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Lie Grupta Bir Eğri Boyunca Sabit Ortalama Eğrilikli Yüzeyler Üzerine

Zühal KÜÇÜKARSLAN YÜZBAŞI[1*], Sevinç TAZE[1]

ÖZET:
Bu çalışmada, bir izoparametrik eğri ve onun Frenet çatısı, 3 boyutlu Lie grubunda bir yüzey oluşturmak üzere lineer olarak birleştirilmiştir. Yüzey, verilen eğri boyunca sabit bir ortalama eğriliğe sahip olduğunda, yeterli koşullar karşılanmıştır. Sonuç olarak, elde ettiklerimiz için örnekler verilmiş ve grafikler çizilmiştir.

Öne Çıkanlar:
- İzoparametrik eğri
- Sabit ortalama eğrilik

Anahtar Kelimeler:
- Lie Grup
- Ortalama eğrilik
- Yüzey

On the Surfaces with Constant Mean Curvature along a Curve in the Lie Group

ABSTRACT:
In this study, an isoparametric curve and its Frenet frame are linearly combined to form a surface in 3-dimensional Lie group. When the surface has a constant mean curvature along the given curve, sufficient conditions have been satisfied. In conclusion, we provide examples of our findings and draw graphs.

Highlights:
- Isoparametric curve
- Constant mean curvature

Keywords:
- Lie Group
- Mean curvature
- Surface

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INTRODUCTION

Numerous scholars have extensively investigated the classification of curves on a surface. Afterwards, as a solution to the reverse problem, researchers concentrated on the building surfaces along a particular curve. Initial studies on these topics were done by (Kasap et al. 2006; Li et al. 2011; Wang et al. 2014; Ergün et al. 2014; Yoon et al. 2017,2019). The method used in these publications is as follows: the isoparametric curve's geodesic, asymptotic, and line-of-curvature conditions have been determined, and the parametric surfaces have been created as a linear combination of that curve and its Frenet frame. A new study on construction surfaces with constant curvatures along a particular curve was recently proposed by (Cosanoglu et al. 2020; Bayram et al. 2022).

We organized our paper as follows in this manuscript: We give some basic information regarding curve and surface theory in the 3-dimensional Lie group in Section 2. When the surface has constant mean curvature along the specified curve, we construct a surface along the curve and then deduce sufficient conditions in Section 3. We provide some instances in the final part to illustrate our findings.

MATERIALS AND METHODS

The Frenet formulas for a unit speed curve $\alpha(s)$ in the Lie group such that G has the Levi-Civita connection D are expressed as follows:

$$\begin{bmatrix} D_T T = T'(s) \\ D_T N = N'(s) \\ D_T B = B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & (\kappa_2 - \overline{\kappa_2}) \\ 0 & -(\kappa_2 - \overline{\kappa_2}) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},$$

where κ_1 and κ_2 are the curvature functions of $\alpha(s)$ and $\overline{\kappa_2} = \frac{1}{2}\langle [T, N], B\rangle$ which was introduced (Çiftçi et al. 2009; Okuyucu et al. 2013; Yoon et al. 2012) is the Lie group torsion of for the bi-invariant metric on G. Here $T = \alpha'$, $\kappa_1 = \|T'\|$ and $\kappa_2 = \|B'\| - \overline{\kappa_2}$.

Remark 2.1 Let G be a 3-dimensional Lie group with a bi-invariant metric. Then the following statements hold in different Lie groups:

(i) $\overline{\kappa_2} = \frac{1}{2}$, if G is $SO(3)$.

(ii) $\overline{\kappa_2} = 1$, if G is $S^3 = SU(2)$.

(iii) $\overline{\kappa_2} = 0$, if G is a commutative group, (Çiftçi et al. 2009; Yoon 2012).

Definition 2.2 The mean curvature of $P = P(s,t)$ surface is defined by

$$H(s,t) = \frac{(\det(P_{ss}, P_s, P_t)\|P_t\|^2}{2(\|P_s\|^2\|P_t\|^2 - \langle P_s, P_t\rangle^2)^{\frac{3}{2}}}(s,t) - \frac{2(\det(P_{st}, P_s, P_t)\langle P_s, P_t\rangle}{2(\|P_s\|^2\|P_t\|^2 - \langle P_s, P_t\rangle^2)^{\frac{3}{2}}}(s,t) + \frac{(\det(P_{tt}, P_s, P_t)\|P_s\|^2}{2(\|P_s\|^2\|P_t\|^2 - \langle P_s, P_t\rangle^2)^{\frac{3}{2}}}(s,t), \quad (1)$$

where $\frac{\partial P}{\partial s} = P_s, \frac{\partial P}{\partial t} = P_t$ and $\frac{\partial^2 P}{\partial s^2} = P_{ss}, \frac{\partial^2 P}{\partial t^2} = P_{tt}$.

