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AN ARITHMETIC-GEOMETRIC MEAN INEQUALITY RELATED TO NUMERICAL RADIUS OF MATRICES

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Abstract. For positive matrices $A, B \in \mathbb{M}_n$ and for all $X \in \mathbb{M}_n$, we show that $\omega(AXA) \leq \frac{1}{2}\omega(A^2X + XA^2)$, and the inequality $\omega(AXB) \leq \frac{1}{2}\omega(A^2X + XB^2)$ does not hold in general, where $\omega(.)$ is the numerical radius.

1. Introduction

Let us denote by $\mathbb{M}_n$ the C^*-algebra of all $n \times n$ complex matrices. For $A \in \mathbb{M}_n$ the numerical radius and the operator norm are defined and denoted, respectively, by

$$\omega(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\},$$

and

$$\|A\| = \max\{|x^*Ay| : x, y \in \mathbb{C}^n, x^*x = y^*y = 1\}.$$

We recall the following results that were proved in [3, 6].

Lemma 1.1. *Let $A \in \mathbb{M}_n$ and let $\omega(.)$ be the numerical radius. Then*
(i) $\omega(.)$ is a norm on $\mathbb{M}_n$,
(ii) $\omega(UAU^*) = \omega(A)$, for all unitary matrices U,
(iii) $\omega(A^k) \leq \omega(A)^k, k = 1, 2, 3, \ldots$ (power inequality)
(iv) $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$.

Moreover, $\omega(.)$ is not a unitarily invariant norm and is not submultiplicative. For positive real numbers a, b, the classical Young inequality says that if $p, q > 1$ such that $1/p + 1/q = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.1}$$

Replacing a, b by their squares, we could write (1.1) in the form

$$(ab)^2 \leq \frac{a^{2p}}{p} + \frac{b^{2q}}{q}. \tag{1.2}$$

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Some authors considered replacing the numbers a, b by positive matrices A, B. But there are some difficulties, for example if A and B are positive matrices, the matrix AB is not positive in general. Hence the authors studied the singular values and the norms of the matrices instead of matrices in some inequalities.
In $\mathbb{M}_n$, beside the usual matrix product, the entrywise product is quite important and interesting. The entrywise product of two matrices A, B is called their Schur (or Hadamard) product and denoted by $A \circ B$. With this multiplication $\mathbb{M}_n$ becomes a commutative algebra, for which the matrix with all entries equal to one is the unit. The linear operator S_A on $\mathbb{M}_n$, called the Schur multiplier operator, is defined by $S_A(X) := A \circ X$. The induced norm of S_A with respect to the spectral norm will be denoted by

$$\|S_A\| = \sup_{X \neq 0} \frac{\|S_A(X)\|}{\|X\|} = \sup_{X \neq 0} \frac{\|A \circ X\|}{\|X\|},$$

and the induced norm of S_A with respect to numerical radius norm will be denoted by

$$\|S_A\|_\omega = \sup_{X \neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X \neq 0} \frac{\omega(A \circ X)}{\omega(X)}.$$

Throughout the paper we use the term positive for a positive semidefinite matrix, and strictly positive for a positive definite matrix. Also we use the notation $A \geq 0$ to mean that A is positive, $A > 0$ to mean it is strictly positive, $|||A|||$ to denote an arbitrary unitarily invariant norm of A. It is known that if $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$ [10, page 8]. Also in [8], we established that, if $p > q > 1$ such that $1/p + 1/q = 1$ and $A \in \mathbb{M}_n$ is a non scalar strictly positive matrix with $1 \in \sigma(A)$, then there exists $X \in \mathbb{M}_n$ such that $\omega(AXA) > \omega(\frac{1}{p}A^pX + \frac{1}{q}XA^q)$. In this paper we consider this inequality for $p = q = 2$.

2. MAIN RESULTS

Bhatia and Kittaneh in 1990 [4] established a matrix mean inequality as follows:

$$|||A^*B||| \leq \frac{1}{2}|||A^*A + B^*B|||, \tag{2.1}$$

for matrices $A, B \in \mathbb{M}_n$.

