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SOME RESULTS ON PSEUDO RICCI SYMMETRIC ALMOST α-COSYMPLECTIC f-MANIFOLDS

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Abstract. In this study, we consider pseudo Ricci symmetric almost α-cosymplectic f-manifolds. We get some results on pseudo Ricci symmetric α-cosymplectic f-manifolds and almost α-cosymplectic f-manifolds verifying (κ, μ, ν)-nullity conditions.

1. Introduction

The notion of f-structure, which is a generalization of almost complex structures and almost contact structures, was firstly introduced by Yano in 1963 [1]. In 1971, Goldberg and Yano defined that globally framed f-structures are f-structures and globally framed manifolds are f pk.-manifolds [2]. Later many authors studied f pk.-manifolds. In 2006, Falcitelli and Pastore defined $(2n+s)$-dimensional almost Kenmotsu f-manifolds [3]. Öztürk et. all defined almost α-cosymplectic f-manifolds [4].

The notion of an almost cosymplectic manifold was introduced by Goldberg and Yano in 1969 [9]. The simplest examples of such manifolds are those being the products (possibly local) of almost Kählerian manifolds and the real line $\mathbb{R}$ or the circle S^1. In particular, cosymplectic manifolds in the sense of Blair [10] are of this type. However the class of almost cosymplectic manifolds is much more wider. There are already many known examples (among others, compact, homogeneous) of such manifolds which are not products (even locally). See Cordero et al. [11] Chinea and Gonzalez [12] and Olszak ([13],[14]).

The topology of cosymplectic manifolds was studied by Blair and Goldberg [15], Chinea et al. ([16], [17]) and others. Most of the results of Libermann [18], Lichnerowicz [19], Fujimoto and Muto [20] also have applications in characterizing of topological and analytical properties of almost cosymplectic manifolds (these authors have used a different terminology).

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Curvature properties of almost cosymplectic manifolds were studied mainly by Golberg and Yano [9] ,Olszak ([13], [14]), Kirichenko [21] and Endo [22]. We relate some of them in a historical order.

As a generalization of Chaki's pseudosymmetric and pseudo Ricci sym-metric manifolds ([28] and [29]), the notion of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by Tamassy and Binh ([30] and [31]). These type manifolds were studied with different structures by several authors ([30], [32] and [33]). Recently in [34], Özgür studied weakly symmetric Kenmotsu manifolds. The notion of special weakly Ricci symmetric manifolds was introduced and studied by Singh, and Khan in [35]. Aktan studied special weakly symmetric Kenmotsu manifolds [36].

In this paper, we have studied some geometric properties of pseudo Ricci-symmetric almost α-cosymplectic f-manifold.

2. Preliminaries

Let M be a real $(2n+s)$-dimensional smooth manifolds. M admits an f-structure [1] if there exists a non-null smooth $(1,1)$ tensor field φ, of the tangent bundle TM, satisfying $\varphi^3+\varphi=0,\ rank\varphi=2n$. An f-structure is a generalization of almost complex $(s=0)$ and almost contact $(s=1)$ structure. In the latter case, M is orientable [25]. Corresponding to two complementary projection operators P and Q applied to TM, defined by $P=-\varphi^2$ and $Q=\varphi^2+I$, where I is the identity operator, there exist two complementary distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that $\dim(\mathcal{D})=2n$ and $\dim(\mathcal{D})=s$. The following relations hold

$$\varphi P=P\varphi=\varphi,\quad \varphi Q=Q\varphi=0,\quad \varphi^2P=-P,\quad \varphi^2Q=0. \tag{2.1}$$

Thus, we have an almost complex distribution $\left(\mathcal{D},\ J=\varphi_{|_D},\ J^2=-I\right)$ and φ acts on $\mathcal{D}^{\perp}$ as a null operator. It follows that

$$TM=\mathcal{D}\oplus\mathcal{D}^{\perp},\quad \mathcal{D}\cap\mathcal{D}^{\perp}=\{0\}. \tag{2.2}$$