RESULTS AND DISCUSSION

We will describe the surfaces with constant mean curvature in the three-dimensional Lie Group. Suppose $\alpha(s)$ be an arc length parametrized curve on a surface $P(s,t)$ in G. Then the curve $\alpha(s)$ is called an isoparametric curve if it is a parameter curve, that is, there exists the parameter t_0 such that $\alpha(s) = P(s, t_0)$.

$P = P(s,t)$ is defined based on $\alpha(s)$ and the Frenet frame in Lie group G as follows

$$P(s,t) = \alpha(s) + f(s,t)T(s) + g(s,t)N(s) + h(s,t)B(s), \quad (2)$$
$$L_1 \leq \mathrm{s} \leq L_2 \quad \text{and} \quad T_1 \leq \mathrm{t} \leq T_2$$

where $f(s,t),\ g(s,t)\ and\ h(s,t)$ are all C^1 functions. These functions are called the marching-scale functions.

Since $\alpha(s)$ is an isoparametric curve on this surface, there exists a parameter $t_0 \in [\mathrm{T}_1, T_2]$ such that $\alpha(s) = P(s,\ t_0)$, that is,

$$f(s,t_0) = g(s,t_0) = h(s,t_0) = 0 \quad (3)$$
$$L_1 \leq \mathrm{s} \leq L_2 \text{ and } t_0 \in [\mathrm{T}_1, T_2].$$

To calculate the mean curvature by using the equation (1), we can easily get

$$P_s = (1 + f_s - g\kappa_1)\mathrm{T} + \left(f\kappa_1 + g_s - h(\kappa_2 - \overline{\kappa_2})\right)\mathrm{N} + (g(\kappa_2 - \overline{\kappa_2}) + h_s)\mathrm{B},$$

P_s (s, t_0) = T,

P_{ss} (s, t_0) = κ_1N,

P_t (s,t) =f_tT +g_t N + h_tB,

P_{tt} (s,t) =f_{tt} T + g_{tt}N +h_{tt} B,

P_{ts}=P_{st} = $(-\kappa_1 g_t)$T + $(\kappa_1 f_t - (\kappa_2 - \overline{\kappa_2})\ h_t)$N+$(g_t(\kappa_2 - \overline{\kappa_2})\ )$ B.

Then one may easily calculate the mean curvature using the surface $P(s,t)$ based on the isoparametric curve $\alpha(s)$.

$$H(s,t_0)=\frac{-\kappa_1 h_t\left(f_t{}^2+g_t{}^2+h_t{}^2\right)+(-g_{tt}h_t+g_t h_{tt}) -2f_t\left[(h_t\kappa_1 f_t+ h_t g_{ts}-g_t h_{ts})-(\ h_t{}^2+g_t{}^2)(\kappa_2-\overline{\kappa_2})\ \right]}{2(g_t{}^2+h_t{}^2)^{\frac{3}{2}}}(\mathrm{s},t_0) \quad (4)$$

Therefore we can give the following main theorem:

Theorem 3.1 Let $P(s,t)$ be the surface given by Equation (1). If the mean curvature of $P(s,t)$ in equation (4) along the isoparametric curve $\alpha(s)$ is a constant, then one of the following five conditions is satisfied:

1. $$\begin{cases} f_t(\mathrm{s},t_0) = const. \neq 0, g_t(\mathrm{s},t_0) = const. \neq 0., \\ f(s,t_0) = g(s,t_0) = h(s,t_0) \\ = h_t(\mathrm{s},t_0) = h_{tt}(\mathrm{s},t_0) \equiv 0 \\ (\kappa_2 - \overline{\kappa_2}) = const. \end{cases}$$

2. $$\begin{cases} f_t(\mathrm{s},t_0) = const. \neq 0, h_t(\mathrm{s},t_0) = const. \neq 0, \\ f(s,t_0) = g(s,t_0) \\ = h(s,t_0) = g_t\ (\mathrm{s},t_0) = g_{tt}\ (\mathrm{s},t_0) \equiv 0, \\ (\kappa_2 - \overline{\kappa_2}) = const. \end{cases}$$

3. $$\begin{cases} f_t(\mathrm{s},t_0) = g_t(\mathrm{s},t_0) = h_t(\mathrm{s},t_0) = const. \neq 0, \\ f(s,t_0) = g(s,t_0) = h(s,t_0) \equiv 0, \\ (\kappa_2 - \overline{\kappa_2}) = \kappa_1 = const. \end{cases}$$