In [3] a generalization of (2.1) was proved, for all $X \in \mathbb{M}_n$,

$$|||A^*XB||| \leq \frac{1}{2}|||AA^*X + XBB^*|||. \tag{2.2}$$

Ando in 1995 [1] established a matrix Young inequality:

$$|||AB||| \leq \left|\left|\left| \frac{A^p}{p} + \frac{B^q}{q} \right|\right|\right| \tag{2.3}$$

for $p, q > 1$ with $1/p + 1/q = 1$ and positive matrices A, B. In [9], we showed that $|||AXB||| \leq \left|\left|\left| \frac{1}{p}A^pX + \frac{1}{q}XB^q \right|\right|\right|$ does not hold in general, and in [8], we considered the inequalities (2.1) and (2.3) with the numerical radius norm as follows:

Proposition 2.1. [8, Proposition 1] *If A, B are $n \times n$ matrices, then*

$$\omega(A^*B) \leq \frac{1}{2}\omega(A^*A + B^*B). \tag{2.4}$$

Also if A and B are positive matrices and $p, q > 1$ with $1/p + 1/q = 1$, then

$$\omega(AB) \leq \omega(\frac{A^p}{p} + \frac{B^q}{q}).$$

Also, in [8] we showed that if $A \in \mathbb{M}_2$ is a non scalar strictly positive matrix such that $1 \in \sigma(A)$, then for all $X \in \mathbb{M}_2$ we have $\omega(AXA) \leq \dfrac{1}{2}\omega(A^2X + XA^2)$. In the following theorem we will generalize this theorem for all $n \times n$ positive matrix A. Therefore we will show that the version of the arithmetic geometric mean inequality with numerical radius holds when $A = B \in \mathbb{M}_n$.

Lemma 2.1. [3, Exercise 1.1.2] *Let $A = [\dfrac{1}{\lambda_i + \lambda_j}] \in \mathbb{M}_n$ be a Cauchy matrix based on positive elements λ_i. Then A is positive.*

Theorem 2.1. *Let $A \in \mathbb{M}_n$ be a positive matrix. Then for all $X \in \mathbb{M}_n$,*

$$\omega(AXA) \leq \frac{1}{2}\omega(A^2X + XA^2). \tag{2.5}$$

Proof. First, we assume that $A = diag(a_1, a_2, \ldots, a_n)$ such that $a_i > 0$ and define $F = [f_{ij}] := \left[\dfrac{2a_ia_j}{a_i^2 + a_j^2}\right]$. Now, let $Y = (a_1, a_2, \ldots, a_n)^t$ and $C := \left[\dfrac{1}{a_i^2 + a_j^2}\right]$ be a Cauchy matrix. Since YY^* and C (using definition of positive matrix and in view of Lemma 2.1) are positive matrices, then $F = 2YY^* \circ C$ is positive; see [10, page 8]. In fact $F = 2YY^* \circ C$, where $Y = (a_1, a_2, \ldots, a_n)^t$ and $C := \left[\dfrac{1}{a_i^2 + a_j^2}\right]$ is a Cauchy matrix, consequently F is positive. By [2, Corollary 4], we have $\|S_F\|_\omega = \max f_{ii} \leq 1$ and hence for all $X \in \mathbb{M}_n$,

$$\omega(AXA) \leq \frac{1}{2}\omega(A^2X + XA^2).$$

Now, assume $A = A_1 \oplus 0$, such that $A_1 \in \mathbb{M}_k$ $(k < n)$ is a strictly positive matrix. Then by the above argument, we obtain $\omega(A_1X_1A_1) \leq \frac{1}{2}\omega(A_1^2X_1 + X_1A_1^2)$, for all $X_1 \in \mathbb{M}_k$. For all $X \in \mathbb{M}_n$, we have $AXA = A_1X_1A_1 \oplus 0$, and

$$\frac{1}{2}(A^2X + XA^2) = \begin{bmatrix} \frac{1}{2}(A_1^2X_1 + X_1A_1^2) & \frac{1}{2}A_1^2X_2 \\ \frac{1}{2}X_3A_1^2 & 0 \end{bmatrix},$$

where $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$. Finally, by [5, Lemma 2.1]

$$\omega(AXA) = \omega(A_1X_1A_1) \leq \frac{1}{2}\omega(A_1^2X_1 + X_1A_1^2) \leq \frac{1}{2}\omega(A^2X + XA^2)$$

and so the inequality (2.5) holds. $\square$

Note that for any matrix F, $\omega\left(\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}\right) = \dfrac{\|F\|}{2}$. So if in the inequality (2.5), A and X are replaced by $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ respectively, then we obtain the following:

Corollary 2.1. *Let $A, B \in \mathbb{M}_n$ be positive matrices. Then for all $X \in \mathbb{M}_n$,*

$$\|AXB\| \leq \frac{1}{2}\|A^2X + XB^2\|.$$

We will show that if $A, B \in \mathbb{M}_n$ are positive matrices, then for all $X \in \mathbb{M}_n$, the inequality

$$\omega(AXB) \leq \frac{1}{2}\omega(A^2X + XB^2) \tag{2.6}$$

does not hold in general (It is clear that for $n = 1$ the inequality (2.6) for all $A, B \geq 0$ and $X \in \mathbb{M}_n$ holds).

Lemma 2.2. [7, Theorem 1] *Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{M}_2$ and $a\bar{c}$ be a real number. Then*

$$\omega(A) = \frac{1}{2}(|a + c| + \sqrt{|b|^2 + |a - c|^2}).$$

Example 2.1. Let $A = I_n(n \geq 2), B = diag(0, 1) \oplus 0_{n-2}$ and $X = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \oplus 0_{n-2}$. Then we have

$$AXB = \begin{bmatrix} 0 & 3 \\ 0 & -2 \end{bmatrix} \oplus 0_{n-2}, A^2X + XB^2 = \begin{bmatrix} 1 & 6 \\ 0 & -4 \end{bmatrix} \oplus 0_{n-2}.$$

Now by Lemma 2.2

$$\omega(AXB) > \frac{1}{2}\omega(A^2X + XB^2). \tag{2.7}$$

In fact for all $A = \alpha I_n(n \geq 2), B = diag(0, \alpha) \oplus 0_{n-2}, (\alpha > 0)$ and $X = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \oplus 0_{n-2}$, the inequality (2.7) holds.

Example 2.2. Let $A = I_2, B = diag((4 \pm \sqrt{12})/2, 1)$ and $X = \begin{bmatrix} 1/(4 \pm \sqrt{12}) & 3 \\ 0 & -2 \end{bmatrix}$. Then $AXB = \begin{bmatrix} 1/2 & 3 \\ 0 & -2 \end{bmatrix}, A^2X + XB^2 = \begin{bmatrix} 2 & 6 \\ 0 & -4 \end{bmatrix}$. Now, by Lemma 2.2 we have

$$\omega(AXB) = 2.7025 > 2.6213 = \frac{1}{2}\omega(A^2X + XB^2).$$

Therefore the inequality (2.7) holds.

Theorem 2.2. *Let $F = \left[\dfrac{2a_ib_j}{a_i^2 + b_j^2}\right]$ be a $n \times n$ matrix such that $a_i, b_j \geq 0$, $(i, j = 1, \ldots, n)$. Then $\|S_F\| \leq 1$.*

Proof. Assume if possible $\|S_F\| > 1$. Then there is $Y \in \mathbb{M}_n$, such that $\|F \circ Y\| > \|Y\|$. Now, if we define the matrices $A := diag(a_1, a_2, \ldots, a_n), B := diag(b_1, b_2, \ldots, b_n)$ and $X := E \circ Y$, where $E = \left[\dfrac{1}{a_i^2 + b_j^2}\right]$, then it is readily seen that $F \circ Y = 2AXB$ and $Y = E^{-1} \circ X = A^2X + XB^2$, where $E^{-1} = \left[a_i^2 + b_j^2\right]$ is inverse of E with respect to the Hadamard product. Thus

$$2\|AXB\| = \|F \circ Y\| > \|Y\| = \|A^2X + XB^2\|.$$

This is a contradiction to Corellary 2.1. □

In the following example, we will show that the converse of Theorem 2.2 does not holds.