Assume that $\mathcal{D}_p^{\perp}$ is spanned by s globally defined orthonormal vectors $\{\xi_i\}$ at each point $p\in M$, $(1\leq i,j,...\leq s)$, with its dual set $\{\eta_i\}$. Then one obtains

$$\varphi^2=-I+\sum_{i=1}^{s}\eta^i\otimes\xi_i. \tag{2.3}$$

In above case, M is called a globally framed manifold (or simply an f-manifold) ([1], [26] and [27]) and we denote its framed structure by $M(\varphi,\xi_i)$. From the conditions one has

$$\varphi\xi_i=0,\quad \eta^i\circ\varphi=0,\quad \eta^i(\xi_j)=\delta_i^j. \tag{2.4}$$

Now, we consider compatible Riemannian metric g on M with an f-structure such that

$$\begin{gathered} g(\phi X,Y)+g(X,\phi Y)=0,\\ g(\phi X,\phi Y)=g(X,Y)-\sum_{i=1}^{s}\eta^i(X)\eta^i(Y),\\ g(X,\xi_i)=\eta^i(X). \end{gathered} \tag{2.5}$$

In the above case, we say that M is a metric f-manifold and its associated structure will be denoted by $M(\phi,\xi_i,\eta^i,g)$.

A framed structure $M(\phi, \xi_i)$ is said to be normal [27] if the torsion tensor N_ϕ of ϕ is zero i.e., if

$$N_\phi = N + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0, \tag{2.6}$$

where N denotes the Nijenhuis tensor field of ϕ.

Define a 2-form Φ on M by $\Phi(X, Y) = g(X, \phi Y)$, for any $X, Y \in \Gamma(TM)$. The Levi-Civita connection ∇ of a metric f-manifold satisfies the following formula [26]:

$$\begin{aligned} 2g\left(\left(\nabla_X \phi\right) Y, Z\right) &= 3d\Phi(X, \phi Y, \phi Z) - 3d\Phi(X, Y, Z) \\ &\quad + g\left(N(Y, Z), \phi X\right) + N_j^2(Y, Z)\eta^j(X) \\ &\quad + 2d\eta^j(\phi Y, X)\eta^j(Z) - 2d\eta^j(\phi Z, X)\eta^j(Y), \end{aligned} \tag{2.7}$$

where the tensor field N_j^2 is defined by $N_j^2\left(X, Y\right) = \left(L_{\phi X}\eta^j\right) Y - \left(L_{\phi Y}\eta^j\right) X = 2d\eta^j(\phi X, Y) - 2d\eta^j(\phi Y, X)$, for each $j \in \{1, ..., s\}$.

Throughout this paper we denote by $\overline{\eta} = \eta^1 + \eta^2 + ... + \eta^s$, $\overline{\xi} = \xi^1 + \xi^2 + ... + \xi^s$ and $\overline{\delta}_i^j = \delta_i^1 + \delta_i^2 + ... + \delta_i^s$.

Definition 2.1. [4] Let $M\left(\varphi, \xi_i, \eta^i, g\right)$ be $(2n + s)$-dimensional a metric f-manifold. For each η^i, $(1 \leq i \leq s)$ 1-forms and each Φ 2-forms, if $d\eta^i = 0$ and $d\Phi = 2\alpha\overline{\eta} \wedge \Phi$ satisfy,then M is called almost α-cosymplectic f-manifold.

The manifold is called generalized almost Kenmotsu f-manifold for $\alpha = 1$ [4].

