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Q-FOURIER LIPSCHITZ FUNCTIONS FOR THE GENERALIZED FOURIER TRANSFORM IN THE SPACE $L^2_Q(\mathbb{R})$

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Abstract. In this paper, we prove the generalization of Titchmarsh's theorem for the generalized Fourier transform for functions satisfying the Q-Fourier Lipschitz condition in the space $L^2_Q(\mathbb{R})$.

1. Introduction and Preliminaries

In [3], E. C. Titchmarsh's characterizes the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

Theorem 1.1. *([3]) Let $\delta \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents*

(i) $\|f(t+h) - f(t)\| = O(h^\delta), \quad as \quad h \to 0,$

(ii) $\displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\delta}) \quad as \quad r \to \infty,$

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we prove the generalization of Theorem 1.1 for the generalized Fourier transform for functions satisfying the Q-Fourier Lipschitz condition in the space $L^2_Q(\mathbb{R})$. For this purpose, we use the generalized dual translation operator.

In this section, we develop some results from harmonic analysis related to the differential-difference operator Λ. Further details can be found in [1]-[2] and [6]. Consider the first-order singular differential-difference operator on the real line defined by

$$\Lambda f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x} + q(x)f(x),$$

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where $\alpha > -\frac{1}{2}$ and q is a C^∞ real-valued odd function on $\mathbb{R}$.
Put

$$Q(x) = \exp\left(-\int_0^x q(t)dt\right).$$

We denote by
• $\mathcal{S}(\mathbb{R})$ the space of C^∞ functions f on $\mathbb{R}$, which are rapidly decreasing together with their derivatives, i.e., such that for all $m, n = 0, 1, ...,$

$$p_{m,n}(f) = \sup_{x\in\mathbb{R}} (1+|x|)^m \left|\frac{d^n}{dx^n} f(x)\right| < \infty.$$

The topology of $\mathcal{S}(\mathbb{R})$ is defined by the semi-norms $p_{m,n}, m, n = 0, 1, ...$
• $\mathcal{S}_Q(\mathbb{R})$ the space of C^∞ functions f on $\mathbb{R}$ such that for all $m, n = 0, 1, ...,$

$$P_{m,n}(f) = p_{m,n}(Qf) < \infty.$$

The topology of $\mathcal{S}_Q(\mathbb{R})$ is defined by the semi-norms $P_{m,n}, m, n = 0, 1, ...$
• $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions on $\mathbb{R}$.
• $\mathcal{S}'_Q(\mathbb{R})$ the topological dual of $\mathcal{S}_Q(\mathbb{R})$.
• $L^2_Q(\mathbb{R})$ be the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{2,Q} = \left(\int_{\mathbb{R}} |f(x)Q(x)|^2 |x|^{2\alpha+1} dx\right)^{1/2} < \infty.$$

The generalized Fourier transform of a function f in $\mathcal{S}_Q(\mathbb{R})$ is defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)Q(x)e_\alpha(-i\lambda x)|x|^{2\alpha+1}dx, \quad \lambda \in \mathbb{R}.$$

where e_α denotes the Dunkl kernel on $\mathbb{R}$ defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz) \quad (z \in \mathbb{C}),$$

j_α being the normalized spherical Bessel function of index α given by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}),$$

The following properties collected from [4]-[5] will play a key role in the sequel.

Lemma 1.1. *(i) For all* $x \in \mathbb{R}$, $\quad |e_\alpha(ix)| \leq 1.$
(ii) For all $x \in \mathbb{R}$,

$$|1 - e_\alpha(ix)| \leq |x|.$$

(iii) There is $c_\alpha > 0$ *such that*

$$|1 - e_\alpha(ix)| \geq c_\alpha,$$

for all $x \in \mathbb{R}$ *with* $|x| \geq 1$.

