Konuralp Journal of Mathematics
Volume 2 No. 1 pp. 90–101 (2014) ©KJM

SIMPSON'S TYPE INEQUALITIES FOR $m-$ AND $(\alpha, m)-$GEOMETRICALLY CONVEX FUNCTIONS

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Abstract. In this paper, we establish Simpson's type inequalities for $m-$ and $(\alpha, m)-$geometrically convex functions using the lemmas.

1. Introduction

The following inequality is well-known in the literature as Simpson's inequality:

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $[a, b]$ and $\left\|f^{(4)}\right\|_{\infty} = \sup_{x\in[a,b]} \left|f^{(4)}(x)\right| < \infty$. Then the folllowing inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

For the recent results based on the above definition see the papers [1], [4] , [7], [14], [18] and [20].

In [6], G.Toader defined the concept of m-convexity as the following;

Definition 1.1. The function $f : [0, b] \to \mathbb{R}$ is said to be $m-$convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the $m-$convex functions on $[0, b]$ for which $f(0) \le 0$.

In [19], Miheşan gave definition of $(\alpha, m)-$convexity as following;

Definition 1.2. The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \le t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

2000 *Mathematics Subject Classification.* Primary 26D15, Secondary 26A51.
Key words and phrases. $m-$ geometrically convex function, $(\alpha, m)-$geometrically convex function, Simpson's type inequality.
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Denote by $K_m^\alpha(b)$ the class of all $(\alpha,m)-$convex functions on $[0,b]$ for which $f(0)\leq 0$. If we choose $(\alpha,m)=(1,m)$, it can be easily seen that $(\alpha,m)-$convexity reduces to $m-$convexity and for $(\alpha,m)=(1,1)$, we have ordinary convex functions on $[0,b]$. For the recent results based on the $m-$ and $(\alpha,m)-$ convexity see the papers [2], [3], [5], [8]-[13] and [15]-[17].

In [2], Xi *et al.* introduced $m-$ and $(\alpha,m)-$geometrically convex functions and give a lemma as following, respectively;

Definition 1.3. Let $f(x)$ be a positive function on $[0,b]$ and $m\in(0,1]$. If

$$f\left(x^t y^{m(1-t)}\right)\leq [f(x)]^t\,[f(y)]^{m(1-t)}$$

holds for all $x,y\in[0,b]$ and $t\in[0,1]$, then we say that the function $f(x)$ is $m-$geometrically convex on $[0,b]$.

It is clear that when $m=1$, $m-$geometrically convex functions become geometrically convex functions.

Definition 1.4. Let $f(x)$ be a positive function on $[0,b]$ and $(\alpha,m)\in(0,1]\times(0,1]$. If

$$f\left(x^t y^{m(1-t)}\right)\leq [f(x)]^{t^\alpha}\,[f(y)]^{m(1-t^\alpha)}$$

holds for all $x,y\in[0,b]$ and $t\in[0,1]$, then we say that the function $f(x)$ is $(\alpha,m)-$geometrically convex on $[0,b]$.

Lemma 1.1. *For $x,y\in[0,\infty)$ and $m,t\in(0,1]$, if $x<y$ and $y\geq 1$, then*

$$x^t y^{m(1-t)}\leq tx+(1-t)\,y.$$

In this paper, we recite two lemmas in the literature, then we obtaine Simpson's type inequalities using the lemmas for $m-$ and $(\alpha,m)-$geometrically convex functions.

2. Results

Lemma 2.1. [[1], *pp.3*] *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I° where $a,b\in I$ with $a<b$. Then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= (b-a)\int_0^1 p(t)\,f'(tb+(1-t)a)\,dt,$$

where

$$p(t)=\begin{cases} t-\frac{1}{6}, & t\in\left[0,\frac{1}{2}\right),\\ t-\frac{5}{6}, & t\in\left(\frac{1}{2},1\right].\end{cases}$$

Theorem 2.1. *Let $f:I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $(\alpha,m)\in(0,1]^2$, then the following inequality holds;*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq (b-a)\,|f'(b)|^m\,M_1(\alpha,m)$$

where
(2.1)
$$M_1(\alpha,m)=\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha}dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha}dt.$$