Let M be an almost α-cosymplectic f-manifold. Since the distribution $\mathcal{D}$ is integrable, we have $L_{\xi_i}\eta^j = 0$, $[\xi_i, \xi_j] \in \mathcal{D}$ and $[X, \xi_j] \in \mathcal{D}$ for any $X \in \Gamma\left(\mathcal{D}\right)$. Then the Levi-Civita connection is given by:

(2.8)

$$2g\left(\left(\nabla_X \varphi\right) Y, Z\right) = 2\alpha\left(\sum_{j=1}^{s}\left(g\left(\varphi X, Y\right)\xi_j - \eta^j\left(Y\right)\varphi X\right), Z\right) + g\left(N\left(Y, Z\right), \varphi X\right),$$

for any $X, Y, Z \in \Gamma\left(TM\right)$. Putting $X = \xi_i$ we obtain $\nabla_{\xi_i}\varphi = 0$ which implies $\nabla_{\xi_i}\xi_j \in \mathcal{D}^{\perp}$ and then $\nabla_{\xi_i}\xi_j = \nabla_{\xi_j}\xi_i$, since $[\xi_i, \xi_j] = 0$.

We put $A_i X = -\nabla_X \xi_i$ and $h_i = \frac{1}{2}\left(L_{\xi_i}\varphi\right)$, where L denotes the Lie derivative operator.

Proposition 2.1. [4] *For any $i \in \{1, ..., s\}$ the tensor field A_i is a symmetric operator such that*

1) $A_i\left(\xi_j\right) = 0$, *for any* $j \in \{1, ..., s\}$.

2) $A_i \circ \varphi + \varphi \circ A_i = -2\alpha\varphi$.

3) $tr(A_i) = -2\alpha n$.

Proposition 2.2. [5] *For any $i \in \{1, ..., s\}$ the tensor field h_i is a symmetric operator and satisfies*

i) $h_i\xi_j = 0$, *for any* $j \in \{1, ..., s\}$.

ii) $h_i \circ \varphi + \varphi \circ h_i = 0$.

iii) $trh_i = 0$.

iv) $tr\varphi h_i = 0$.

Proposition 2.3. [4] ∇_{φ} *satisfies the following relation:*
(2.9)

$$\left(\nabla_{X\varphi}\right) Y+\left(\nabla_{\varphi X} \varphi\right) \varphi Y=\sum_{i=1}^{s}\left[-\alpha\left(\eta^{i}(Y) \varphi X+2 g(X, \varphi Y) \xi_{i}\right)-\eta^{i}(Y) h_{i} X\right].$$

Definition 2.2. [4] Let M be an almost α-cosymplectic f-manifold, κ, μ and ν are real constants. We say that M verifies the (κ, μ, ν)-nullity condition if and only if for each $i \in \{1, ..., s\}$, $X, Y \in \Gamma(TM)$ the following identity holds

$$\begin{aligned} R(X, Y) \xi_{i}= & \kappa\left(\overline{\eta}(X) \varphi^{2} Y-\overline{\eta}(Y) \varphi^{2} X\right) \\ & +\mu\left(\overline{\eta}(Y) h_{i} X-\overline{\eta}(X) h_{i} Y\right) \\ & +\nu\left(\overline{\eta}(Y) \varphi h_{i} X-\overline{\eta}(X) \varphi h_{i} Y\right). \end{aligned} \tag{2.10}$$

Lemma 2.1. [4] *Let M be an almost α-cosymplectic f-manifold verifying (κ, μ, ν)-nullity condition. Then we have*

(i) $h_i \circ h_j = h_j \circ h_i$ *for each* $i, j \in \{1, 2, ..., s\}$

(ii) $\kappa \leq -\alpha^2$

(iii) If $\kappa < -\alpha^2$ *then, for each* $i \in \{1, 2, ..., s\}$, h_i *has eingen values* $0, \pm\sqrt{-(\kappa+\alpha^2)}$.