4. $$\begin{cases} g_t(\mathrm{s}, t_0) \neq 0, \\ f(s,t_0) = g(s,t_0) = h(s,t_0) \\ = f_t(\mathrm{s}, t_0) = h_t(\mathrm{s}, t_0) = h_{tt}(\mathrm{s}, t_0) = 0. \end{cases}$$

5. $$\begin{cases} f(s,t_0) = g(s,t_0) = h(s,t_0) \\ = f_t(\mathrm{s}, t_0) = g_t(\mathrm{s}, t_0) = g_{tt}(\mathrm{s}, t_0) = 0, \\ h_t(\mathrm{s}, t_0) = const. \neq 0, \\ \kappa_1 = const. \end{cases}$$

Proof: 1. Since the given curve isoparametric then we have $f(s,t_0) = g(s,t_0) = h(s,t_0) = 0$.

Considering $f_t(\mathrm{s}, t_0) = g_t(\mathrm{s}, t_0) \neq 0 = h_t(\mathrm{s}, t_0) = h_{tt}(\mathrm{s}, t_0)$ and subsituting these values into equation (4), we get

$\mathrm{H}(\mathrm{s}, t_0) = \frac{(\kappa_2 - \overline{\kappa_2})\, f_t}{g_t}$.

Therefore, we obtain $(\kappa_2 - \overline{\kappa_2}) = const.$

The proof for the rest of the cases can be done similarly to case 1.

Remark 3.2 Let $\alpha(s)$ be the isoparametric curve, then

i) For the first and second conditions of Theorem 3.1, an anti-Salkowski curve can be given as an example of this type of curve.

ii) For the third condition of Theorem 3.1, helices can be given as an example of this type of curve.

iii) For the fifth condition of Theorem 3.1, a Salkowski curve can be given as an example of this type of curve.

Example 3.3 Let $\alpha(\mathrm{s})$ be a parametrized by

$$\alpha(\mathrm{s}) = (\mathrm{sins}, \mathrm{coss}, 0).$$

Then the Frenet vectors in the three dimensional Lie Group are given as

$$T = (coss, -sins, 0),$$

$$N = (-sins, -coss, 0), B = (0,0,-1),$$

here one may get $\kappa_1 = 1, \kappa_2 = 0$ and $\overline{\kappa_2} = \frac{1}{2}$ by using $\kappa_1 = \|T'\|\ and\ \kappa_2 = \|B'\| - \overline{\kappa_2}$.

Case 1. Considering the first condition of the Theorem 3.1 we can choose $f(s,t) = -t, g(s,t) = t, h(s,t) = t^3$ and $t_0 = 0$. So, the surface $P_1(s,t)$ given by (2) in the Lie group is expressed as $P_1(s,t) = \big((1-t)sins - tcoss, (1-t)coss + tsins, -t^3\big)$,

is plotted in Figure 1, where $-1 \le s \le 1$ and $-2 \le t \le 2$ with constant mean curvature $\mathrm{H}(\mathrm{s}, t_0) = \frac{1}{2}$.

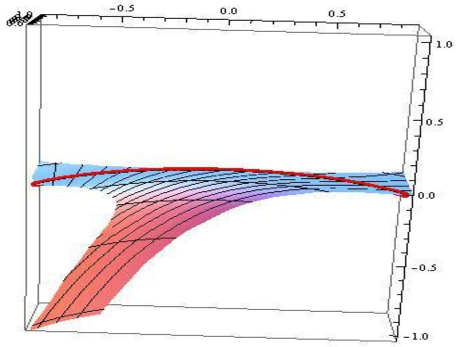

Figure 1. The surface $P_1(s,t)$ with constant mean curvature along the curve α(s)

Case 2. Considering the second condition of Theorem 3.1 we can choose $f(s,t) = t$, $g(s,t) = st^3$, $h(s,t)$=$ssint$ and $t_0 = 0$. So, the surface $P_2(s,t)$ given by (2) in the Lie group is expressed as

$$P_2(s,t) = ((1 - st^3)sins + tcoss, (1 - st^3)\ coss - tsins, -ssint\ )$$

is plotted in Figure 2, where $-1 \leq s \leq 1$ and $-2 \leq t \leq 2$ with constant mean curvature H(s, t_0) $= \frac{-3}{2}$.