Proof. From Lemma 2, Lemma 1 and since f is decreasing, then

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;& (b-a)\int_0^1|p(t)|\,|f'(tb+(1-t)a)|\,dt\\ \leq\;& (b-a)\int_0^1|p(t)|\left|f'\left(a^{1-t}b^{mt}\right)\right|dt.\end{aligned}$$

Using the $(\alpha,m)-$geometrically convexity of $|f'(x)|$, we have,

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;& (b-a)\int_0^1|p(t)|\,|f'(a)|^{(1-t)^\alpha}\,|f'(b)|^{m(1-(1-t)^\alpha)}\,dt\\ =\;& (b-a)\,|f'(b)|^m\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha}dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha}dt\right\}.\end{aligned}$$

So, the proof is completed. □

Corollary 2.1. *Let $f: I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|$ is decreasing and $m-$geometrically convex on $[\min\{1,a\},b]$ for $b\geq 1$, and for $m\in(0,1]$, then the following inequality holds;*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq(b-a)\,|f'(b)|^m\,M_1(1,m)$$

where $M_1(1,m)$ is the term in (2.1).

Theorem 2.2. *Let $f: I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|^{\frac{p}{p-1}}$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $(\alpha,m)\in(0,1]^2$, $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds;*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;& (b-a)\,|f'(b)|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}M_2(\alpha,m,p)\end{aligned}$$

where

(2.2)

$$M_2(\alpha,m,p)=\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}+\left(\int_{\frac{1}{2}}^1\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}.$$

Proof. By using Lemma 2 and Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
\leq\ &(b-a)\int_0^1|p(t)|\,|f'(tb+(1-t)a)|\,dt\\
=\ &(b-a)\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right||f'(tb+(1-t)a)|\,dt+(b-a)\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right||f'(tb+(1-t)a)|\,dt\\
\leq\ &(b-a)\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'(tb+(1-t)a)|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}\\
&+(b-a)\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|f'(tb+(1-t)a)|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Since $|f'(x)|$ is decreasing by using Lemma 1 and $(\alpha,m)-$geometrically convex, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
\leq\ &(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}\left|f'\left(a^{1-t}b^{mt}\right)\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}+\left(\int_{\frac{1}{2}}^1\left|f'\left(a^{1-t}b^{mt}\right)\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}\right\}\\
\leq\ &(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}|f'(b)|^m\\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}+\left(\int_{\frac{1}{2}}^1\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}\right\}.
\end{aligned}$$

So, the desired result is obtained. □

Corollary 2.2. *Let $f:I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|^{\frac{p}{p-1}}$ is decreasing and $m-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $m\in(0,1]$, $p>1$ with*

$\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds;*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;&(b-a)\left|f'(b)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}M_2(1,m,p)\end{aligned}$$

where $M_2(1,m,p)$ is the term in (2.2).

Theorem 2.3. *Let $f:I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|^q$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $(\alpha,m)\in(0,1]^2$, $q\geq 1$, then the following inequality holds;*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;&(b-a)\left|f'(b)\right|^m\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[M_3(\alpha,m,q)\right]^{\frac{1}{q}}\end{aligned}$$

where

(2.3)

$$M_3(\alpha,m,q)=\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^{\alpha}q}dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^{\alpha}q}dt.$$

Proof. From Lemma 2 and using the well-known power mean integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\;&(b-a)\int_0^1|p(t)|\left|f'(tb+(1-t)a)\right|dt\\ \leq\;&(b-a)\left(\int_0^1|p(t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|p(t)|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\ \leq\;&(b-a)\left(\int_0^1|p(t)|\,dt\right)^{1-\frac{1}{q}}\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\right.\\ &\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\right\}^{\frac{1}{q}}.\end{aligned}$$

Since $|f'(x)|^q$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\le\;& (b-a)\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'\left(a^{1-t}b^{mt}\right)\right|^q dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'\left(a^{1-t}b^{mt}\right)\right|^q dt\right\}^{\frac{1}{q}} \\
\le\;& (b-a)\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(a)\right|^{(1-t)^\alpha q}\left|f'(b)\right|^{m(1-(1-t)^\alpha)q} dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(a)\right|^{(1-t)^\alpha q}\left|f'(b)\right|^{m(1-(1-t)^\alpha)q} dt\right\}^{\frac{1}{q}} \\
=\;& (b-a)\left|f'(b)\right|^m\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha q} dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(\frac{|f'(a)|}{|f'(b)|^m}\right)^{(1-t)^\alpha q} dt\right\}.
\end{aligned}$$

