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GENERALISED ITERATION OF ENTIRE FUNCTIONS WITH INDEX-PAIR $[p, q]$

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Abstract. Using generalised iteration [2] of two entire functions we extend the results of Hong-Yan Xu et.al [16] for generalised iterated entire functions with index-pair $[p, q]$

1. Introduction and Definitions

For two transcendental entire functions $f(z)$ and $g(z)$, Clunie [4] showed that $\lim_{r\to\infty} \frac{T(r,f\circ g)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{T(r,f\circ g)}{T(r,g)} = \infty$. After this several authors {see; [8], [9], [10], [13], [17]} made close investigations on growth properties of composition of two entire functions with finite order and achieved great results. In 2009, Jin Tu et.al [15] investigated the growth of two composite entire functions of finite iterated order. Recently Banerjee and Mandal [1] using the idea of generalised iteration defined by Banerjee and Mondal [2] extend the results of Jin Tu et.al [15] for generalised iterated entire functions. In 2013, H. Y. Xu et.al [16] investigated some growth properties of two composite entire functions of finite $[p, q]$ order. The purpose of the present paper is to extend the results of H. Y. Xu et.al [16] for generalised iterated entire functions of $[p, q]$-order and lower $[p, q]$-order.

Definition 1.1.[3,7] The iterated i order $\rho_i(f)$ of an entire function f is defined by

$$\rho_i(f) = \limsup_{r\to\infty} \frac{\log^{[i+1]} M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log^{[i]} T(r,f)}{\log r} \ (i \in \mathbb{N}).$$

Similarly, the iterated i lower order $\lambda_i(f)$ of an entire function f is defined by

$$\lambda_i(f) = \liminf_{r\to\infty} \frac{\log^{[i+1]} M(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log^{[i]} T(r,f)}{\log r} \ (i \in \mathbb{N}),$$

where following Sato [12] we write $\log^{[0]} x = x$, $\exp^{[0]} x = x$, and for positive integer m, $\log^{[m]} x = \log(\log^{[m-1]} x)$, $\exp^{[m]} x = \exp(\exp^{[m-1]} x)$. Also we denote $\exp^{[-1]} x = \log x$.

Definition 1.2.[3,7] The finiteness degree of the order of an entire function f is defined by

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$$i(f) = \begin{cases} 0 & \text{if } f(z) \text{ is a polynomial;} \\ \min\{k \in \{1,2,\ldots\}, \rho_k(f) < \infty\} & \text{if } f(z) \text{ is transcendental;} \\ \infty & \text{when } \rho_k(f) = \infty \text{ for all } k. \end{cases}$$

In [6], Juneja, Kapoor and Bajpai introduced the concept of $[p,q]$-order and lower $[p,q]$-order of an entire function as follows.

Definition 1.3.[6] If $f(z)$ is a transcendental entire function, the $[p,q]$-order of $f(z)$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r} = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}.$$

Similarly, the lower $[p,q]$-order of $f(z)$ is defined by

$$\lambda_{[p,q]}(f) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r} = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}$$

where p, q are positive integers satisfying $p \geq q \geq 1$.

Remark 1.1. It is obvious from Definition 1.3 that $\rho_{[1,1]}(f) = \rho(f)$, $\rho_{[p,1]}(f) = \rho_p(f)$, $\lambda_{[1,1]}(f) = \lambda(f)$ and $\lambda_{[p,1]}(f) = \lambda_p(f)$.

Recently Xu, Tu and Yi [16] introduced the idea of index-pair of an entire function and derived some interesting properties on comparative growth as follows.

Definition 1.4.[16] A transcendental entire function $f(z)$ is said to have index-pair $[p,q]$, if $0 < \rho_{[p,q]}(f) < \infty$ and $\rho_{[p-1,q-1]}(f)$ is not a nonzero finite number.

Definition 1.5.[16] Let f_1, f_2 be two entire functions such that $\rho_{[p_1,q_1]}(f_1) = \rho_1$, $\rho_{[p_2,q_2]}(f_2) = \rho_2$ and $p_1 \leq p_2$. Then the following results about their comparative growth can be easily deducted:

(i) If $p_2 - p_1 > q_2 - q_1$, then the growth of f_1 is slower than the growth of f_2;

(ii) If $p_2 - p_1 < q_2 - q_1$, then f_1 grows faster than f_2;

(iii) If $p_2 - p_1 = q_2 - q_1 > 0$, then the growth of f_1 is slower than the growth of f_2 if $\rho_2 \geq 1$ while the growth of f_1 is faster than the growth of f_2 if $\rho_2 < 1$;

(iv) If $p_2 - p_1 = q_2 - q_1 = 0$, then f_1, f_2 are of the same index-pair $[p_1, q_1]$. If $\rho_1 > \rho_2$, then f_1 grows faster than f_2, and if $\rho_1 < \rho_2$, then f_1 grows slower than f_2. If $\rho_1 = \rho_2$, Definition 1.3 does not give any precise estimate about the relative growth of f_1 and f_2.

In [14], A. P. Singh showed that for $0 < r < R$, $\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f)$, where $\mu(r,f)$ be the maximum term of an entire function $f(z)$ on $|z| = r$. So the $[p,q]$-order and lower $[p,q]$-order of $f(z)$ are defined as follows.

Definition 1.6.[16] The $[p,q]$-order and lower $[p,q]$-order of $f(z)$ are defined by

$$\rho_{[p,q]}(f) = \limsup_{r\to\infty} \frac{\log^{[p+1]} \mu(r,f)}{\log^{[q]} r} \quad \text{and} \quad \lambda_{[p,q]}(f) = \liminf_{r\to\infty} \frac{\log^{[p+1]} \mu(r,f)}{\log^{[q]} r},$$

where p, q are positive integers satisfying $p \geq q \geq 1$.

Let $f(z)$ and $g(z)$ be two entire functions and $\alpha \in (0,1]$ be any real number. In [2], Banerjee and Mondal introduced the idea of generalised iteration of $f(z)$ with respect to $g(z)$ as follows.

$$\begin{aligned} f_{1,g}(z) &= (1-\alpha)z + \alpha f(z) \\ f_{2,g}(z) &= (1-\alpha)g_{1,f}(z) + \alpha f(g_{1,f}(z)) \\ f_{3,g}(z) &= (1-\alpha)g_{2,f}(z) + \alpha f(g_{2,f}(z)) \\ &\vdots \\ f_{n,g}(z) &= (1-\alpha)g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z)) \end{aligned}$$

and so

$$\begin{aligned} g_{1,f}(z) &= (1-\alpha)z + \alpha g(z) \\ g_{2,f}(z) &= (1-\alpha)f_{1,g}(z) + \alpha g(f_{1,g}(z)) \end{aligned}$$

$$g_{3,f}(z) = (1-\alpha)f_{2,g}(z) + \alpha g(f_{2,g}(z))$$
$$\vdots$$
$$g_{n,f}(z) = (1-\alpha)f_{n-1,g}(z) + \alpha g(f_{n-1,g}(z))\ .$$

Clearly all $f_{n,g}(z)$ and $g_{n,f}(z)$ are entire functions.