From [6], we have two following theorems

Theorem 1.2. *(i) The dual operator of* Λ*, defined by*

$$\widetilde{\Lambda} f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x} - q(x)f(x),$$

is a linear bounded operator from $S_Q(\mathbb{R})$ *into itself.*
(ii) The generalized Fourier transform $\mathcal{F}$ *is a topological isomorphism from* $S_Q(\mathbb{R})$ *onto* $\mathcal{S}(\mathbb{R})$*. The inverse transform is given by*

$$\mathcal{F}^{-1}(g)(\lambda) = \frac{1}{Q(x)} \int_{\mathbb{R}} g(\lambda) e_\alpha(i\lambda x) d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = \frac{|\lambda|^{2\alpha+1}}{2^{2\alpha+2}(\Gamma(\alpha+1))^2} d\lambda.$$

Theorem 1.3. *(i) For every* $f \in S_Q(\mathbb{R})$ *we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 (Q(x))^2 |x|^{2\alpha+1} dx = \int_{\mathbb{R}} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda). \tag{1.1}$$

(ii) The generalized Fourier transform $\mathcal{F}$ *extends uniquely to an isometric isomorphism from* $L^2_Q(\mathbb{R})$ *onto* $L^2(\mathbb{R}, \sigma)$*.*

The generalized Fourier transform of a distribution $S \in S'_Q(\mathbb{R})$ is defined by

$$\langle \mathcal{F}(S), \psi \rangle = \langle S, \mathcal{F}^{-1}(\psi) \rangle, \quad \psi \in S(\mathbb{R}).$$

Theorem 1.4. *([6]) (i) The generalized Fourier transform* $\mathcal{F}$ *is one-to-one from* $S'_Q(\mathbb{R})$ *onto* $S'(\mathbb{R})$*.*
(ii) If $f \in L^2_Q(\mathbb{R})$*, then the functional*

$$\langle T_f, \psi \rangle = \int_{\mathbb{R}} f(x) \psi(x) |x|^{2\alpha+1} dx, \quad \psi \in S(\mathbb{R}),$$

is a tempered distribution $\mathbb{R}$*. Moreover,*

$$\mathcal{F}(T_{Q^2 f}) = T_g,$$

with

$$g(\lambda) = \frac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2} \mathcal{F}(f)(-\lambda).$$

In all what follows assume $m = 1, 2, ...$. Let $\mathcal{W}^m_{2,Q}$ be the Sobolev type space constructed by the differential-difference operator $\widetilde{\Lambda}$, i.e.,

$$\mathcal{W}^m_{2,Q} = \{ f \in L^2_Q(\mathbb{R}) : \widetilde{\Lambda}^j f \in L^2_Q(\mathbb{R}), j = 1, 2, ..., m \}.$$

More explicitly, $f \in \mathcal{W}^m_{2,Q}$ if and only if for each $j = 1, 2, ..., m$, there is a function in $L^2_Q(\mathbb{R})$ abusively denoted by $\widetilde{\Lambda}^j f$, such that $\widetilde{\Lambda}^j T_f = T_{\widetilde{\Lambda}^j f}$.

Proposition 1.1. *([6]) For* $f \in \mathcal{W}^m_{2,Q}$ *we have*

$$\mathcal{F}(\widetilde{\Lambda}^m f)(\lambda) = (i\lambda)^m \mathcal{F}(f)(\lambda). \tag{1.2}$$

Recall that the Dunkl translation operators τ^x, $x \in \mathbb{R}$ are defined by

$$\begin{aligned} \tau^x f(y) &= \frac{1}{2} \int_{-1}^{1} f(\sqrt{x^2+y^2-2xyt}) \left(1 + \frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right) A_\alpha(t) dt \\ &+ \frac{1}{2} \int_{-1}^{1} f(-\sqrt{x^2+y^2-2xyt}) \left(1 - \frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right) A_\alpha(t) dt, \end{aligned}$$

where

$$A_\alpha(t) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}(1+t)(1-t^2)^{\alpha-1/2}.$$

