions of Corollary 2.2, we have

$$(2.15) \quad |\mathcal{T}(f, g)| \leq \frac{1}{2^p} H_f (b-a)^p \bigvee_a^b(g),$$

which improves the first inequality in (2.12), where  $H_f > 0$  and  $p \in (0, 1]$  are given.

**Theorem 2.4.** *Let  $g, u : [a, b] \rightarrow \mathbb{R}$  be such that  $g$  is of  $q$ - $H_g$ -Hölder type on  $[a, b]$ , and  $u$  be  $L_u$ -Lipschitzian on  $[a, b]$ , then we have*

$$(2.16) \quad |\mathcal{T}(f, g; u)| \leq L_u^2 H_g \cdot \begin{cases} \frac{(b-a)^{q+2}}{(q+1)(q+2)(u(b)-u(a))^2} \cdot \bigvee_a^b(f), & \text{if } f \text{ is of bounded variation} \\ \frac{H_f (b-a)^{p+q+2}}{2^p (q+1)(q+2)(u(b)-u(a))^2}, & \text{if } f \text{ is } H_f - p\text{-Hölder} \end{cases}$$

where,  $L_u, H_g, H_f > 0$  and  $p, q \in (0, 1]$  are given.

*Proof.* Assume that  $g$  is of  $q$ - $H_g$ -Hölder type on  $[a, b]$  and  $f$  is of bounded variation on  $[a, b]$ . Using (1.4), then we may write

$$\begin{aligned}
|\mathcal{T}(f, g; u)| &\leq \frac{L_u}{|u(b) - u(a)|} \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| \\
&\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\
&= \frac{L_u}{(u(b) - u(a))^2} \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| \left| \int_a^b [g(t) - g(s)] du(s) \right| dt \\
&\leq \frac{L_u^2}{(u(b) - u(a))^2} \int_a^b \left[ \left| f(t) - \frac{f(a) + f(b)}{2} \right| \cdot \int_a^b |g(t) - g(s)| ds \right] dt \\
(2.17) \quad &\leq \frac{L_u^2}{(u(b) - u(a))^2} \cdot \sup_{t \in [a, b]} \left| f(t) - \frac{f(a) + f(b)}{2} \right| \cdot \int_a^b \left[ \int_a^b |g(t) - g(s)| ds \right] dt \\
&\leq \frac{L_u^2 H_g}{2(u(b) - u(a))^2} \cdot \bigvee_a^b(f) \cdot \int_a^b \left[ \int_a^b |t - s|^q ds \right] dt \\
&= \frac{L_u^2 H_g}{2(u(b) - u(a))^2} \cdot \bigvee_a^b(f) \cdot \int_a^b \left[ \int_a^t (s - a)^q ds + \int_t^b (b - s)^q ds \right] dt \\
&= \frac{L_u^2 H_g}{2(u(b) - u(a))^2} \cdot \bigvee_a^b(f) \cdot \int_a^b \left[ \frac{(t - a)^{q+1} + (b - t)^{q+1}}{q + 1} \right] dt \\
&= \frac{L_u^2 H_g}{(u(b) - u(a))^2} \cdot \bigvee_a^b(f) \cdot \frac{(b - a)^{q+2}}{(q + 1)(q + 2)},
\end{aligned}$$

which proves the first inequality in (2.16).

To prove the second inequality in (2.16), assume that  $f$  is of  $p$ - $H_f$ -Hölder type on  $[a, b]$ , then by (2.17) we have

$$(2.18) \quad \sup_{t \in [a, b]} \left| f(t) - \frac{f(a) + f(b)}{2} \right| \leq H_f \left( \frac{b - a}{2} \right)^p$$

which together with (2.17) proves the second part of (2.16), and thus the proof is established.  $\square$

**Corollary 2.3.** *Let  $g, u : [a, b] \rightarrow \mathbb{R}$  be respectively;  $L_g, L_u$ -Lipschitzian on  $[a, b]$ , then we have*

$$(2.19) \quad |\mathcal{T}(f, g; u)| \leq L_u^2 L_g \cdot \begin{cases} \frac{(b-a)^3}{6(u(b)-u(a))^2} \cdot \bigvee_a^b(f), & \text{if } f \text{ is of bounded variation} \\ \frac{H_f (b-a)^{p+3}}{2^{p+1} \cdot 3(u(b)-u(a))^2}, & \text{if } f \text{ is } H_f - p - \text{Holder} \end{cases}$$

where,  $H_g, H_f > 0$  and  $p, q \in (0, 1]$  are given.

*Remark 2.4.* Under the assumptions of Theorem 2.4, we have

$$(2.20) \quad |\mathcal{T}(f, g)| \leq H_g \cdot \begin{cases} \frac{(b-a)^q}{(q+1)(q+2)} \cdot V_a^b(f), & \text{if } f \text{ is of bounded variation} \\ \frac{H_f(b-a)^{p+q}}{2^p(q+1)(q+2)}, & \text{if } f \text{ is } H_f - p - \text{Holder} \end{cases}$$

where,  $H_f > 0$  and  $p, q \in (0, 1]$  are given. In particular, if  $g$  is  $L_g$ -Lipschitzian, then

$$(2.21) \quad |\mathcal{T}(f, g)| \leq L_g \cdot \begin{cases} \frac{1}{6}(b-a) \cdot V_a^b(f), & \text{if } f \text{ is of bounded variation} \\ \frac{1}{12}L_f(b-a)^2, & \text{if } f \text{ is } L_f - \text{Lipschitzian} \end{cases}$$

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