Throughout the paper, whenever we deal with any entire function f having index-pair $[p, q]$ we mean that f has positive lower $[p, q]$-order and finite $[p, q]$-order. If the index-pair of f and g are $[p_1, q_1]$ and $[p_2, q_2]$ then we denote $\lambda_{[p_1,q_1]}(f)$, $\lambda_{[p_2,q_2]}(g)$, $\rho_{[p_1,q_1]}(f)$ and $\rho_{[p_2,q_2]}(g)$ by A_l, B_l, A and B respectively. It is obvious from our assumption that $n(\in \mathbb{N}) \geq 2$. Also we use the standard notations and definitions of the theory of meromorphic functions which are available in [5].

2. Known Lemmas

In this section, we state some known results in the form of lemmas which will be needed to prove our main results.

Lemma 2.1. [10] *Let f, g be entire functions. If $M(r,g) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for any $\varepsilon > 0$, then*
$$T(r, f\circ g) < (1+\varepsilon)T(M(r,g), f).$$
In particular if $g(0) = 0$, then
$$T(r, f\circ g) \leq T(M(r,g), f),$$
for all $r > 0$.

Lemma 2.2. *[4] Let f, g be entire functions with $g(0) = 0$. Let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for $r > 0$,*
$$M(M(r,g), f) \geq M(r, f\circ g) \geq M(c(\beta)M(\beta r, g), f).$$
Furthermore if $\beta = \frac{1}{2}$, for sufficiently large r,
$$M(r, f\circ g) \geq M(\tfrac{1}{8}M(\tfrac{r}{2}, g), f).$$

Lemma 2.3. *[14] Let f and g be entire functions with $g(0) = 0$. Let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Also let $0 < \delta < 1$. Then*
$$\mu(r, f\circ g) \geq (1-\delta)\mu(c(\beta)\mu(\beta\delta r, g), f).$$
And if g is any entire function with $\beta = \delta = \frac{1}{2}$, for sufficiently large r,
$$\mu(r, f\circ g) \geq \tfrac{1}{2}\mu(\tfrac{1}{8}\mu(\tfrac{r}{4}, g), f).$$

Lemma 2.4. *[5] Let f and g be transcendental entire functions. Then*
$$\frac{T(r,g)}{T(r,f\circ g)} \to 0 \ \text{as}\ r \to \infty.$$

Lemma 2.5. *Let f and g be transcendental entire functions. Then*
$$\frac{\log M(r,g)}{\log M(r,f\circ g)} \to 0 \ \text{as}\ r \to \infty.$$

3. Main Results

In this section, we state and prove the main results of this paper.

Theorem 3.1. *Let f and g be entire functions having index-pair $[p_1, q_1]$ and $[p_2, q_2]$ respectively.*

(I) If n is odd, then
$$\rho_{[\frac{n+1}{2}p_1 + \frac{n-1}{2}(p_2-q_1-q_2+2), q_1]}(f_{n,g}) = \rho_{[p_1,q_1]}(f) = A$$
and (II) if n is even and

(a) $\frac{n-2}{2}(p_1-q_2) + \frac{n}{2}(p_2-q_1) + (n-1) > 0$, then
$$\rho_{[\frac{n}{2}(p_1+p_2-q_1) - \frac{n-2}{2}q_2 + (n-1), q_2]}(f_{n,g}) = \rho_{[p_2,q_2]}(g) = B;$$
(b) $\frac{n-2}{2}(p_1-q_2) + \frac{n}{2}(p_2-q_1) + (n-1) = 0$, then
$$A_l B \leq \rho_{[p_1,q_2]}(f_{n,g}) \leq AB;$$

(c) $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)<0$ *and* $\frac{n}{2}(q_1+q_2-p_2)-\frac{n-2}{2}p_1-(n-1)\geq 1$, *then*

$$A_l \leq \rho_{[p_1,\frac{n}{2}(q_1+q_2-p_2)-\frac{n-2}{2}p_1-(n-1)]}(f_{n,g}) \leq A.$$

Proof. Case (I). When n is odd.

From definition, for large r and given any ε (>0) we get

$$\begin{cases} T(r,f) \leq \exp^{[p_1]}\{(A+\varepsilon)\log^{[q_1]} r\}; \\ \log M(r,g) \leq \exp^{[p_2]}\{(B+\varepsilon)\log^{[q_2]} r\}. \end{cases} \tag{3.1}$$

By Lemma 2.1, Lemma 2.4, Lemma 2.5 and using (3.1), for sufficiently large r, we have

$$\begin{aligned} T(r,f_{n,g}) &\leq T(r,g_{n-1,f})+T(r,f(g_{n-1,f}))+O(1) \\ &= (1+o(1))T(r,f(g_{n-1,f})) \\ &\leq 2T(M(r,g_{n-1,f}),f) \\ &\leq 2\exp^{[p_1]}\{(A+\varepsilon)\log^{[q_1]} M(r,g_{n-1,f})\} \\ &\leq \exp^{[p_1]}\{(A+2\varepsilon)\log^{[q_1]} M(r,g_{n-1,f})\} \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\log^{[q_1-1]}\{\log M(r,f_{n-2,g})+\log M(r,g(f_{n-2,g})) \\ &\qquad +O(1)\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\log^{[q_1-1]}\{(1+o(1))\log M(r,g(f_{n-2,g}))\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\log^{[q_1-1]}\{(1+o(1))\log M(M(r,f_{n-2,g}),g)\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\log^{[q_1-1]}\{(1+o(1))\exp^{[p_2]} \\ &\qquad \{(B+\varepsilon)\log^{[q_2]} M(r,f_{n-2,g})\}\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[p_2-q_1+1]}\{(B+2\varepsilon)\log^{[q_2]} M(r,f_{n-2,g})\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[p_2-q_1+1]}\{(B+2\varepsilon)\log^{[q_2-1]} \\ &\qquad \{(1+o(1))\exp^{[p_1]}\{(A+\varepsilon)\log^{[q_1]} M(r,g_{n-3,f})\}\}\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[p_2-q_1+1]}\{(B+2\varepsilon)\log^{[q_2-1]} \\ &\qquad \{\exp^{[p_1]}\{(A+2\varepsilon)\log^{[q_1]} M(r,g_{n-3,f})\}\}\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[(p_1-q_2)+(p_2-q_1)+2]} \\ &\qquad \{(A+3\varepsilon)\log^{[q_1]} M(r,g_{n-3,f})\}] \qquad (3.2) \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[(p_1-q_2)+2(p_2-q_1)+3]}\{(B+3\varepsilon)\log^{[q_2]} M(r,f_{n-4,g})\}] \\ &\qquad \cdots \qquad \cdots \qquad \cdots \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]} \\ &\qquad \{(B+3\varepsilon)\log^{[q_2]} M(r,f_{1,g})\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]} \\ &\qquad \{(B+3\varepsilon)\log^{[q_2-1]}\{(1+o(1))\log M(r,f)\}\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]} \\ &\qquad \{(B+3\varepsilon)\log^{[q_2-1]}\{(1+o(1))\exp^{[p_1]}\{(A+\varepsilon)\log^{[q_1]} r\}\}\}] \\ &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[\frac{n-1}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-1)]} \\ &\qquad \{(A+3\varepsilon)\log^{[q_1]} r\}]. \end{aligned}$$