So, the proof is completed. □

Corollary 2.3. *Let $f: I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|^q$ is decreasing and $m-$geometrically convex on $[\min\{1,a\},b]$, for $b\ge 1$, and for $m\in(0,1]$, $q\ge 1$, then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\le\;& (b-a)\left|f'(b)\right|^m\left(\frac{5}{36}\right)^{1-\frac{1}{q}}\left[M_3(1,m,q)\right]^{\frac{1}{q}}
\end{aligned}$$

where $M_3(1,m,q)$ is the term in (2.3).

Now, we obtain Simpson's type inequalities for twice differentiable functions using the following lemma.

Lemma 2.2. [[14], *pp.2*]*Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable mapping on I° such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx \\
=\;& (b-a)^2\int_0^1 k(t)\,f''(tb+(1-t)a)\,dt
\end{aligned}$$

where

$$k(t)=\begin{cases}\frac{t}{2}\left(\frac{1}{3}-t\right), & t\in\left[0,\frac{1}{2}\right),\\ (1-t)\left(\frac{t}{2}-\frac{1}{3}\right), & t\in\left(\frac{1}{2},1\right].\end{cases}$$

Theorem 2.4. *Let $f: I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f''(x)|$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, for $b\geq 1$, and for $(\alpha,m)\in(0,1]^2$, then the following inequality holds;*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\leq (b-a)^2\left|f''(b)\right|^m\left[M_4(\alpha,m)\right]\end{aligned}$$

where

$$\begin{aligned}M_4(\alpha,m) &= \int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{(1-t)^{\alpha}}dt\\ &\quad+\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{(1-t)^{\alpha}}dt.\end{aligned}\tag{2.4}$$

Proof. From Lemma 3, Lemma 1 and since $|f''(x)|$ is decreasing, then

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\leq (b-a)^2\int_0^1|k(t)|\left|f''(tb+(1-t)a)\right|dt\\ &\leq (b-a)^2\int_0^1|k(t)|\left|f''\left(a^{1-t}b^{mt}\right)\right|dt.\end{aligned}$$

Using the $(\alpha,m)-$geometrically convexity of $|f''(x)|$, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\leq (b-a)^2\int_0^1|k(t)|\left|f''(a)\right|^{(1-t)^{\alpha}}\left|f''(b)\right|^{m(1-(1-t)^{\alpha})}dt\\ &= (b-a)^2\left|f''(b)\right|^m\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{(1-t)^{\alpha}}dt\right.\\ &\quad\left.+\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{(1-t)^{\alpha}}dt\right\}.\end{aligned}$$

So, the proof is completed. □

Corollary 2.4. *Let $f: I\subset[0,\infty)\to(0,\infty)$ be a differentiable mapping on I° such that $f''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f''(x)|$ is decreasing and*

$m-$geometrically convex on $[\min\{1,a\},b]$ for $b \geq 1$, and for $m \in (0,1]$, then the following inequality holds;

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq (b-a)^2\left|f''(b)\right|^m M_4(1,m)$$

where $M_4(1,m)$ is the term in (2.4).

Theorem 2.5. *Let $f : I \subset [0,\infty) \to (0,\infty)$ be a differentiable mapping on I° such that $f'' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f''(x)|^q$ is decreasing and $(\alpha,m)-$geometrically convex on $[\min\{1,a\},b]$, for $b \geq 1$, and for $(\alpha,m) \in (0,1]^2$, $q \geq 1$, then the following inequality holds;*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq\ & (b-a)^2\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left(M_6(\alpha,m,q)^{\frac{1}{q}}+M_7(\alpha,m,q)^{\frac{1}{q}}\right)\end{aligned}$$

where

$$M_6(\alpha,m,q)=\left|f''(b)\right|^{mq}\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{q(1-t)^\alpha}dt$$

and

$$M_7(\alpha,m,q)=\left|f''(b)\right|^{mq}\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{q(1-t)^\alpha}dt.$$

