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SOME RL-INTEGRAL INEQUALITIES FOR THE WEIGHTED AND THE EXTENDED CHEBYSHEV FUNCTIONALS

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Abstract. In this paper, we use the Riemann-Liouville fractional integrals to establish some new integral inequalities related to the weighted and the extended Chebyshev functionals. Some classical results are also presented.

1. Introduction

Let us consider the weighted Chebyshev functional [2]:

$$T(f,g,p) := \int_a^b p(x) \int_a^b f(x)g(x)p(x) - \int_a^b f(x)p(x) \int_a^b g(x)p(x), \tag{1.1}$$

where f and g are two integrable functions on $[a,b]$ and p is a positive and integrable function on $[a,b]$.
In [8], N. Elezovic et al. proved that

$$\begin{aligned} |T(f,g,p)| &\leq \frac{1}{2}\left(\int_a^b \int_a^b p(x)p(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \left|\int_y^x |f'(t)|^{\alpha}\,dt\right|^{\frac{\gamma}{\alpha}} dxdy\right)^{\frac{1}{\gamma}} \\ &\quad \times \left(\int_a^b \int_a^b p(x)p(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \left|\int_y^x |g'(t)|^{\beta}\,dt\right|^{\frac{\gamma'}{\beta}} dxdy\right)^{\frac{1}{\gamma'}} \\ &\leq \frac{1}{2}||f'||_{\alpha}||g'||_{\beta}\left(\int_a^b \int_a^b p(x)p(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\,dxdy\right), \end{aligned} \tag{1.2}$$

where $f' \in L^{\alpha}([a,b])$ and $g' \in L^{\beta}([a,b]); \alpha > 1, \beta > 1, \gamma > 1, \alpha^{-1} + {\alpha'}^{-1} = 1, \beta^{-1} + {\beta'}^{-1} = 1, \gamma^{-1} + {\gamma'}^{-1} = 1.$

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Then, in [5], Z. Dahmani et al. proved the following result in fractional integral case :

$$\begin{aligned} & 2\left|J^{\delta}p(t)J^{\delta}pfg(t)-J^{\delta}pf(t)J^{\delta}pg(t)\right| \\ (1.3) \qquad & \leq \frac{||f'||_{\alpha}\ ||g'||_{\beta}}{\Gamma^{2}(\delta)}\int_{0}^{t}\int_{0}^{t}(t-x)^{\delta-1}(t-y)^{\delta-1}\left|x-y\right|p(x)p(y)dxdy, \end{aligned}$$

where $f'\in L^{\alpha}([o,\infty[)$ and $g'\in L^{\beta}([o,\infty[);\alpha>1,\beta>1,\alpha^{-1}+\beta^{-1}=1$.

Furthermore, taking q a positive and integrable function on $[a,b]$, we consider the extended Chebyshev's functional [3, 10]:

$$\begin{aligned} (1.4)\ \tilde{T}(f,g,p,q) \quad : \quad = & \int_{a}^{b}q(x)\int_{a}^{b}p(x)f(x)g(x)+\int_{a}^{b}p(x)\int_{a}^{b}q(x)f(x)g(x) \\ & -\int_{a}^{b}p(x)f(x)\int_{a}^{b}q(x)g(x)-\int_{a}^{b}q(x)f(x)\int_{a}^{b}p(x)g(x). \end{aligned}$$

Many researchers have given considerable attention to the functionals (1.1) and (1.4). For more details, we refer the reader to [1, 4, 6, 7, 11, 12, 13, 14] and the references therein.

The main purpose of this paper is to establish some new inequalities for (1.1) and (1.4) by using the Riemann-Liouville fractional integrals. For our results, some classical inequalities can be deduced as some particular cases. We also present and prove an auxiliary classical result related to the extended Chebyshev functional.