Proposition 2.4. [4] *Let M be an almost α-cosymplectic f-manifold verifying (κ, μ, ν)-nullity condition. Then*

$$h_1 = ... = h_s. \tag{2.11}$$

Remark 2.1. [4] Throughout all this paper whenever (2.10) holds we put $h := h_1 = ... = h_s$. Then (2.10) becomes

$$\begin{aligned} R(X, Y) \xi_{i}= & \kappa\left(\overline{\eta}(X) \varphi^{2} Y-\overline{\eta}(Y) \varphi^{2} X\right) \\ & +\mu\left(\overline{\eta}(Y) h X-\overline{\eta}(X) h Y\right) \\ & +\nu\left(\overline{\eta}(Y) \varphi h X-\overline{\eta}(X) \varphi h Y\right). \end{aligned} \tag{2.12}$$

Furthermore, using (2.12), the symmetry properties of the curvature tensor and the symmetry of φ^2 and h, we get

$$\begin{aligned} R(\xi_{i}, X) Y= & \kappa\left(\overline{\eta}(Y) \varphi^{2} X-g\left(X, \varphi^{2} Y\right) \overline{\xi}\right) \\ & +\mu\left(g(X, h Y) \overline{\xi}-\overline{\eta}(Y) h X\right) \\ & +\nu\left(g(\varphi h X, Y) \overline{\xi}-\overline{\eta}(Y) \varphi h X\right) \end{aligned} \tag{2.13}$$

Remark 2.2. Let M be an almost α-cosymplectic f-manifold verifying (κ, μ, ν)-nullity condition, with $\kappa \neq -\alpha^2$. We denote by $\mathcal{D}_+$ and $\mathcal{D}_-$ the n-dimensional distributions of the eigenspaces of $\lambda = \sqrt{-(\kappa+\alpha^2)}$ and $-\lambda$, respectively. We have that $\mathcal{D}_+$ and $\mathcal{D}_-$ are mutually orthogonal. Moreover, since φ anticommutes with h, we have $\varphi(\mathcal{D}_+) = \mathcal{D}_-$ and $\varphi(\mathcal{D}_-) = \mathcal{D}_+$. In other words, $\mathcal{D}_+$ is a Legendrian distribution and $\mathcal{D}_-$ is the conjugate Legendrian distribution of $\mathcal{D}_+$.

3. Curvature Properties

Let $M\left(\varphi,\xi_i,\eta^i,g\right)$ be a $(2n+s)$-dimensional almost Kenmotsu f-manifold. We consider the $(1,1)$-tensor fields defined by

$$l_{ij}\left(X\right)=R_{X\xi_i}\xi_j$$

for each $X\in\Gamma\left(TM\right),\ i,j\in\{1,...,s\}$ and put $l_i=l_{ii}$.

Proposition 3.1. [4] *Let M be an almost α-cosymplectic f-manifold. Then we have*

$$\begin{aligned}(3.1)\qquad R\left(X,Y\right)\xi_i &= \alpha^2\sum_{k=1}^{s}\left[\eta^k\left(Y\right)\varphi^2X-\eta^k\left(X\right)\varphi^2Y\right]\\ &-\alpha\sum_{k=1}^{s}\left[\eta^k\left(X\right)\left(\varphi\circ h_k\right)Y-\eta^k\left(Y\right)\left(\varphi\circ h_k\right)X\right]\\ &+\left(\nabla_Y\left(\varphi\circ h_i\right)\right)X-\left(\nabla_X\left(\varphi\circ h_i\right)\right)Y,\end{aligned}$$

for each $X\in\Gamma\left(TM\right)$.