Since $p_i \geq q_i \geq 1$ for $i=1,2$ and $n \geq 3$; $\frac{n-1}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-1)>0$ always. Therefore,

$$T(r,f_{n,g}) \leq \exp^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+(n-1)]}\{(A+4\varepsilon)\log^{[q_1]} r\}. \tag{3.3}$$

On the other hand, since $A>0$, there exists a sequence $\{r_m\}$ tending to infinity such that for given ε $[0<\varepsilon<A]$ and for sufficiently large r_m, we have

$$\log M(r_m,f) \geq \exp^{[p_1]}\{(A-\varepsilon)\log^{[q_1]} r_m\}. \tag{3.4}$$

We denote $\{r_m\}$, a sequence, tending to infinity, not necessarily the same at each occurrence. Since $A_l>0, B_l>0$ and by the same reasoning as K. Niino and C.C.

Yang [11], for sufficiently large r_m and for chosen ε $(0 < 4\varepsilon < \min\{A_l, B_l\})$, using Lemma 2.2, Lemma 2.4 and (3.4), we have

$$
\begin{aligned}
T(r_m, f_{n,g}) &\geq (1+o(1))T(r_m, f(g_{n-1,f})) \\
&\geq \tfrac{1}{3}(1+o(1))\log M(\tfrac{1}{9}M(\tfrac{r_m}{2^2}, g_{n-1,f}), f) \qquad (3.5)\\
&\geq \tfrac{1}{3}(1+o(1))\exp^{[p_1]}[(A_l-\varepsilon)\log^{[q_1]}\{\tfrac{1}{9}M(\tfrac{r_m}{2^2}, g_{n-1,f})\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\log^{[q_1]}\{\tfrac{1}{9}M(\tfrac{r_m}{2^2}, g_{n-1,f})\}] \\
&= \exp^{[p_1]}[(A_l-2\varepsilon)\log^{[q_1-1]}\{(1+o(1))\log M(\tfrac{r_m}{2^2}, g(f_{n-2,g}))\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\log^{[q_1-1]}\{(1+o(1)) \\
&\qquad \log M(\tfrac{1}{9}M(\tfrac{r_m}{2^3}, f_{n-2,g}), g)\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\log^{[q_1-1]}\{(1+o(1))\exp^{[p_2]} \\
&\qquad \{(B_l-\varepsilon)\log^{[q_2]}(\tfrac{1}{9}M(\tfrac{r_m}{2^3}, f_{n-2,g}))\}\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[p_2-q_1+1]}\{(B_l-2\varepsilon)\log^{[q_2]} M(\tfrac{r_m}{2^3}, f_{n-2,g})\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[p_2-q_1+1]}\{(B_l-2\varepsilon)\log^{[q_2-1]} \\
&\qquad \{(1+o(1))\log M(\tfrac{r_m}{2^3}, f(g_{n-3,f}))\}\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[p_2-q_1+1]}\{(B_l-2\varepsilon)\log^{[q_2-1]} \\
&\qquad \{(1+o(1))\log M(\tfrac{1}{9}M(\tfrac{r_m}{2^4}, g_{n-3,f}), f)\}\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[p_2-q_1+1]}\{(B_l-2\varepsilon)\log^{[q_2-1]} \\
&\qquad \{(1+o(1))\exp^{[p_1]}\{(A_l-\varepsilon)\log^{[q_1]}(\tfrac{1}{9}M(\tfrac{r_m}{2^4}, g_{n-3,f}))\}\}\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[p_2-q_1+1]}\{(B_l-2\varepsilon)\exp^{[p_1-q_2+1]} \\
&\qquad \{(A_l-2\varepsilon)\log^{[q_1]} M(\tfrac{r_m}{2^4}, g_{n-3,f})\}\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[(p_1-q_2)+(p_2-q_1)+2]} \qquad (3.6)\\
&\qquad \{(A_l-3\varepsilon)\log^{[q_1]} M(\tfrac{r_m}{2^4}, g_{n-3,f})\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[(p_1-q_2)+2(p_2-q_1)+3]} \qquad (3.7)\\
&\qquad \{(B_l-3\varepsilon)\log^{[q_2]} M(\tfrac{r_m}{2^5}, f_{n-4,g})\}] \\
&\qquad \cdots \qquad \cdots \qquad \cdots \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]} \\
&\qquad \{(B_l-3\varepsilon)\log^{[q_2]} M(\tfrac{r_m}{2^n}, f_{1,g})\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]}\{(B_l-3\varepsilon) \\
&\qquad \log^{[q_2-1]}\{(1+o(1))\log M(\tfrac{r_m}{2^n}, f)\}\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]}\{(B_l-3\varepsilon) \\
&\qquad \log^{[q_2-1]}\{(1+o(1))\exp^{[p_1]}\{(A-\varepsilon)\log^{[q_1]}(\tfrac{r_m}{2^n})\}\}\}] \\
&\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[\frac{n-1}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-1)]} \\
&\qquad \{(A-3\varepsilon)\log^{[q_1]} r_m\}].
\end{aligned}
$$

Since $p_i \geq q_i \geq 1$ for $i = 1, 2$ and $n \geq 3$; $\frac{n-1}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-1) > 0$ always. Therefore,

$$T(r_m, f_{n,g}) \geq \exp^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+(n-1)]}\{(A-4\varepsilon)\log^{[q_1]} r_m\}. \qquad (3.8)$$

Now from (3.3) and (3.8), since ε (>0) is arbitrary,

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2+2)]} T(r, f_{n,g})}{\log^{[q_1]} r} = A$$

$$\text{i.e.,} \quad \rho_{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2+2),q_1]}(f_{n,g}) = A = \rho_{[p_1,q_1]}(f).$$

Case (II). When n is even.