The generalized translation operators $\mathcal{T}^x$, $x \in \mathbb{R}$, tied to Λ are defined by

$$\mathcal{T}^x(y) = Q(x)Q(y)\tau^x(f/Q)(y).$$

The generalized dual translation operators are given by

$${}^t\mathcal{T}^x f(y) = \frac{Q(x)}{Q(y)}\tau^{-x}(Qf)(y).$$

Proposition 1.2. *([6]) (i) Let $f \in L^2_Q(\mathbb{R})$, Then for all $x \in \mathbb{R}$, ${}^t\mathcal{T}^x f \in L^2_Q(\mathbb{R})$ and*

$$\|{}^t\mathcal{T}^x f\|_{2,Q} \leq 2Q(x)\|f\|_{2,Q}.$$

(ii) For $f \in L^2_Q(\mathbb{R})$ we have

$$\mathcal{F}({}^t\mathcal{T}^x f)(\lambda) = Q(x)e_\alpha(-i\lambda x)\mathcal{F}(f)(\lambda). \tag{1.3}$$

Let $f \in L^2_Q(\mathbb{R})$. We define the differences of the orders m with a step $h > 0$ by

$$\Delta_h^m f(x) = ({}^t\mathcal{T}^h - Q(h)I)^m f(x),$$

where I is the unit operator in $L^2_Q(\mathbb{R})$.

2. Main Results

Lemma 2.1. *For all $f \in \mathcal{W}^m_{2,Q}$ and $h > 0$ we have*

$$\|\Delta_h^m \widetilde{\Lambda}^k f\|^2_{2,Q} = (Q(h))^{2m}\int_{\mathbb{R}} |1 - e_\alpha(-i\lambda h)|^{2m}|\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda),$$

where $k = 0, 1, ..., m$.

Proof. The result follows readily by using (1.1), (1.2), (1.3) and an induction on m. ☐

Definition 2.1. Let $\delta > 1$ and $\beta > 0$. A function $f \in \mathcal{W}^m_{2,Q}$ is said to be in the Q-Fourier Lipschitz class, denoted by $Lip(Q, \delta, \beta)$, if

$$\|\Delta_h^m \widetilde{\Lambda}^k f\|_{2,Q} = O((Q(h))^m h^\delta \psi(h^\beta)) \quad \text{as} \quad h \to 0,$$

where
(a) $k = 0, 1, ..., m$ and ψ is a continuous increasing function on $[0, \infty)$,
(b) $\psi(0) = 0$, $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$,
(c) and

$$\int_0^{1/h} s^{1-2\delta}\psi(s^{-2\beta})ds = O(h^{2\delta-2}\psi(h^{2\beta})), \quad h \to 0.$$

Theorem 2.1. *Let $f \in \mathcal{W}^m_{2,Q}$. Then the following are equivalents*

(i) $f \in Lip(Q, \delta, \beta)$,
(ii) $\displaystyle\int_{|\lambda| \geq r} |\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}\psi(r^{-2\beta})), \quad as \quad r \to \infty.$

Proof. $(i)\Rightarrow (ii)$ Let $f \in Lip(Q,\delta,\beta)$. Then we have

$$\|\Delta_h^m \widetilde{\Lambda}^k f\|_{2,Q} = O((Q(h))^m h^\delta \psi(h^\beta)) \quad \text{as} \quad h \to 0.$$

If $|\lambda| \in [\frac{1}{h}, \frac{2}{h}]$, then $|\lambda h| \geq 1$ and (ii) of Lemma 1.1 implies that

$$1 \leq \frac{1}{c_\alpha^{2m}} |1 - e_\alpha(-i\lambda h)|^{2m}.$$

Then

$$\begin{aligned}
\int_{\frac{1}{h}\leq|\lambda|\leq\frac{2}{h}} |\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) &\leq \frac{1}{c_\alpha^{2m}} \int_{\frac{1}{h}\leq|\lambda|\leq\frac{2}{h}} |1 - e_\alpha(-i\lambda h)|^{2m}|\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\
&\leq \frac{1}{c_\alpha^{2m}} \int_{\mathbb{R}} |1 - e_\alpha(-i\lambda h)|^{2m}|\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\
&\leq \frac{1}{(Q(h))^{2m} c_\alpha^{2m}} \|\Delta_h^m \widetilde{\Lambda}^k f\|_{2,Q}^2 \\
&= O(h^{2\delta}\psi(h^{2\beta})).
\end{aligned}$$