Lemma 3.1. [4] *For an almost α-cosymplectic f-manifold with the f-structure $\left(\varphi,\xi_i,\eta^i,g\right)$, the following relations hold*

(3.2)

$$l_{ji}\left(X\right)=\sum_{k=1}^{s}\delta_j^k\left[\alpha^2\varphi^2X+\alpha\left(\varphi\circ h_k\right)X\right]+\alpha\left(\varphi\circ h_i\right)X-\left(h_i\circ h_j\right)X+\varphi\left(\nabla_{\xi_j}h_i\right)X,$$

$$R\left(\xi_j,X\right)\xi_i-\varphi R\left(\xi_j,\varphi X\right)\xi_i=2\left[-\alpha^2\varphi^2X+\left(h_i\circ h_j\right)X\right],\tag{3.3}$$

$$\left(\nabla_{\xi_j}h_i\right)X=-\varphi l_{ji}\left(X\right)-\alpha^2\varphi X-\alpha h_iX-\alpha h_jX-\left(\varphi\circ h_i\circ h_j\right)X,\tag{3.4}$$

$$S\left(X,\xi_i\right)=-2n\alpha^2\sum_{k=1}^{s}\eta^k\left(X\right)-\left(div\left(\varphi\circ h_i\right)\right)X,\tag{3.5}$$

$$S\left(\xi_i,\xi_j\right)=-2n\alpha^2-tr\left(h_j\circ h_i\right),\tag{3.6}$$

$$\left(\nabla_{\xi_i}h_i\right)X=-\varphi l_{ii}\left(X\right)-\alpha^2\varphi X-2\alpha h_iX-\left(\varphi\circ h_i^2\right)X\tag{3.7}$$

for each $X\in\Gamma\left(TM\right)$.

Remark 3.1. Let M be an almost α-cosymplectic f-manifold verifying (κ,μ,ν)-nullity condition. Then for each $i,j\in\{1,...,s\}$ we have

$$l_{ji}=-\kappa\varphi^2+\mu h+\nu\varphi h.\tag{3.8}$$

It follows that all the l_{ji}'s coincide. We put $l=l_{ji}$.

Lemma 3.2. *Let M be an almost α-cosymplectic f-manifold verifying (κ,μ,ν)-nullity condition. Then for each $i\in\{1,...,s\}$ and $X\in\Gamma\left(TM\right)$ we have*

$$\left(\nabla_{\xi_i}h\right)X=-\mu\varphi hX+\nu hX-2\alpha hX,\tag{3.9}$$

$$l\varphi-\varphi l=2\mu h\varphi+2\nu h,\tag{3.10}$$

$$l\varphi+\varphi l=2\kappa\varphi,\tag{3.11}$$

$$Q\xi_i=2n\kappa\overline{\xi}.\tag{3.12}$$

Proof. From (3.7), using (3.8) and (**??**) we get (3.9). (3.10) and (3.11) follow directly from (3.8) using $h \circ \varphi = -\varphi \circ h$. For the proof of (3.12) we fix $x \in M$ and $\{E_1, ..., E_{2n+s}\}$ a local φ-basis around x with $E_{2n+1} = \xi_1, ..., E_{2n+s} = \xi_s$. Then using (2.13) and $trace\,(h) = 0$ and $trace\,(\varphi h) = 0$ we get $Q\xi_i = \sum_{j=1}^{2n} R_{\xi_i E_j} E_j = \sum_{j=1}^{2n} \kappa g\left(\varphi^2 E_j, E_j\right) \overline{\xi} = \kappa \sum_{j=1}^{2n} \delta_{jj} \overline{\xi}$. □

4. On Pseudo Ricci-Symmetric Almost α-Cosymplectic f-Manifolds

Definition 4.1. A $(2n+s)$-dimensional almost α-cosymplectic f-manifold (M, g) is called a pseudo Ricci-symmetric manifold $(SWRS)_{(2n+s)}$ if

$$(\nabla_X S)(Y, Z) = 2\beta(X) S(Y, Z) + \beta(Y) S(X, Z) + \beta(Z) S(Y, X), \tag{4.1}$$

for any $X, Y, Z \in \Gamma(TM)$, where β is a 1-form and is defined by

$$\beta(X) = g(X, \rho), \tag{4.2}$$

where ρ is associated vector field. This notion was introduced by Chaki [29].