From (3.2), for sufficiently large r, we have

$$
\begin{aligned}
T(r, f_{n,g}) &\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]} \\
&\qquad \{(A+3\varepsilon)\log^{[q_1]} M(r, g_{1,f})\}]
\end{aligned}
$$

$$\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A+3\varepsilon)\log^{[q_1-1]}\{(1+o(1))\log M(r,g)\}\}]$$
$$\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A+3\varepsilon)\log^{[q_1-1]}\{(1+o(1))\exp^{[p_2]}\{(B+\varepsilon)\log^{[q_2]} r\}\}\}]$$
$$\leq \exp^{[p_1]}[(A+2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)]}\{(B+3\varepsilon)\log^{[q_2]} r\}]. \tag{3.9}$$

By similar argument as in *Case (I)* and from (3.6), we have

$$T(r_m,f_{n,g}) \geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A_l-3\varepsilon)\log^{[q_1]} M(\tfrac{r_m}{2^n},g_{1,f})\}]$$
$$\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A_l-3\varepsilon)\log^{[q_1-1]}\{(1+o(1))\log M(\tfrac{r_m}{2^n},g)\}\}]$$
$$\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A_l-3\varepsilon)\log^{[q_1-1]}\{(1+o(1))\exp^{[p_2]}\{(B-\varepsilon)\log^{[q_2]}(\tfrac{r_m}{2^n})\}\}\}]$$
$$\geq \exp^{[p_1]}[(A_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)]}\{(B-3\varepsilon)\log^{[q_2]} r_m\}]. \tag{3.10}$$

From (3.9) and (3.10), we get

(a) if $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)>0$, then

$$B-4\varepsilon \leq \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+(n-1)]} T(r,f_{n,g})}{\log^{[q_2]} r} \leq B+4\varepsilon.$$

Now since $\varepsilon\ (>0)$ is arbitrary,

$$\limsup_{r\to\infty} \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+(n-1)]} T(r,f_{n,g})}{\log^{[q_2]} r} = B$$

i.e., $\rho_{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+(n-1),q_2]}(f_{n,g}) = B = \rho_{[p_2,q_2]}(g).$

Again from (3.9) and (3.10), we get

(b) if $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)=0$, then

$$(A_l-2\varepsilon)(B-3\varepsilon) \leq \frac{\log^{[p_1]} T(r,f_{n,g})}{\log^{[q_2]} r} \leq (A+2\varepsilon)(B+3\varepsilon).$$

Now since $\varepsilon\ (>0)$ is arbitrary,

$$A_lB \leq \rho_{[p_1,q_2]}(f_{n,g}) \leq AB.$$

Finally from (3.9) and (3.10), we get

(c) if $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)<0$ and $\frac{n}{2}(q_1+q_2-p_2)-\frac{n-2}{2}p_1-(n-1)\geq 1$, then

$$A_l-2\varepsilon \leq \frac{\log^{[p_1]} T(r,f_{n,g})}{\log^{[\frac{n}{2}(q_1+q_2-p_2)-\frac{n-2}{2}p_1-(n-1)]} r} \leq A-2\varepsilon.$$

Now since $\varepsilon\ (>0)$ is arbitrary,

$$A_l \leq \rho_{[p_1,\frac{n}{2}(q_1+q_2-p_2)-\frac{n-2}{2}p_1-(n-1)]}(f_{n,g}) \leq A. \qquad \square$$

Remark 3.1. Since $n\geq 2$ and $p_i\geq q_i\geq 1$, for $i=1,2$; we always have $\frac{n-1}{2}(p_1+p_2-q_1-q_2+2)>0$. Therefore when n is odd, case (b) and (c) has no relevance.

Remark 3.2. When $q_i=1$ $(i=1,2)$, the result obtained in (a) is quite similar to Theorem 3.1 of Banerjee and Mandal [1].

Theorem 3.2. *Let f and g be entire functions having index-pair $[p_1,q_1]$ and $[p_2,q_2]$ respectively. If n is even and $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)=0$, then*

$$AB_l \leq \rho_{[p_1,q_2]}(f_{n,g}) \leq AB.$$

Proof. For sufficiently large r and for chosen ε $(0<3\varepsilon<\min\{A_l,B_l\})$, we have

$$\begin{cases} M(r,f) \geq \exp^{[p_1+1]}\{(A_l-\varepsilon)\log^{[q_1]} r\}; \\ M(r,g) \geq \exp^{[p_2+1]}\{(B_l-\varepsilon)\log^{[q_2]} r\}. \end{cases} \tag{3.11}$$

So, there exists a sequence $\{r_m\}$, tending to infinity such that for all sufficiently large r_m, using Lemma 2.2 and (3.11), we get

$$\begin{aligned}
M(r_m, f_{n,g}) &\geq \exp\{(1+o(1))\log M(r_m, f(g_{n-1,f}))\}\\
&\geq \exp\{(1+o(1))\log M(\tfrac{1}{9}M(\tfrac{r_m}{2}, g_{n-1,f}), f)\}\\
&\geq \exp[(1+o(1))\exp^{[p_1]}\{(A-\varepsilon)\log^{[q_1]}(\tfrac{1}{9}M(\tfrac{r_m}{2}, g_{n-1,f}))\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\log^{[q_1-1]}\{(1+o(1))\log M(\tfrac{r_m}{2}, g(f_{n-2,g}))\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\log^{[q_1-1]}\{(1+o(1))\{\exp^{[p_2]}\\
&\qquad \{(B_l-\varepsilon)\log^{[q_2]}(\tfrac{1}{9}M(\tfrac{r_m}{2^2}, f_{n-2,g}))\}\}\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[p_2-q_1+1]}\{(B_l-2\varepsilon)\log^{[q_2]}(\tfrac{1}{9}M(\tfrac{r_m}{2^2}, f_{n-2,g}))\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[p_2-q_1+1]}\{(B_l-2\varepsilon)\log^{[q_2-1]}\\
&\qquad \{(1+o(1))\log M(\tfrac{r_m}{2^2}, f(g_{n-3,f}))\}\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[p_2-q_1+1]}\{(B_l-2\varepsilon)\log^{[q_2-1]}\\
&\qquad \{(1+o(1))\log M(\tfrac{1}{9}M(\tfrac{r_m}{2^3}, g_{n-3,f}))\}\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[p_2-q_1+1]}\{(B_l-2\varepsilon)\log^{[q_2-1]}\{(1+o(1))\\
&\qquad \exp^{[p_1]}\{(A_l-\varepsilon)\log^{[q_1]}(\tfrac{1}{9}M(\tfrac{r_m}{2^3}, g_{n-3,f}))\}\}\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[(p_1-q_2)+(p_2-q_1)+2]}\{(A_l-3\varepsilon)\\
&\qquad \log^{[q_1]} M(\tfrac{r_m}{2^3}, g_{n-3,f})\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[(p_1-q_2)+2(p_2-q_1)+3]}\{(B_l-3\varepsilon)\\
&\qquad \log^{[q_2]} M(\tfrac{r_m}{2^4}, f_{n-4,g})\}]\\
&\qquad \cdots \qquad \cdots \qquad \cdots\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A_l-3\varepsilon)\\
&\qquad \log^{[q_1]} M(\tfrac{r_m}{2^{n-1}}, g_{1,f})\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A_l-3\varepsilon)\\
&\qquad \log^{[q_1-1]}\{(1+o(1))\log M(\tfrac{r_m}{2^{n-1}}, g)\}\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A_l-3\varepsilon)\\
&\qquad \log^{[q_1-1]}\{(1+o(1))\exp^{[p_2]}\{(B_l-\varepsilon)\log^{[q_2]}(\tfrac{r_m}{2^{n-1}})\}\}\}]\\
&\geq \exp^{[p_1+1]}[(A-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)]}\\
&\qquad \{(B_l-3\varepsilon)\log^{[q_2]} r_m\}]. \qquad (3.12)
\end{aligned}$$