2. Preliminaries on Riemann-Liouville Integration

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\delta\geq 0$, for a continuous function f on $[a,b]$ is defined as

$$\begin{aligned} (2.1) \qquad J_{a}^{\delta}f\left(t\right) & = \frac{1}{\Gamma\left(\delta\right)}\int_{a}^{t}\left(t-\tau\right)^{\delta-1}f\left(\tau\right)d\tau,\ \ \delta>0,\ \ a<t\leq b, \\ J_{a}^{0}f\left(t\right) & = f\left(t\right), \end{aligned}$$

For $\delta>0$, $\lambda>0$, we have:

$$(2.2) \qquad J_{a}^{\delta}J_{a}^{\lambda}f\left(t\right)=J_{a}^{\delta+\lambda}f\left(t\right)$$

which implies the commutative property

$$(2.3) \qquad J_{a}^{\delta}J_{a}^{\lambda}f\left(t\right)=J_{a}^{\lambda}J_{a}^{\delta}f\left(t\right).$$

For more details, one can consult [9].

3. Main Results

We begin by proving the following theorem which generalises one of the result in [5]. We have:

Theorem 3.1. *Let f and g be two differentiable functions on $[0,\infty[$ and let p be positive and integrable function on $[0,\infty[$. If $f'\in L^{\alpha}([0,\infty[)$ and $g'\in L^{\beta}([0,\infty[);\alpha>1,\beta>1,\gamma>1$ with $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$, $\frac{1}{\beta}+\frac{1}{\beta'}=1$ and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0,\delta>0$*

we have the inequality

$$2\left|J^{\delta}p(t)J^{\delta}pfg(t)-J^{\delta}pf(t)J^{\delta}pg(t)\right| \tag{3.1}$$
$$\leq \left(\frac{||f'||_{\alpha}^{\gamma}}{\Gamma(\delta)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)\left|x-y\right|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dxdy\right)^{\frac{1}{\gamma}}$$
$$\times\left(\frac{||g'||_{\beta}^{\gamma'}}{\Gamma(\delta)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)\left|x-y\right|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dxdy\right)^{\frac{1}{\gamma'}}$$
$$\leq \frac{||f'||_{\alpha}||g'||_{\beta}}{\Gamma^2(\delta)}\left(\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}p(x)p(y)dxdy\right).$$

Proof. Let f and g be two functions that satisfy the conditions of Theorem 3.1 and let p be a positive and integrable function on $[0,\infty[$.

We consider the quantity H (already introduced in [5] and in some other references):

$$H(x,y):=(f(x)-f(y))(g(x)-g(y)).\quad x,y\in(0,t),t>0. \tag{3.2}$$

Multiplying (3.2) by $\frac{(t-x)^{\delta-1}}{\Gamma(\delta)}p(x), x\in(0,t)$, and integrating the resulting identity with respect to x over $(0,t)$, we can state that

$$\frac{1}{\Gamma(\delta)}\int_0^t (t-x)^{\delta-1}p(x)H(x,y)dx \tag{3.3}$$
$$= J^{\delta}pfg(t)-g(y)J^{\delta}pf(t)-f(y)J^{\delta}pg(t)+f(y)g(y)J^{\delta}p(t).$$

With the same arguments, we obtain:

$$\frac{1}{\Gamma^2(\delta)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)H(x,y)dxdy \tag{3.4}$$
$$= 2(J^{\delta}p(t)J^{\delta}pfg(t)-J^{\delta}pf(t)J^{\delta}pg(t)).$$

On the other hand, it is clear that

$$H(x,y)=\int_y^x\int_y^x f'(\tau)g'(\mu)d\tau d\mu. \tag{3.5}$$

Thanks to Hölder's inequality, it follows that:

$$|f(x)-f(y)| \leq |x-y|^{\frac{1}{\alpha'}}\left|\int_y^x |f'(\tau)|^{\alpha}d\tau\right|^{\frac{1}{\alpha}}. \tag{3.6}$$
$$|g(x)-g(y)| \leq |x-y|^{\frac{1}{\beta'}}\left|\int_y^x |g'(\mu)|^{\beta}d\mu\right|^{\frac{1}{\beta}}.$$