Theorem 4.1. *If a pseudo Ricci-symmetric almost α-cosymplectic f-manifold admits a cyclic parallel Ricci tensor then the* 1*-form β must be vanish.*

Proof. Let (4.1) and (4.2) be satisfied in an almost α-cosymplectic f-manifold. Taking the cyclic sum in (4.1) we get,

$$\begin{aligned} &(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \\ &= 4\left(\beta(X) S(Y, Z) + \beta(Y) S(X, Z) + \beta(Z) S(X, Y)\right) \end{aligned} \tag{4.3}$$

Let M admit a cyclic Ricci tensor. Then (4.3) reduces to

$$\left(\beta(X) S(Y, Z) + \beta(Y) S(X, Z) + \beta(Z) S(X, Y)\right) = 0. \tag{4.4}$$

Taking $Z = \xi_i$ in (4.4) and using (3.5) and (4.2), we obtain

$$\begin{aligned} &\beta(X)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(Y) - (div(\varphi \circ h_i)) Y\right] \\ &+\beta(Y)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(X) - (div(\varphi \circ h_i)) X\right] \\ &+\eta_i(\rho) S(X, Y) = 0. \end{aligned} \tag{4.5}$$

Now putting $Y = \xi_i$ in (4.5) and using (2.4), (3.5) (4.2) and $(div(\varphi \circ h_i)) \xi_i$, we get

$$\begin{aligned} &\beta(X)\left[-2n\alpha^2\right] \\ &+\eta_i(\rho)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(X) - (div(\varphi \circ h_i)) X\right] \\ &+\eta_i(\rho)\left[-2n\alpha^2 \textstyle\sum_{k=1}^{s} \eta^k(X) - (div(\varphi \circ h_i)) X\right] = 0. \end{aligned} \tag{4.6}$$

Taking $X = \xi_i$ (4.6) and using (2.4), (3.5) (4.2) and $(div(\varphi \circ h_i)) \xi_i$, we have

$$\eta_i(\rho) = 0. \tag{4.7}$$

So by use of (4.7) in (4.6), since $\alpha \neq 0$, we get $\beta(X) = 0$, for any vector field X on M. This completes the proof of the theorem. □

Theorem 4.2. *If a pseudo Ricci-symmetric almost α-cosymplectic f-manifold verifying (κ, μ, ν)-nullity condition, admits a cyclic parallel Ricci tensor then the* 1*-form β must be vanish.*

Proof. Let (4.1) and (4.2) be satisfied in an almost α-cosymplectic f-manifold verifying (κ, μ, ν)-nullity condition, . Taking the cyclic sum in (4.1) we get,

$$\begin{aligned} &(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \\ &= 4\left(\beta(X) S(Y, Z) + \beta(Y) S(X, Z) + \beta(Z) S(X, Y)\right) \end{aligned} \tag{4.8}$$

Let M admit a cyclic Ricci tensor. Then (4.8) reduces to

$$(\beta(X)S(Y,Z)+\beta(Y)S(X,Z)+\beta(Z)S(X,Y))=0. \tag{4.9}$$

Taking $Z=\xi_i$ in (4.9) and using (3.12) and (4.2), we obtain

$$\beta(X)[2n\kappa\overline{\eta}(Y)]+\beta(Y)[2n\kappa\overline{\eta}(X)]+\eta_i(\rho)S(X,Y)=0. \tag{4.10}$$

Now putting $Y=\xi_i$ in (4.10) and using (2.4), (3.12) and (4.2), we get

$$\beta(X)[2n\kappa]+\eta_i(\rho)[2n\kappa\overline{\eta}(X)]+\eta_i(\rho)[2n\kappa\overline{\eta}(X)]=0. \tag{4.11}$$

Taking $X=\xi_i$ (4.11) and using (2.4), (3.12) and (4.2), we have

$$\eta_i(\rho)=0. \tag{4.12}$$

So by use of (4.12) in (4.11), we get $\beta(X)=0$, for any vector field X on M. This completes the proof of the theorem. □