If $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)=0$, then from (3.9) and (3.12), we get

$$AB_l \leq \rho_{[p_1,q_2]}(f_{n,g}) \leq AB. \qquad \square$$

Corollary 3.1. *Under the hypothesis of Theorem 3.1, if n is even and $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)=0$, then*

$$\max\{A_lB, AB_l\} \leq \rho_{[p_1,q_2]}(f_{n,g}) \leq AB.$$

Theorem 3.3. *Let f and g be entire functions having index-pair $[p_1,q_1]$ and $[p_2,q_2]$ respectively.*

(I) When n is odd, then $\frac{A_l}{A} \leq \lim_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \leq \frac{A}{A_l}$

and (II) when n is even,

(a) if $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)>0$ *and*

(i) $q_1>q_2$, then $\lim_{r\to\infty} \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} = \infty;$

(ii) $q_1=q_2$, then $\frac{B_l}{A} \leq \lim_{r\to\infty} \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \leq \frac{B}{A_l};$

(iii) $q_1<q_2$, then $\lim_{r\to\infty} \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} = 0;$

(b) if $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)=0$ *and*

(i) $q_1 > q_2$, *then* $\lim_{r\to\infty} \frac{\log^{[p_1+1]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} = \infty$;

(ii) $q_1 = q_2$, *then* $\frac{A_l B_l}{A} \leq \lim_{r\to\infty} \frac{\log^{[p_1+1]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \leq \frac{AB}{A_l}$;

(iii) $q_1 < q_2$, *then* $\lim_{r\to\infty} \frac{\log^{[p_1+1]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} = 0$.

(c) if $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1) < 0$, *then* $\lim_{r\to\infty} \frac{\log^{[p_1+1]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} = \infty$.

Proof. Case (I). When n is odd.

For sufficiently large r and for chosen ε $(0 < 4\varepsilon < \min\{A_l, B_l\})$, from (3.7) we get

$$\begin{aligned} M(r,f_{n,g}) &\geq \exp^{[p_1+1]}[(A_l-2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]}\{(B_l-3\varepsilon) \\ &\quad \log^{[q_2]} M(\tfrac{r}{2^n}, f_{1,g})\}] \\ &\geq \exp^{[p_1+1]}[(A_l-2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]}\{(B_l-3\varepsilon) \\ &\quad \log^{[q_2-1]}\{(1+o(1))\log M(\tfrac{r}{2^n}, f)\}\}] \\ &\geq \exp^{[p_1+1]}[(A_l-2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]}\{(B_l-3\varepsilon) \\ &\quad \log^{[q_2-1]}\{(1+o(1))\exp^{[p_1]}\{(A_l-\varepsilon)\log^{[q_1]}(\tfrac{r}{2^n})\}\}\}] \end{aligned}$$

$$\begin{aligned} &\geq \exp^{[p_1+1]}[(A_l-2\varepsilon)\exp^{[\frac{n-1}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-1)]} \\ &\quad \{(A_l-3\varepsilon)\log^{[q_1]} r\}] \end{aligned} \tag{3.13}$$

$$\geq \exp^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]}\{(A_l-4\varepsilon)\log^{[q_1]} r\}. \tag{3.14}$$

By Lemma 2.2, Lemma 2.5 and using (3.1), for sufficiently large r, we have

$$\begin{aligned} M(r,f_{n,g}) &\leq \exp\{(1+o(1))\log M(r, f(g_{n-1,f}))\} \\ &\leq \exp[(1+o(1))\exp^{[p_1]}\{(A+\varepsilon)\log^{[q_1]} M(r, g_{n-1,f})\}] \\ &\leq \exp^{[p_1+1]}[(A+2\varepsilon)\log^{[q_1-1]}\{(1+o(1))\log M(r, g(f_{n-2,g}))\}] \\ &\leq \exp^{[p_1+1]}[(A+2\varepsilon)\log^{[q_1-1]}\{(1+o(1)) \\ &\quad \exp^{[p_2]}\{(B+\varepsilon)\log^{[q_2]} M(r, f_{n-2,g})\}\}] \\ &\leq \exp^{[p_1+1]}[(A+2\varepsilon)\exp^{[p_2-q_1+1]} \\ &\quad \{(B+2\varepsilon)\log^{[q_2]} M(r, f_{n-2,g})\}] \end{aligned}$$

$$\begin{aligned} &\leq \exp^{[p_1+1]}[(A+2\varepsilon)\exp^{[p_1-q_2+p_2-q_1+2]} \\ &\quad \{(A+3\varepsilon)\log^{[q_1]} M(r, g_{n-3,f})\}] \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\quad \cdots \qquad \cdots \qquad \cdots \\ &\leq \exp^{[p_1+1]}[(A+2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]} \\ &\quad \{(B+3\varepsilon)\log^{[q_2]} M(r, f_{1,g})\}] \\ &\leq \exp^{[p_1+1]}[(A+2\varepsilon)\exp^{[\frac{n-3}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-2)]} \\ &\quad \{(B+3\varepsilon)\log^{[q_2-1]}\{(1+o(1))M(r,f)\}\}] \\ &\leq \exp^{[p_1+1]}[(A+2\varepsilon)\exp^{[\frac{n-1}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-1)]} \\ &\quad \{(A+3\varepsilon)\log^{[q_1]} r\}] \end{aligned}$$

$$\leq \exp^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]}\{(A+4\varepsilon)\log^{[q_1]} r\}. \tag{3.16}$$

Form (3.11), (3.14) and (3.16), we have

$$\begin{aligned} &\frac{\log^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]}[\exp^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]}\{(A_l-4\varepsilon)\log^{[q_1]} r\}]}{\log^{[p_1+1]}\{\exp^{[p_1+1]}\{(A+\varepsilon)\log^{[q_1]} r\}\}} \\ &\leq \frac{\log^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \\ &\leq \frac{\log^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]}[\exp^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]}\{(A+4\varepsilon)\log^{[q_1]} r\}]}{\log^{[p_1+1]}\{\exp^{[p_1+1]}\{(A_l-\varepsilon)\log^{[q_1]} r\}\}} \end{aligned}$$

i.e., $\frac{(A_l-4\varepsilon)\log^{[q_1]} r}{(A+\varepsilon)\log^{[q_1]} r} \leq \frac{\log^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \leq \frac{(A+4\varepsilon)\log^{[q_1]} r}{(A_l-\varepsilon)\log^{[q_1]} r}$.

Since ε (>0) is arbitrary,

$$\frac{A_l}{A} \le \lim_{r\to\infty} \frac{\log^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \le \frac{A}{A_l}.$$

Case (II). When n is even.

