ON SOME SINGULAR VALUE INEQUALITIES FOR MATRICES

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Abstract. Some singular value inequalities for matrices are given. Among other inequalities it is proved that if \( f \) and \( g \) be nonnegative functions on \([0, \infty)\) which are continuous and satisfying the relation \( f(t)g(t) = t \), for all \( t \in [0, \infty) \), then
\[
s_j(A^*_1XB_1 + A^*_2XB_2) \leq s_j(\|A^*_1f_2(\|X^*\|)A_1 + A^*_2f_2(\|X^*\|)A_2\| \oplus (B_1^*g_2(\|X\|)B_1 + B_2^*g_2(\|X\|)B_2)),
\]
for \( j = 1, 2, \ldots, n \), where \( A_1, A_2, B_1, B_2, X \) are square matrices. Our results in this article generalize some existing singular value inequalities of matrices.

1. Introduction

Let \( M_{m,n} \) be the space of \( m \times n \) complex matrices and \( M_n = M_{n,n} \). Let \( \| \cdot \| \) stand for any unitarily invariant norm on \( M_n \), i.e., a norm with the property that \( \| UAV \| = \| A \| \) for all \( A \in M_n \) and for all unitary matrices \( U, V \in M_n \). Any matrix \( A \in M_n \) is called positive semidefinite, denoted as \( A \geq 0 \) if for all \( x \in \mathbb{C}^n \), \( x^*Ax \geq 0 \) and it is called positive definite if for all nonzero \( x \in \mathbb{C}^n \), \( x^*Ax > 0 \) and it is denoted as \( A > 0 \). The singular values of matrix \( A \) are the eigenvalues of positive semidefinite matrix \( |A| = (AA^*)^{\frac{1}{2}} \), enumerated as \( s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A) \) and repeated according to multiplicity. The direct sum \( A \oplus B \) represent the block diagonal matrix \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \).

The well-known classical arithmetic-geometric mean inequality for \( a, b \geq 0 \) defined as
\[
a \frac{1}{2}b \frac{1}{2} \leq \frac{a + b}{2}.
\]

Arithmetic-geometric mean inequality is important in matrix theory, functional analysis, electrical networks, etc. For \( A, B, X \in M_n \), such that \( A, B \geq 0 \), R. Bhatia and F. Kittaneh formulated some matrix versions of this inequality in [3,4] one of

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which is the following
\[ \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \frac{1}{2}\|AX + XB\|. \]  
From (1.2), for \( X = I \) we have the following inequality for positive semidefinite matrices.
\[ \|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|A + B\|, \]
R. Bhatia and F. Kittaneh also have proved in [5] that if \( A, B \in M_n \) such that \( A, B \geq 0 \), then
\[ \|A^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|(A + B)^2\| + \frac{1}{2}\|A + B\|. \]
From (1.3), (1.4) and also by triangle inequality, we obtain the following inequality
\[ \|A^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|(A + B)^2\| + \frac{1}{2}\|A + B\|. \]
In [2] L. Zou and Y. Jiang proved that for positive semidefinite matrices \( A, B \in M_n \) and \( 1 \leq j \leq n \), the following inequality also holds
\[ 2s_j(A^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}) \leq s_j((A + B)^2 + (A + B)), \]
and consequently,
\[ \|A^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}} + A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{1}{2}\|(A + B)^2\| + \frac{1}{2}\|A + B\|. \]
The inequality (1.7) is an improvement of the inequality (1.5).
One another interesting inequality for sum and direct sum of matrices proved by R. Bhatia and F. Kittaneh [6] is
\[ s_j(A^*B + B^*A) \leq s_j((A^*A + B^*B) \oplus (A^*A + B^*B)), \]
where \( A, B \in M_n \) and \( 1 \leq j \leq n \).
In Section 2, we give generalized form of the inequality (1.6) and also, we obtain the X-version of the inequality (1.8).