Theorem 4.3. *A pseudo Ricci-symmetric almost α-cosymplectic f-manifold can not be an Einstein manifold if the 1-form $\beta=0$.*

Proof. For an Einstein manifold $(\nabla_X S)(Y,Z)$ and $S(Y,Z)=kg(Y,Z)$ $(k\in\mathbb{R})$ then (4.1) gives

$$2\beta(X)S(Y,Z)+\beta(Y)S(X,Z)+\beta(Z)S(Y,X)=0. \tag{4.13}$$

Taking $Z=\xi_i$ in (4.13) and using (3.5) and (4.2), we obtain

$$\begin{aligned}&2\beta(X)\left[-2n\alpha^2\textstyle\sum_{k=1}^s\eta^k(Y)-(div(\varphi\circ h_i))Y\right]\\&+\beta(Y)\left[-2n\alpha^2\textstyle\sum_{k=1}^s\eta^k(X)-(div(\varphi\circ h_i))X\right]\\&+\eta_i(\rho)S(Y,X)=0.\end{aligned} \tag{4.14}$$

Now putting $Y=\xi_i$ in (4.14) and using (2.4), (3.5) (4.2) and $(div(\varphi\circ h_i))\xi_i$, we get

$$\begin{aligned}&2\beta(X)\left[-2n\alpha^2\right]\\&+\eta_i(\rho)\left[-2n\alpha^2\textstyle\sum_{k=1}^s\eta^k(X)-(div(\varphi\circ h_i))X\right]\\&+\eta_i(\rho)\left[-2n\alpha^2\textstyle\sum_{k=1}^s\eta^k(X)-(div(\varphi\circ h_i))X\right]=0\end{aligned} \tag{4.15}$$

Taking $X=\xi_i$ in (4.15) and using (2.4), (3.5) (4.2) and $(div(\varphi\circ h_i))\xi_i$, we have

$$\eta_i(\rho)=0. \tag{4.16}$$

So by use of (4.16) in (4.15), since $\alpha\neq 0$, we get $\beta(X)=0$, for any vector field X on M. This completes the proof of the theorem. □

Theorem 4.4. *A pseudo Ricci-symmetric almost α-cosymplectic f-manifold verifying (κ,μ,ν)-nullity condition,can not be an Einstein manifold if the 1-form $\beta\neq 0$.*

Proof. From the previous theorem, taking $Z=\xi_i$ in (4.13) and using (3.5) and (4.2), we obtain

$$2\beta(X)[2n\kappa\overline{\eta}(Y)]+\beta(Y)[2n\kappa\overline{\eta}(X)]+\eta_i(\rho)S(Y,X)=0. \tag{4.17}$$

Now putting $Y=\xi_i$ in (4.17) and using (2.4), (3.12) and (4.2), we get

$$2\beta(X)[2n\kappa]+\eta_i(\rho)[2n\kappa\overline{\eta}(X)]+\eta_i(\rho)[2n\kappa\overline{\eta}(X)]=0 \tag{4.18}$$

Taking $X=\xi_i$ in (4.18) and using (2.4), (3.12) and (4.2), we have

$$\eta_i(\rho)=0. \tag{4.19}$$

So by use of (4.19) in (4.18), we get $\beta(X)=0$, for any vector field X on M. This completes the proof of the theorem. □

Theorem 4.5. *The Ricci tensor of a pseudo Ricci-symmetric almost α-cosymplectic f-manifold is parallel.*

Proof. Taking $Z=\xi_i$ in (4.1), we get

$$(\nabla_X S)(Y,\xi_i)=2\beta(X)S(Y,\xi_i)+\beta(Y)S(X,\xi_i)+\beta(\xi_i)S(Y,X). \tag{4.20}$$