From (3.15), for sufficiently large r, we have

$$\begin{aligned} M(r,f_{n,g}) &\le \exp^{[p_1+1]}[(A+2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]} \\ &\quad \{(A+3\varepsilon)\log^{[q_1]}\{M(r,g_1)\}\}] \\ &\le \exp^{[p_1+1]}[(A+2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)]} \\ &\quad \{(B+3\varepsilon)\log^{[q_2]} r\}]. \end{aligned} \tag{3.17}$$

From (3.6), for sufficiently large r, we have

$$\begin{aligned} M(r,f_{n,g}) &\ge \exp^{[p_1+1]}[(A_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A_l-3\varepsilon) \\ &\quad \log^{[q_1]} M(\tfrac{r}{2^n},g_{1,f})\}] \\ &\ge \exp^{[p_1+1]}[(A_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A_l-3\varepsilon) \\ &\quad \log^{[q_1-1]}\{(1+o(1))\log M(\tfrac{r}{2^n},g)\}\}] \\ &\ge \exp^{[p_1+1]}[(A_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n-2}{2}(p_2-q_1)+(n-2)]}\{(A_l-3\varepsilon) \\ &\quad \log^{[q_1-1]}\{(1+o(1))\exp^{[p_2]}\{(B_l-\varepsilon)\log^{[q_2]}(\tfrac{r}{2^n})\}\}\}] \\ &\ge \exp^{[p_1+1]}[(A_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)]} \\ &\quad \{(B_l-3\varepsilon)\log^{[q_2]} r\}]. \end{aligned} \tag{3.18}$$

Form (3.11), (3.17) and (3.18), we have

(3.19)

$$\begin{aligned} &\frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]}[\exp^{[p_1+1]}\{(A_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)]}\{(B_l-3\varepsilon)\log^{[q_2]} r\}\}]}{\log^{[p_1+1]}\{\exp^{[p_1+1]}\{(A+\varepsilon)\log^{[q_1]} r\}\}} \\ &\le \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \\ &\le \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]}[\exp^{[p_1+1]}\{(A+2\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)]}\{(B+3\varepsilon)\log^{[q_2]} r\}\}]}{\log^{[p_1+1]}\{\exp^{[p_1+1]}\{(A_l-\varepsilon)\log^{[q_1]} r\}\}}. \end{aligned}$$

(a) If $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)>0$, then from (3.19), we get

$$\frac{(B_l-4\varepsilon)\log^{[q_2]} r}{(A+\varepsilon)\log^{[q_1]} r} \le \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \le \frac{(B+4\varepsilon)\log^{[q_2]} r}{(A_l-\varepsilon)\log^{[q_1]} r}.$$

(i) $q_1>q_2$, $\lim_{r\to\infty} \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} = \infty$;

(ii) $q_1=q_2$, $\frac{B_l}{A} \le \lim_{r\to\infty} \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \le \frac{B}{A_l}$;

(iii) $q_1<q_2$, $\lim_{r\to\infty} \frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} = 0$.

(b) If $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)=0$, then from (3.11), (3.17) and (3.18), we get

$$\frac{(A_l-2\varepsilon)(B_l-3\varepsilon)\log^{[q_2]} r}{(A+\varepsilon)\log^{[q_1]} r} \le \frac{\log^{[p_1+1]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \le \frac{(A+2\varepsilon)(B+3\varepsilon)\log^{[q_2]} r}{(A_l-\varepsilon)\log^{[q_1]} r}.$$

(i) $q_1>q_2$, $\lim_{r\to\infty} \frac{\log^{[p_1+1]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} = \infty$;

(ii) $q_1=q_2$, $\frac{A_lB_l}{A} \le \lim_{r\to\infty} \frac{\log^{[p_1+1]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \le \frac{AB}{A_l}$;

(iii) $q_1<q_2$, $\lim_{r\to\infty} \frac{\log^{[p_1+1]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} = 0$.

(c) If $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)<0$, then $\frac{n}{2}(q_1+q_2-p_2)-\frac{n-2}{2}p_1-(n-1) = q_1+\frac{n-2}{2}(q_1-p_1)+\frac{n}{2}(q_2-p_2)-(n-1)<q_1$. Therefore from (3.18), we get

$$\lim_{r\to\infty} \frac{\log^{[p_1+1]} M(r,f_{n,g})}{\log^{[p_1+1]} M(r,f)} \ge \lim_{r\to\infty} \frac{(A_l-2\varepsilon)\log^{[\frac{n}{2}(q_1+q_2-p_2)-\frac{n-2}{2}p_1-(n-1)]} r}{(A+\varepsilon)\log^{[q_1]} r} \to \infty. \qquad \square$$

Remark 3.3. Similar results can be obtained for $M(r,f_{n,g})$ and $M(r,g)$ using similar arguments.

Theorem 3.4. *Let f and g be entire functions and g have index-pair $[p_2,q_2]$. If $f_{n,g}$ have index-pair $[p,q]$ and $0<\lambda_{[p,q]}(f_{n,g})=C_l<\infty$, then $\lambda_{[p,q]}(f)=0$.*

Proof. Let us assume that f have index-pair $[p_1,q_1]$.

From definition, there exists a sequence $\{r_m\}$ tending to infinity such that for given $\varepsilon\ (>0)$ and for sufficiently large r_m, from (3.5) we get

$$\tfrac{1}{3}(1+o(1))\log M(\tfrac{1}{9}M(\tfrac{r_m}{2^2},g_{n-1,f}),f)\le T(r_m,f_{n,g})\le \exp^{[p]}\{(C_l+\varepsilon)\log^{[q]} r_m\}. \tag{3.20}$$

Case (I). When n is odd.

Since $(n-1)$ is even and so from (3.18), for chosen ε $[0<4\varepsilon<\min\{A_l,B_l\}]$ and for sufficiently large r_m, we get

$$\begin{aligned}\tfrac{1}{9}M(\tfrac{r_m}{2^2},g_{n-1,f}) &\ge \tfrac{1}{9}\exp^{[p_2+1]}[(B_l-2\varepsilon)\exp^{[\frac{n-3}{2}(p_2-q_1)+\frac{n-1}{2}(p_1-q_2)+(n-2)]}\\ &\qquad\{(A_l-3\varepsilon)\log^{[q_1]} r_m\}]\\ &\ge \exp^{[p_2+1]}[(B_l-3\varepsilon)\exp^{[\frac{n-3}{2}(p_2-q_1)+\frac{n-1}{2}(p_1-q_2)+(n-2)]}\\ &\qquad\{(A_l-3\varepsilon)\log^{[q_1]} r_m\}].\end{aligned}$$