Proof. Suppose that $q \geq 1$. From Lemma 3 and using the well-known power mean integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & (b-a)^2\int_0^1 |k(t)|\left|f''(tb+(1-t)a)\right|dt \\
=\ & (b-a)^2\left\{\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|f''(tb+(1-t)a)\right|dt\right. \\
& \left.+\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|f''(tb+(1-t)a)\right|dt\right\} \\
\leq\ & (b-a)^2\left\{\left(\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|dt\right)^{1-\frac{1}{q}}\right. \\
& \times\left(\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|dt\right)^{1-\frac{1}{q}} \\
& \left.\times\left(\int_{\frac{1}{2}}^1\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right|\left|f''(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f''(x)|^q$ is decreasing using Lemma 1 and $(\alpha, m)-$geometrically convex on $[\min\{1,a\}, b]$, we have

$$\begin{aligned}
(2.5)\qquad & \int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|f''(tb+(1-t)a)\right|^q dt \\
\leq\ & \int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|f''\left(a^{1-t}b^{mt}\right)\right|^q dt \\
\leq\ & \int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left|f''(a)\right|^{q(1-t)^\alpha}\left|f''(b)\right|^{mq(1-(1-t)^\alpha)}dt \\
=\ & \left|f''(b)\right|^{mq}\int_0^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right|\left(\frac{|f''(a)|}{|f''(b)|^m}\right)^{q(1-t)^\alpha}dt \\
=\ & M_6(\alpha,m,q)
\end{aligned}$$

and

$$\begin{aligned} (2.6) \quad & \int_{\frac{1}{2}}^{1} \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \left|f''\left(tb+(1-t)a\right)\right|^{q} dt \\ \leq & \int_{\frac{1}{2}}^{1} \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \left|f''\left(a^{1-t}b^{mt}\right)\right|^{q} dt \\ \leq & \int_{\frac{1}{2}}^{1} \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \left|f''(a)\right|^{q(1-t)^{\alpha}} \left|f''(b)\right|^{mq(1-(1-t)^{\alpha})} dt \\ = & \left|f''(b)\right|^{mq} \int_{\frac{1}{2}}^{1} \left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| \left(\frac{\left|f''(a)\right|}{\left|f''(b)\right|^{m}}\right)^{q(1-t)^{\alpha}} dt \\ = & M_{7}(\alpha, m, q) \end{aligned}$$

From (2.5) and (2.6), we have

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b} f(x)\,dx\right| \\ \leq & (b-a)^{2}\left\{\left(\int_{0}^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| dt\right)^{1-\frac{1}{q}} M_{6}(\alpha, m, q)^{\frac{1}{q}} \right. \\ & \left. +\left(\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| dt\right)^{1-\frac{1}{q}} M_{7}(\alpha, m, q)^{\frac{1}{q}}\right\} \\ = & (b-a)^{2}\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left(M_{6}(\alpha, m, q)^{\frac{1}{q}}+M_{7}(\alpha, m, q)^{\frac{1}{q}}\right) \end{aligned}$$

where we use the fact that

$$\int_{0}^{\frac{1}{2}}\left|\frac{t}{2}\left(\frac{1}{3}-t\right)\right| dt=\int_{\frac{1}{2}}^{1}\left|(1-t)\left(\frac{t}{2}-\frac{1}{3}\right)\right| dt=\frac{1}{162}.$$

So, the proof is completed. □

Corollary 2.5. *Let $f : I \subset [0,\infty) \to (0,\infty)$ be a differentiable mapping on I° such that $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f''(x)\right|^{q}$ is decreasing and $m-$geometrically convex on $[\min\{1,a\}, b]$, for $b \geq 1$, and for $m \in (0,1]$, $q \geq 1$, then the following inequality holds;*

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b} f(x)\,dx\right| \\ \leq & (b-a)^{2}\left(\frac{1}{162}\right)^{1-\frac{1}{q}}\left(M_{6}(1, m, q)^{\frac{1}{q}}+M_{7}(1, m, q)^{\frac{1}{q}}\right) \end{aligned}$$

where $M_6(1,m,q)$ and $M_7(1,m,q)$ are in the Theorem 5.

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