2. Singular values inequalities for matrices

In this section, we generalize the inequalities (1.6) and also, we obtain X-version of the inequality (1.8). Our results based on Several lemmas. First two lemmas have been given by F. Kittaneh in [1] and Lemma 2.3 can be found in [8, Theorem 1].

**Lemma 2.1.** Let \( T \in M_n \), then the block matrix \( \begin{pmatrix} |T| & T^* \\ T & |T^*| \end{pmatrix} \geq 0 \).

**Lemma 2.2.** Let \( A, B, C \in M_n \), such that \( A \) and \( B \) are positive semidefinite, \( BC = CA \) and let \( f \) and \( g \) be nonnegative functions on \([0, \infty)\) which are continuous and satisfying the relation \( f(t)g(t) = t \), for all \( t \in [0, \infty) \). If the block matrix \( \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \geq 0 \), then so \( \begin{pmatrix} f^2(A) & C^* \\ C & g^2(B) \end{pmatrix} \geq 0 \).

**Lemma 2.3.** Let \( A, B, C \in M_n \) such that \( \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \), then
\[ 2s_j(B) \leq s_j \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}, \quad j = 1, 2, \ldots, n. \]
The following Lemma was proved in [7].

**Lemma 2.4.** Let $A, B, C \in M_n$, such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \succeq 0$, then

\[(2.2) \quad s_j(B) \leq s_j(A \oplus C), \quad j = 1, 2, \ldots, n.\]

To give the general form of (1.6), first we prove the following result.

**Theorem 2.5.** Let $A, B \in M_n$ be any two matrices and $r$ be a positive integer, then

\[2s_j(A(\|A\|^2 + \|B\|^2)^{r-1}B^* + AB^*) \leq s_j((\|A\|^2 + \|B\|^2)^r + (\|A\|^2 + \|B\|^2)), \]

for $j = 1, 2, \ldots, n$.

**Proof.** Let $X = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$. Then,

\[X^*X = \begin{pmatrix} A^*A + B^*B & 0 \\ 0 & 0 \end{pmatrix}, \quad XX^* = \begin{pmatrix} AA^* & AB^* \\ BA^* & BB^* \end{pmatrix}.\]

So, we have

\[(X^*X)^r = \begin{pmatrix} (A^*A + B^*B)^r & 0 \\ 0 & 0 \end{pmatrix},\]

and

\[(XX^*)^r = X(X^*X)^{(r-1)}X^* = \begin{pmatrix} A(A^*A + B^*B)^{(r-1)}A^* & A(A^*A + B^*B)^{(r-1)}B^* \\ B(A^*A + B^*B)^{(r-1)}A^* & B(A^*A + B^*B)^{(r-1)}B^* \end{pmatrix}.\]

Therefore, we obtain

\[(X^*X)^r + X^*X = \begin{pmatrix} (A^*A + B^*B)^r + A^*A + B^*B & 0 \\ 0 & 0 \end{pmatrix},\]

and

\[(XX^*)^r + XX^* = \begin{pmatrix} A(A^*A + B^*B)^{(r-1)}A^* + AA^* & A(A^*A + B^*B)^{(r-1)}B^* + AB^* \\ B(A^*A + B^*B)^{(r-1)}A^* + BA^* & B(A^*A + B^*B)^{(r-1)}B^* + BB^* \end{pmatrix}.\]

So, by Lemma 2.3, from the positive semidefinite block matrix $(XX^*)^r + XX^*$, we have

\[2s_j(A(A^*A + B^*B)^{(r-1)}B^* + AB^*) \leq s_j((XX^*)^r + XX^*) \leq s_j((X^*X)^r + X^*X) = s_j((A^*A + B^*B)^r + (A^*A + B^*B)),\]

for $j = 1, 2, \ldots, n$.