Then, H can be estimated as follows:

$$|H(x,y)|\leq|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x |f'(\tau)|^{\alpha}d\tau\right|^{\frac{1}{\alpha}}\left|\int_y^x |g'(\mu)|^{\beta}d\mu\right|^{\frac{1}{\beta}}. \tag{3.7}$$

Hence, we can write

$$2|J^{\delta}p(t)J^{\delta}pfg(t)-J^{\delta}pf(t)J^{\delta}pg(t)|$$
$$\leq \frac{1}{\Gamma^2(\delta)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x |f'(\tau)|^{\alpha}d\tau\right|^{\frac{1}{\alpha}}\left|\int_y^x |g'(\mu)|^{\beta}d\mu\right|^{\frac{1}{\beta}}dxdy.$$

By Hölder's inequality for double integral, we obtain:

$$\begin{aligned} &2|J^\delta p(t)J^\delta pfg(t) - J^\delta pf(t)J^\delta pg(t)| \\ \leq\ & \frac{1}{\Gamma^2(\delta)}\left(\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x |f'(\tau)|^\alpha d\tau\right|^{\frac{\gamma}{\alpha}} dxdy\right)^{\frac{1}{\gamma}} \\ & \times\left(\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x |g'(\mu)|^\beta d\mu\right|^{\frac{y'}{\beta}} dxdy\right)^{\frac{1}{\gamma'}}. \end{aligned}$$

Using the following properties

$$\left|\int_y^x |f'(\tau)|^\alpha d\tau\right| \leq ||f'||_\alpha^\alpha \ , \quad \left|\int_y^x |g'(\mu)|^\beta d\mu\right| \leq ||g'||_\beta^\beta \ , \tag{3.8}$$

we observe that

$$\begin{aligned} &2|J^\delta p(t)J^\delta pfg(t) - J^\delta pf(t)J^\delta pg(t)| \\ \leq\ & \left(\frac{||f'||_\alpha^\gamma}{\Gamma^\gamma(\delta)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dxdy\right)^{\frac{1}{\gamma}} \\ & \times\left(\frac{||g'||_\beta^{\gamma'}}{\Gamma^{\gamma'}(\delta)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dxdy\right)^{\frac{1}{\gamma'}}. \end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned} &2|J^\delta p(t)J^\delta pfg(t) - J^\delta pf(t)J^\delta pg(t)| \\ \leq\ & \frac{||f'||_\alpha ||g'||_\beta}{\Gamma^2(\delta)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}p(x)p(y)dxdy. \end{aligned} \tag{3.10}$$

Theorem 3.1 is thus proved. □

Remark 3.1. Taking $\delta = 1$ in Theorem 3.1, we obtain the inequality (1.2) on $[0, t]$.

Now, we prove the following classical result:

Theorem 3.2. *Let f and g be two differentiable functions on $[a, b]$ and let p, q be two positive and integrable functions on $[a, b]$. If $f' \in L^\alpha([a, b])$ and $g' \in L^\beta([a, b])$; $\alpha > 1, \beta > 1, \gamma > 1$ with $\frac{1}{\alpha}+\frac{1}{\alpha'} = 1, \frac{1}{\beta}+\frac{1}{\beta'} = 1$ and $\frac{1}{\gamma}+\frac{1}{\gamma'} = 1$, then we have:*

$$\begin{aligned} \left|\tilde{T}(f,g,p,q)\right| \leq\ & \left(\int_a^b\int_a^b p(x)q(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x |f'(\tau)|^\alpha\, d\tau\right|^{\frac{\gamma}{\alpha}} dxdy\right)^{\frac{1}{\gamma}} \\ & \times\left(\int_a^b\int_a^b p(x)q(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x |g'(\mu)|^\beta\, d\mu\right|^{\frac{\gamma'}{\beta}} dxdy\right)^{\frac{1}{\gamma'}} \\ \leq\ & ||f'||_\alpha ||g'||_\beta\left(\int_a^b\int_a^b p(x)q(y)\,|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dxdy\right). \end{aligned} \tag{3.11}$$