We obtain

$$\int_{r\leq|\lambda|\leq 2r} |\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \leq C r^{-2\delta}\psi(r^{-2\beta}), \quad r \to \infty,$$

where C is a positive constant. Now,

$$\begin{aligned}
\int_{|\lambda|\geq r} |\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) &= \sum_{i=0}^{\infty} \int_{2^i r\leq|\lambda|\leq 2^{i+1} r} |\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\
&\leq C r^{-2\delta}\psi(r^{-2\beta}) \sum_{i=0}^{\infty} (2^{-2\delta}\psi(2^{-2\beta}))^i \\
&\leq C C_{\delta,\beta} r^{-2\delta}\psi(r^{-2\beta}),
\end{aligned}$$

where $C_{\delta,\beta} = (1 - 2^{-2\delta}\psi(2^{-2\beta})))^{-1}$ since $2^{-2\delta}\psi(2^{-2\beta}) < 1$.
Consequently

$$\int_{|\lambda|\geq r} |\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}\psi(r^{-2\beta})), \quad as \quad r \to \infty.$$

$(ii)\Rightarrow (i)$. Suppose now that

$$\int_{|\lambda|\geq r} |\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}\psi(r^{-2\beta})), \quad as \quad r \to \infty,$$

and write

$$\|\Delta_h^m \widetilde{\Lambda}^k f\|_{2,Q}^2 = (Q(h))^{2m}(I_1 + I_2),$$

where

$$I_1 = \int_{|\lambda|<\frac{1}{h}} |1 - e_\alpha(-i\lambda h)|^{2m}|\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda),$$

and

$$I_2 = \int_{|\lambda|\geq\frac{1}{h}} |1 - e_\alpha(-i\lambda h)|^{2m}|\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda).$$

Let us estimate the summands I_1 and I_2 from above. To estimate I_1, we use both the first two estimates of e_α in Lemma 1.1. Therefore

$$\begin{aligned} I_1 &= \int_{|\lambda|<\frac{1}{h}} |1-e_\alpha(-i\lambda h)|^{2m}|\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\ &= \int_{|\lambda|<\frac{1}{h}} |1-e_\alpha(-i\lambda h)|^{2m-2}|1-e_\alpha(-i\lambda h)|^2|\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\ &\leq 2^{2m-2}\int_{|\lambda|<\frac{1}{h}} |1-e_\alpha(-i\lambda h)|^2|\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) \\ &\leq 2^{2m-2}h^2\int_{|\lambda|<\frac{1}{h}} |\lambda|^{2k+2}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda). \end{aligned}$$

Now, we apply integration by parts for a function

$$\phi(x) = \int_x^{+\infty} |\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda),$$

to get

$$\begin{aligned} \int_0^x \lambda^{2k+2}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) &= \int_0^x -\lambda^2\phi'(\lambda)d\lambda = -x^2\phi(x) + 2\int_0^x \lambda\phi(\lambda)d\lambda \\ &\leq C_1\int_0^x \lambda^{1-2\delta}\psi(\lambda^{-2\beta})d\lambda = O(x^{2-2\delta}\psi(x^{-2\beta})), \end{aligned}$$

where C_1 is a positive constant. Hence

$$I_1 = O(h^{2\delta}\psi(h^{2\beta})), \quad as \quad h \to 0.$$

On the other hand, it follows from the first inequality of Lemma 1.1 that

$$I_2 \leq 4^m \int_{|\lambda|\geq\frac{1}{h}} |\lambda|^{2k}|\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) = O(h^{2\delta}\psi(h^{2\beta})), \quad as \quad h \to 0.$$

Consequently,

$$\|\Delta_h^m \widetilde{\Lambda}^k f\|_{2,Q} = O((Q(h))^m h^\delta \psi(h^\beta)) \quad \text{as} \quad h \to 0.$$

and this ends the proof of the theorem. □

Corollary 2.1. *Let $f \in \mathcal{W}^m_{2,Q}$. If*

$$\|\Delta_h^m \widetilde{\Lambda}^k f\|_{2,Q} = O((Q(h))^m h^\delta \psi(h^\beta)) \quad as \quad h \to 0,$$

then

$$\int_{|\lambda|\geq r} |\mathcal{F}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2k-2\delta}\psi(r^{-2\beta})), \quad as \quad r \to \infty,$$

where $k = 0, 1, ..., m$.

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