Example 2.3. Let $F = I_2$. Then by [2, Corollary 4], $\|S_F\| = 1$.

Proposition 2.2. *Let* $F = \left[\dfrac{2a_ib_j}{a_i^2+b_j^2}\right]$ *be an* $n\times n$ *matrix such that* $a_i, b_j \geq 0$, $(i,j=1,\dots,n)$ *and* $\|S_F\|_\omega \leq 1$. *Then there are matrices* $A, B \geq 0$, *such that for all* $X \in \mathbb{M}_n$ *the reverse of inequality (2.7) holds.*

Proof. If $\|S_F\|_\omega \leq 1$, then by definition we have $\omega(F\circ X) \leq \omega(X)$, for all $X \in \mathbb{M}_n$. Replacing X by $C\circ X$, where $C = [a_i^2+b_j^2]$, we have $\omega(F\circ C\circ X) \leq \omega(C\circ X)$ which is equivalent to $\omega(AXB) \leq \dfrac{1}{2}\omega(A^2X + XB^2)$, such that $A := diag(a_1, a_2, \dots, a_n), B := diag(b_1, b_2, \dots, b_n)$. Hence we get the required result. □

Lemma 2.3. *Let* $F = [f_{ij}] \in \mathbb{M}_2$ *such that* $0 < |f_{ij}| \leq 1$. *Then*

$$\prod_{i,j=1}^{2} |f_{ij}| = \prod_{\substack{i,j=1\\ \epsilon_{ij}\in\{1,-1\}}}^{2} (1+\epsilon_{ij}\sqrt{1-|f_{ij}|^2}),$$

(for at least one of the 16 *possible cases) if and only if there exist positive numbers* a_i *and* $b_i (i = 1,2)$ *such that* $|f_{ij}| = \dfrac{2a_ib_j}{a_i^2+b_j^2}$, *for all* $i,j = 1,2$.

Proof. First we define $T_{ij}^{\pm} := \dfrac{1\pm\sqrt{1-|f_{ij}|^2}}{|f_{ij}|}$ for all $i,j = 1,2$. Easy computation shows that

$$|f_{ij}| = \frac{2a_ib_j}{a_i^2+b_j^2} \Longleftrightarrow a_i = b_jT_{ij}^{\pm} \text{ and } b_j = a_iT_{ij}^{\pm}. \tag{2.8}$$

($\Rightarrow$) Without loss of generality, assume that $b_1 = 1$ and

$$\left(\frac{1+\sqrt{1-|f_{11}|^2}}{|f_{11}|}\right)\left(\frac{1-\sqrt{1-|f_{12}|^2}}{|f_{12}|}\right)\left(\frac{1+\sqrt{1+|f_{21}|^2}}{|f_{21}|}\right)\left(\frac{1-\sqrt{1-|f_{22}|^2}}{|f_{22}|}\right) = 1.$$

Now define $a_1 := b_1T_{11}^+, b_2 := a_1T_{12}^-, a_2 := b_2T_{22}^-$ and consequently, $a_2T_{21}^+ = 1 = b_1$. By using the above definitions and (2.8), we obtain that $|f_{ij}| = \dfrac{2a_ib_j}{a_i^2+b_j^2}$, for all $i,j = 1,2$. The other cases are in the same way.

($\Leftarrow$) Let $|f_{ij}| = \dfrac{2a_ib_j}{a_i^2+b_j^2}$, for all $(i,j=1,2)$. If we define $S_{11} := \dfrac{a_1}{b_1}$, $S_{21} := \dfrac{b_1}{a_2}, S_{22} := \dfrac{a_2}{b_2}$ and $S_{12} := \dfrac{b_2}{a_1}$, then by (2.8) it is easy to show that $S_{ij} = T_{ij}^+$ or $S_{ij} = T_{ij}^-$ and

$$1 = \prod_{i,j=1}^{2} S_{ij} = \prod_{\substack{i,j=1\\ \epsilon_{ij}\in\{1,-1\}}}^{2} \frac{\left(1+\epsilon_{ij}\sqrt{1-|f_{ij}|^2}\right)}{|f_{ij}|}.$$

Therefore,

$$\prod_{i,j=1}^{2} |f_{ij}| = \prod_{\substack{i,j=1\\ \epsilon_{ij}\in\{1,-1\}}}^{2} \left(1+\epsilon_{ij}\sqrt{1-|f_{ij}|^2}\right).$$

□

The following example shows that, we cannot remove the condition $|f_{ij}| > 0$ in Lemma 2.3.