Proof. We have:

$$\begin{aligned}\left|\tilde{T}(f,g,p,q)\right| &\leq \int_a^b\int_a^b p(x)q(y)\left|H(x,y)\right|dxdy\\ &\leq \int_a^b\int_a^b p(x)q(y)\left|x-y\right|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x\left|f'(t)\right|^{\alpha}dt\right|^{\frac{1}{\alpha}}\left|\int_y^x\left|g'(t)\right|^{\beta}dt\right|^{\frac{1}{\beta}}dxdy.\end{aligned}$$

Applying Hölder inequality for double integral, we get

$$\begin{aligned}\left|\tilde{T}(f,g,p,q)\right| &\leq \left(\int_a^b\int_a^b p(x)q(y)\left|x-y\right|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x\left|f'(\tau)\right|^{\alpha}d\tau\right|^{\frac{\gamma}{\alpha}}dxdy\right)^{\frac{1}{\gamma}}\\ &\times\left(\int_a^b\int_a^b p(x)q(y)\left|x-y\right|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x\left|g'(\mu)\right|^{\beta}d\mu\right|^{\frac{\gamma'}{\beta}}dxdy\right)^{\frac{1}{\gamma'}}.\end{aligned}$$

Consequently,

$$\left|\tilde{T}(f,g,p,q)\right| \leq\leq ||f'||_{\alpha}||g'||_{\beta}\left(\int_a^b\int_a^b p(x)q(y)\left|x-y\right|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dxdy\right).$$

□

We present also the following theorem which generalizes Theorem 3.2.

Theorem 3.3. *Let f and g be two differentiable functions on $[0,\infty[$ and let p,q be two positive and integrable functions on $[0,\infty[$. If $f'\in L^{\alpha}([0,\infty[)$ and $g'\in L^{\beta}([0,\infty[);\alpha>1,\beta>1,\gamma>1$ with $\frac{1}{\alpha}+\frac{1}{\alpha'}=1,\frac{1}{\beta}+\frac{1}{\beta'}=1$ and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0,\delta>0$, we have:*

$$\begin{aligned}&\left|J^{\delta}q(t)J^{\delta}pfg(t)+J^{\delta}p(t)J^{\delta}qfg(t)-J^{\delta}pf(t)J^{\delta}qg(t)-J^{\delta}qf(t)J^{\delta}pg(t)\right|\\ &\leq \frac{||f'||_{\alpha}||g'||_{\beta}}{\Gamma^2(\delta)}\int_0^t\int_0^t(t-x)^{\delta-1}(t-y)^{\delta-1}|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}p(x)q(y)dxdy.\end{aligned}\tag{3.12}$$

Proof. Multiplying (3.3) by $\frac{(t-y)^{\delta-1}}{\Gamma(\delta)}q(y),y\in(0,t)$, and integrating the resulting identity with respect to y over $(0,t)$, yields the following inequality

$$\begin{aligned}&\left|J^{\delta}q(t)J^{\delta}pfg(t)+J^{\delta}p(t)J^{\delta}qfg(t)-J^{\delta}pf(t)J^{\delta}qg(t)-J^{\delta}qf(t)J^{\delta}pg(t)\right|\\ \leq\ &\frac{1}{\Gamma^2(\delta)}\int_0^t\int_0^t(t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)\left|x-y\right|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x|f'(\tau)|^{\alpha}d\tau\right|^{\frac{1}{\alpha}}\left|\int_y^x|g'(\mu)|^{\beta}d\mu\right|^{\frac{1}{\beta}}dxdy.\end{aligned}$$

Using the same arguments as in the proof of Theorem 3.1, we obtain Theorem 3.3. □

Remark 3.2. Taking $\delta=1$ in Theorem 3.3, we obtain Theorem 3.2 on $[0,t]$.

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