The left-hand side can be written in the form

$$(\nabla_X S)(Y,\xi_i)=\nabla_X S(Y,\xi_i)-S(\nabla_X Y,\xi_i)-S(Y,\nabla_X\xi_i). \tag{4.21}$$

Then, in view of (3.5), (4.2) and (4.21), equation (4.20) becomes

$$\begin{aligned}&\nabla_X S(Y,\xi_i)-S(\nabla_X Y,\xi_i)-S(Y,\nabla_X\xi_i)=\\&2\beta(X)\left[-2n\alpha^2\textstyle\sum_{k=1}^{s}\eta^k(Y)-(div(\varphi\circ h_i))Y\right]\\&+\beta(Y)\left[-2n\alpha^2\textstyle\sum_{k=1}^{s}\eta^k(X)-(div(\varphi\circ h_i))X\right]\\&+\eta_i(\rho)S(Y,X).\end{aligned} \tag{4.22}$$

Now putting $Y=\xi_i$ in (4.22) and using (2.3), (2.4) (**??**) (3.5), (4.2) and $(div(\varphi\circ h_i))\xi_i$, we get

$$\begin{aligned}&0=2\beta(X)\left[-2n\alpha^2\right]\\&+\eta_i(\rho)\left[-2n\alpha^2\textstyle\sum_{k=1}^{s}\eta^k(X)-(div(\varphi\circ h_i))X\right]\\&+\eta_i(\rho)\left[-2n\alpha^2\textstyle\sum_{k=1}^{s}\eta^k(X)-(div(\varphi\circ h_i))X\right].\end{aligned} \tag{4.23}$$

Taking $X=\xi_i$ in (4.23)

$$\eta_i(\rho)=0. \tag{4.24}$$

Using (4.24) in (4.23)

$$\beta(X)=0 \tag{4.25}$$

for any vector field X on M. Hence in view of (4.25) in (4.1), we obtain $\nabla_X S=0$, which is proves the result. □

Theorem 4.6. *The Ricci tensor of a pseudo Ricci-symmetric almost α-cosymplectic f-manifold is parallel.*

Proof. Taking $Z=\xi_i$ in (4.1), we get

$$(\nabla_X S)(Y,\xi_i)=2\beta(X)S(Y,\xi_i)+\beta(Y)S(X,\xi_i)+\beta(\xi_i)S(Y,X). \tag{4.26}$$

The left-hand side can be written in the form

$$(\nabla_X S)(Y,\xi_i)=\nabla_X S(Y,\xi_i)-S(\nabla_X Y,\xi_i)-S(Y,\nabla_X\xi_i). \tag{4.27}$$

Then, in view of (3.12), (4.2) and (4.27), equation (4.26) becomes

$$\begin{aligned}&\nabla_X S(Y,\xi_i)-S(\nabla_X Y,\xi_i)-S(Y,\nabla_X\xi_i)=\\&2\beta(X)\left[2n\kappa\overline{\eta}(Y)\right]+\beta(Y)\left[2n\kappa\overline{\eta}(X)\right]\\&+\eta_i(\rho)S(Y,X).\end{aligned} \tag{4.28}$$

Now putting $Y=\xi_i$ in (4.28) and using (2.3), (2.4) (**??**) (3.12) and (4.2) we get

$$0=2\beta(X)\left[2n\kappa\right]+\eta_i(\rho)\left[2n\kappa\overline{\eta}(X)\right]+\eta_i(\rho)\left[2n\kappa\overline{\eta}(X)\right]. \tag{4.29}$$

Taking $X=\xi_i$ in (4.29)

$$\eta_i(\rho)=0. \tag{4.30}$$

Using (4.30) in (4.29)

$$\beta(X)=0 \tag{4.31}$$

for any vector field X on M. Hence in view of (4.31) in (4.1), we obtain $\nabla_X S = 0$, which is proves the result. $\square$

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