Example 2.4. Let $F = [f_{ij}] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then $\prod_{i,j=1}^{2} |f_{ij}| = \prod_{i,j=1}^{2}(1-\sqrt{1-|f_{ij}|^2})$, but there are not any $a_1, b_1 > 0$ such that $|f_{11}| = \dfrac{2a_1b_1}{a_1^2+b_1^2}$.

Theorem 2.3. *The following are equivalent:*
(a) There is $F = [f_{ij}] \in \mathbb{M}_2$ *with positive entries such that* $\|S_F\|_\omega > 1 \geq \|S_F\|$ *and*

$$\prod_{i,j=1}^{2} f_{ij} = \prod_{\substack{i,j=1\\ \epsilon_{ij}\in\{1,-1\}}}^{2} \left(1+\epsilon_{ij}\sqrt{1-f_{ij}^2}\right),$$

for at least one of the 16 *possible cases*
(b) There are 2×2 *matrices* A, B *and* X *such that* $AB = BA$ *and* $A, B > 0$ *and the inequality (2.7) holds.*

Proof. We define the matrices $D := [2a_ib_j]$, $E := \left[\dfrac{1}{a_i^2+b_j^2}\right]$ and $C := \left[a_i^2+b_j^2\right]$.
$(a) \Longrightarrow (b)$ Since $\|S_F\|_\omega > 1 \geq \|S_F\|$, there exists $Y \in \mathbb{M}_2$ such that $\omega(F\circ Y) > \omega(Y)$ and $|f_{ij}| \leq 1$. In view of Lemma 2.3 there exist $a_i, b_j > 0 (i,j = 1,2)$ such that $f_{ij} = \dfrac{2a_ib_j}{a_i^2+b_j^2}$. Now, define the matrix $X := E\circ Y$. Then $\omega(D\circ X) = \omega(F\circ Y) > \omega(Y) = \omega(C\circ X)$. Hence if we define $A := diag(a_1, a_2)$ and $B := diag(b_1, b_2)$, then $\omega(D\circ X) = 2\omega(AXB)$ and $\omega(C\circ X) = \omega(A^2X + XB^2)$ and hence the inequality (2.7) holds.
$(b) \Longrightarrow (a)$ Without loss of generality, we assume that $A = diag(a_1, a_2)$ and $B = diag(b_1, b_2)$ and $\omega(AXB) > \frac{1}{2}\omega(A^2X + XB^2)$. Now, define $F = [f_{ij}] := \left[\dfrac{2a_ib_j}{a_i^2+b_j^2}\right] \in \mathbb{M}_2$. by Lemma 2.3 we have for at least one of the 16 possible cases

$$\prod_{i,j=1}^{2} f_{ij} = \prod_{\substack{i,j=1\\ \epsilon_{ij}\in\{1,-1\}}}^{2} (1+\epsilon_{ij}\sqrt{1-f_{ij}^2}).$$

Assume if possible, $\|S_F\|_\omega \leq 1$, then for all $Y \in \mathbb{M}_2$, we have $\omega(F\circ Y) \leq \omega(Y)$. Let $Y = C\circ X$. Then $\omega(D\circ X) \leq \omega(C\circ X)$. Since $D\circ X = 2AXB$ and $C\circ X = A^2X + XB^2$, then we have $2\omega(AXB) \leq \omega(A^2X + XB^2)$, a contradiction. Hence $\|S_F\|_\omega > 1$. Also by the inequality (2.2), we know that $\|S_F\| \leq 1$. Then we conclude that $\|S_F\|_\omega > 1 \geq \|S_F\|$. □

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