The proof is completed. □

When $A, B \in M_n$ be positive semidefinite in Theorem 2.5 and $A$ is replaced by $A^2$ and $B$ is replaced by $B^2$, then we obtain the following promised generalization of the inequality (1.6).
Corollary 2.6. Let $A, B \in M_n$ be positive semidefinite and $r$ be a positive integer. Then,
\[ 2s_j(A^\frac{r}{2}(A + B)^{(r-1)}B^\frac{r}{2} + A^\frac{r}{2}B^\frac{r}{2}) \leq s_j((A + B)^r + (A + B)), \]
for $j = 1, 2, \ldots, n$.

Remark 2.7. When we take $r = 2$ in Corollary 2.6, then we obtain the inequality (1.6).

To give the $X$-version of the inequality (1.8), first we obtain the following result.

Theorem 2.8. Let $A_1, A_2, B_1, B_2, X \in M_n$. If $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$, for all $t \in [0, \infty)$, then
\[
s_j(A_1^*XB_1 + A_2^*XB_2) \leq s_j((A_1^*f^2(\|X^*\|)A_1 + A_2^*f^2(\|X^*\|)A_2) \oplus (B_1^*g^2(\|X\|)B_1 + B_2^*g^2(\|X\|)B_2)), \]
for $j = 1, 2, \ldots, n$.

Proof. Let $T_1 = \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}$, $T_2 = \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}$.

Since the block matrix $\begin{pmatrix} \|X^*\| & X \\ X^* & |X| \end{pmatrix}$ is positive semidefinite (by Lemma 2.1) and the block matrix $Y = \begin{pmatrix} f^2(\|X^*\|) & X \\ X^* & g^2(\|X\|) \end{pmatrix}$ is positive semidefinite (by Lemma 2.2), so, $T_1^*YT_1 = \begin{pmatrix} A_1^*f^2(\|X^*\|)A_1 & A_1^*XB_1 \\ B_1^*X^*A_1 & B_1^*g^2(\|X\|)B_1 \end{pmatrix} \geq 0$ and also,

$T_2^*YT_2 = \begin{pmatrix} A_2^*f^2(\|X^*\|)A_2 & A_2^*XB_2 \\ B_2^*X^*A_2 & B_2^*g^2(\|X\|)B_2 \end{pmatrix} \geq 0$.
That is, we have
\[
T_1^*YT_1 + T_2^*YT_2 = \begin{pmatrix} A_1^*f^2(\|X^*\|)A_1 + A_2^*f^2(\|X^*\|)A_2 & A_1^*XB_1 + A_2^*XB_2 \\ B_1^*X^*A_1 + B_2^*X^*A_2 & B_1^*g^2(\|X\|)B_1 + B_2^*g^2(\|X\|)B_2 \end{pmatrix} \geq 0.
\]
So, our desired result now follows by invoking inequality (2.2). The proof is completed.

Following is our desired X-version of the inequality (1.8).

Corollary 2.9. Let $A, B, X \in M_n$, then
\[
s_j(A^*XB + B^*XA) \leq s_j((A^* | X^* | A + B^* | X^* | B) \oplus (A^* | X | A + B^* | X | B)), \]
for $j = 1, 2, \ldots, n$.

Proof. By taking $f(t) = g(t) = t^\frac{r}{2}$, $A_1 = B_2 = A$ and $A_2 = B_1 = B$ in Theorem 2.8, we get the desired result.
The proof is completed.

One another important case follows from Corollary 2.9 for normal matrices.
Corollary 2.10. Let $A, B, X \in M_n$ such that $X$ is normal matrix, then
\[
s_j(A^*XB + B^*XA) 
\leq s_j(A^*|X|A + B^*|X|B) \oplus (A^*|X|A + B^*|X|B),
\]
for $j = 1, 2, ..., n$.
In particular, when $X$ is positive semidefinite matrix, then
\[
s_j(A^*XB + B^*XA) 
\leq s_j((A^*XA + B^*XB) \oplus (A^*XA + B^*XB)),
\]
for $j = 1, 2, ..., n$.

References


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