Set $R_m=\frac{1}{9}M(\frac{r_m}{2^2},g_{n-1,f})$, then

$$r_m\le\exp^{[q_1]}[\tfrac{1}{A_l-3\varepsilon}\log^{[\frac{n-3}{2}(p_2-q_1)+\frac{n-1}{2}(p_1-q_2)+(n-2)]}\{\tfrac{1}{B_l-3\varepsilon}\log^{[p_2+1]} R_m\}] \tag{3.21}$$

and from (3.20) and (3.21), we have for large R_m

$$\begin{aligned}\log M(R_m,f) &\le 3(1+o(1))\exp^{[p]}[(C_l+\varepsilon)\log^{[q]}\{\exp^{[q_1]}\{\tfrac{1}{A_l-3\varepsilon}\\ &\qquad\log^{[\frac{n-3}{2}(p_2-q_1)+\frac{n-1}{2}(p_1-q_2)+(n-2)]}\{\tfrac{1}{B_l-3\varepsilon}\log^{[p_2+1]} R_m\}\}\}]\\ &\le \exp^{[p]}[(C_l+2\varepsilon)\exp^{[q_1-q]}\{\tfrac{1}{A_l-3\varepsilon}\\ &\qquad\log^{[\frac{n-3}{2}(p_2-q_1)+\frac{n-1}{2}(p_1-q_2)+(n-2)]}\{\tfrac{1}{B_l-3\varepsilon}\log^{[p_2+1]} R_m\}\}]\\ &\le \exp^{[p+q_1-q]}[\tfrac{1}{A_l-4\varepsilon}\log^{[\frac{n-3}{2}(p_2-q_1)+\frac{n-1}{2}(p_1-q_2)+(n-2)]}\\ &\qquad\{\tfrac{1}{B_l-3\varepsilon}\log^{[p_2+1]} R_m\}].\end{aligned} \tag{3.22}$$

Since $0<4\varepsilon<\min\{A_l,B_l\}$ and $q+\frac{n-1}{2}(p_2+p_1-q_1-q_2)+(n-1)>q$, so for sufficiently large R_m we have from (3.22)

$$\frac{\log^{[p+1]} M(R_m,f)}{\log^{[q]} R_m}\le\frac{\exp^{[q_1-q]}[\frac{1}{A_l-4\varepsilon}\log^{[\frac{n-3}{2}(p_2-q_1)+\frac{n-1}{2}(p_1-q_2)+(n-2)]}\{\frac{1}{B_l-3\varepsilon}\log^{[p_2+1]} R_m\}]}{\log^{[q]} R_m}\to 0.$$

Therefore,

$$\lim_{R_m\to\infty}\frac{\log^{[p+1]} M(R_m,f)}{\log^{[q]} R_m}=0\qquad\text{i.e.,}\qquad \lambda_{[p,q]}(f)=0.$$

Case (II). When n is even.

Since $(n-1)$ is odd so from (3.13), for chosen ε $[0<4\varepsilon<\min\{A_l,B_l\}]$ and for sufficiently large r_m, we get

$$\begin{aligned}\tfrac{1}{9}M(\tfrac{r_m}{2^2},g_{n-1,f}) &\ge \tfrac{1}{9}\exp^{[p_2+1]}[(B_l-2\varepsilon)\exp^{[\frac{n-2}{2}(p_2-q_1)+\frac{n-2}{2}(p_1-q_2)+(n-2)]}\\ &\qquad\{(B_l-3\varepsilon)\log^{[q_2]} r_m\}]\\ &\ge \exp^{[p_2+1]}[(B_l-3\varepsilon)\exp^{[\frac{n-2}{2}(p_2-q_1)+\frac{n-2}{2}(p_1-q_2)+(n-2)]}\\ &\qquad\{(B_l-3\varepsilon)\log^{[q_2]} r_m\}].\end{aligned}$$

Set $R_m=\frac{1}{9}M(\frac{r_m}{2^2},g_{n-1,f})$, then

$$r_m\le\exp^{[q_2]}[\tfrac{1}{B_l-3\varepsilon}\log^{[\frac{n-2}{2}(p_2-q_1)+\frac{n-2}{2}(p_1-q_2)+(n-2)]}\{\tfrac{1}{B_l-3\varepsilon}\log^{[p_2+1]} R_m\}] \tag{3.23}$$

and from (3.20) and (3.23), we have for large R_m

$$\begin{aligned}\log M(R_m,f) &\le 3(1+o(1))\exp^{[p]}[(C_l+\varepsilon)\log^{[q]}\{\exp^{[q_2]}\\ &\qquad\{\tfrac{1}{B_l-3\varepsilon}\log^{[\frac{n-2}{2}(p_2-q_1)+\frac{n-2}{2}(p_1-q_2)+(n-2)]}\{\tfrac{1}{B_l-3\varepsilon}\log^{[p_2+1]} R_m\}\}\}]\\ &\le \exp^{[p]}[(C_l+2\varepsilon)\exp^{[q_2-q]}\end{aligned}$$

$$\{\tfrac{1}{B_l-3\varepsilon}\log^{[\frac{n-2}{2}(p_2-q_1)+\frac{n-2}{2}(p_1-q_2)+(n-2)]}\{\tfrac{1}{B_l-3\varepsilon}\log^{[p_2+1]}R_m\}\}]$$

$$\le \exp^{[p+q_2-q]}[\tfrac{1}{B_l-4\varepsilon}\log^{[\frac{n-2}{2}(p_2-q_1)+\frac{n-2}{2}(p_1-q_2)+(n-2)]} \tag{3.24}$$

$$\{\tfrac{1}{B_l-3\varepsilon}\log^{[p_2+1]}R_m\}].$$

Since $0<4\varepsilon<\min\{A_l,B_l\}$ and $q+\frac{n}{2}(p_2-q_2)+\frac{n-2}{2}(p_1-q_1)+(n-1)>q$ so for sufficiently large R_m, we have from (3.24)

$$\frac{\log^{[p+1]}M(R_m,f)}{\log^{[q]}R_m}\le\frac{\exp^{[q_2-q]}[\frac{1}{B_l-4\varepsilon}\log^{[\frac{n-2}{2}(p_2-q_1)+\frac{n-2}{2}(p_1-q_2)+(n-2)]}\{\frac{1}{B_l-3\varepsilon}\log^{[p_2+1]}R_m\}]}{\log^{[q]}R_m}\to 0.$$

Therefore,

$$\lim_{R_m\to\infty}\frac{\log^{[p+1]}M(R_m,f)}{\log^{[q]}R_m}=0\qquad\text{i.e.,}\qquad\lambda_{[p,q]}(f)=0.\qquad\square$$

Theorem 3.5. *Let f and g be entire functions and g have index-pair $[p_2,q_2]$. If $f_{n,g}$ have index-pair $[p_1,q_1]$ and $0<\rho_{[p_1,q_1]}(f_{n,g})=C<\infty$, then $\rho_{[p_1,q_1]}(f)=0$.*