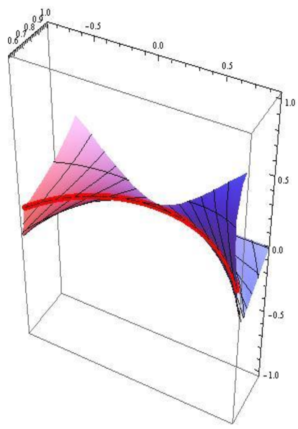

Figure 2. The surface $P_2(s,t)$ with constant mean curvature along the curve α(s)

Case 3. Considering the third condition of Theorem 3.1 we can choose $-f(s,t) = -h(s,t) = g(s,t)$=$e^s t$ and $t_0 = 0$. So, the surface $P_3(s,t)$ given by (2) in the Lie group is expressed as

$$P_3(s,t) = \left((1 - e^s t\ )sins - e^s t\ coss, (1 - e^s t\ )coss + e^s t\ sins, e^s t\ \right)$$

is shown in Figure 3, where $-1 \leq s \leq 1$ and $-2 \leq t \leq 2$ with constant mean curvature H(s, t_0) $= \frac{7}{4}$.

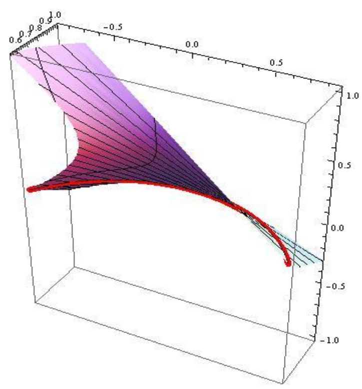

Figure 3. The surface $P_3(s,t)$ with constant mean curvature along the curve α(s)

Case 4. Considering the fourth condition of Theorem 3.1 we can choose $f(s,t) = t^2, g(s,t) = t$, $h(s,t)$=t^3 and $t_0 = 0$. So, the surface $P_4(s,t)$ given by (2) in the Lie group is expressed as

$$P_4(s,t) = ((1-t)sins + t^2coss, (1-t)\ coss - t^2sins, -t^3\ )$$

is shown in Figure 4, where $-1 \leq s \leq 1$ and $-2 \leq t \leq 2$ with constant mean curvature $\mathrm{H}(\mathrm{s}, t_0) = 0$.

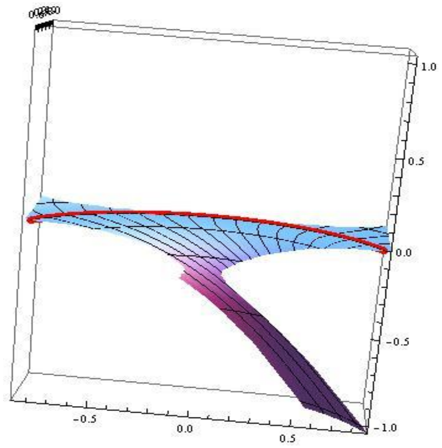

Figure 4. The surface $P_4(s,t)$ with constant mean curvature along the curve $\alpha(s)$

Case 5. Considering the fifth condition of Theorem 3.1 we can choose $f(s,t) = t^2$, $g(s,t) = t^3$, $h(s,t)$=$-t$ and $t_0 = 0$. So, the surface $P_5(s,t)$ given by (2) in the Lie group is expressed as

$$P_5(s,t) = ((1-t^3)sins + t^2\ coss, (1-t^3)\ coss - t^2\ sins, t\ )$$

is shown in Figure 5, where $-1 \leq s \leq 1$ and $-2 \leq t \leq 2$ with constant mean curvature $\mathrm{H}(\mathrm{s}, t_0) = \frac{1}{2}$.

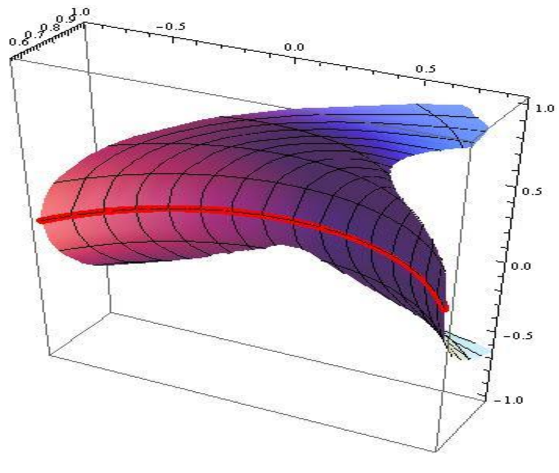

Figure 5. The surface $P_5(s,t)$ with constant mean curvature along the curve $\alpha(s)$

CONCLUSION

We build a surface along the curve and then derive the sufficient conditions when the surface has a constant mean curvature along the given curve. In the conclusion, we provided some examples to illustrate our findings.

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Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author's Contributions

The authors declare that they have contributed equally to the article.

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