The proof of this theorem is very much analogous to Theorem 3.4. In this case instead of (3.20) we have for given $\varepsilon\ (>0)$ and for sufficiently large r,

$$\tfrac{1}{3}(1+o(1))\log M(\tfrac{1}{9}M(\tfrac{r}{2^2},g_{n-1,f}),f)\le T(r,f_{n,g})\le\exp^{[p_1]}\{(C+\varepsilon)\log^{[q_1]}r\}$$

and proceed as before to get the result. $\square$

Theorem 3.6. *Let f and g be entire functions and p, q be two positive integers such that $p\ge q\ge 1$ and $\lambda_{[p,q]}(f_{n,g})=\lambda_1<\lambda_{[p,q]}(g)=\lambda_2<\infty$, then $\lambda(f)=0$.*

Proof. Let us assume that f have index-pair $[p_1,q_1]$. By the same reasoning as K. Niino and C. C. Yang [11], there exists a sequence $\{r_m\}$ tending to infinity such that for given $\varepsilon\ (>0)$ and for sufficiently large r_m, we get

$$\tfrac{1}{3}(1+o(1))\log M(\tfrac{1}{9}M(\tfrac{r_m}{2^2},g_{n-1,f}),f)\le T(r_m,f_{n,g})\le\exp^{[p]}\{(\lambda_1+\varepsilon)\log^{[q]}r_m\}. \tag{3.25}$$

Case (I). When n is odd.

For chosen ε $[0<4\varepsilon<\min\{A_l,\lambda_2\}]$ and for sufficiently large r_m, we get

$$\tfrac{1}{9}M(\tfrac{r_m}{2^2},g_{n-1,f})\ge\exp^{[p+1]}[(\lambda_2-3\varepsilon)\exp^{[\frac{n-3}{2}(p-q_1)+\frac{n-1}{2}(p_1-q)+(n-2)]}\{(A_l-3\varepsilon)\log^{[q_1]}r_m\}].$$

Set $R_m=\frac{1}{9}M(\frac{r_m}{2^2},g_{n-1,f})$, then

$$r_m\le\exp^{[q_1]}[\tfrac{1}{A_l-3\varepsilon}\log^{[\frac{n-3}{2}(p-q_1)+\frac{n-1}{2}(p_1-q)+(n-2)]}\{\tfrac{1}{\lambda_2-3\varepsilon}\log^{[p+1]}R_m\}]. \tag{3.26}$$

Now from (3.25) and (3.26), we have for sufficiently large R_m

$$\log M(R_m,f)\le\exp^{[p+q_1-q]}[\tfrac{1}{A_l-4\varepsilon}\log^{[\frac{n-3}{2}(p-q_1)+\frac{n-1}{2}(p_1-q)+(n-2)]}\{\tfrac{1}{\lambda_2-3\varepsilon}\log^{[p+1]}R_m\}]. \tag{3.27}$$

Since $0<4\varepsilon<\min\{A_l,\lambda_2\}$ and $\frac{n-3}{2}(p-q)+\frac{n-1}{2}(p_1-q_1)+(n-1)\ge 2$ so for sufficiently large R_m , we have from (3.27)

$$\frac{\log^{[2]}M(R_m,f)}{\log R_m}\le\frac{\log^{[3]}R_m}{\log R_m}\to 0.$$

Therefore,

$$\liminf_{R_m\to\infty}\frac{\log^{[2]}M(R_m,f)}{\log R_m}=0,\qquad\text{i.e.,}\qquad\lambda(f)=0.$$

Case (II). When n is even.

Now for chosen ε $[0<4\varepsilon<\min\{A_l,\lambda_2\}]$ and for sufficiently large r_m, we get

$$\tfrac{1}{9}M(\tfrac{r_m}{2^2},g_{n-1,f})\ge\exp^{[p+1]}[(\lambda_2-3\varepsilon)\exp^{[\frac{n-2}{2}(p-q_1)+\frac{n-2}{2}(p_1-q)+(n-2)]}\{(\lambda_2-3\varepsilon)\log^{[q]}r_m\}].$$

Set $R_m=\frac{1}{9}M(\frac{r_m}{2^2},g_{n-1,f})$, then

$$r_m\le\exp^{[q]}[\tfrac{1}{\lambda_2-3\varepsilon}\log^{[\frac{n-2}{2}(p-q_1)+\frac{n-2}{2}(p_1-q)+(n-2)]}\{\tfrac{1}{\lambda_2-3\varepsilon}\log^{[p+1]}R_m\}]. \tag{3.28}$$

Now from (3.25) and (3.28), we have for large R_m

$$\log M(R_m,f) \le \exp^{[p-q+q]}[\tfrac{1}{\lambda_2-4\varepsilon}\log^{[\frac{n-2}{2}(p-q_1)+\frac{n-2}{2}(p_1-q)+(n-2)]}\{\tfrac{1}{\lambda_2-3\varepsilon}\log^{[p+1]}R_m\}]$$

$$\le \exp^{[p]}[\tfrac{1}{\lambda_2-4\varepsilon}\log^{[\frac{n-2}{2}(p-q_1)+\frac{n-2}{2}(p_1-q)+(n-2)]}\{\tfrac{1}{\lambda_2-3\varepsilon}\log^{[p+1]}R_m\}]. \tag{3.29}$$

Since $0<4\varepsilon<\lambda_2$, so for sufficiently large R_m we have from (3.29)

$$\frac{\log^{[2]}M(R_m,f)}{\log R_m} \le \frac{\log^{[2]}R_m}{\log R_m} \to 0.$$

Therefore,

$$\liminf_{R_m\to\infty}\frac{\log^{[2]}M(R_m,f)}{\log R_m}=0, \qquad \text{i.e.,} \qquad \lambda(f)=0. \qquad \square$$

Theorem 3.7. *Let f and g be entire functions having index-pair $[p_1,q_1]$ and $[p_2,q_2]$ respectively. Then we can get the following conclusions :*

(I) When n is odd, then $\lim_{r\to\infty}\frac{\log^{[\frac{n+1}{2}p_1+\frac{n-1}{2}(p_2-q_1-q_2)+n]}\mu(r,f_{n,g})}{\log^{[p_2+1]}\mu(r,f)} \ge \frac{A_l}{A}$

and (II) when n is even,

(a) if $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)>0$ *and*

(i) $q_1>q_2$, *then* $\lim_{r\to\infty}\frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]}\mu(r,f_{n,g})}{\log^{[p_1+1]}\mu(r,f)}=\infty$;

(ii) $q_1=q_2$, *then* $\lim_{r\to\infty}\frac{\log^{[\frac{n}{2}(p_1+p_2-q_1)-\frac{n-2}{2}q_2+n]}\mu(r,f_{n,g})}{\log^{[p_1+1]}\mu(r,f)} \ge \frac{B_l}{A}$;

(b) if $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)=0$ *and*

(i) $q_1>q_2$, *then* $\lim_{r\to\infty}\frac{\log^{[p_1+1]}\mu(r,f_{n,g})}{\log^{[p_1+1]}\mu(r,f)}=\infty$;

(ii) $q_1=q_2$, *then* $\lim_{r\to\infty}\frac{\log^{[p_1+1]}\mu(r,f_{n,g})}{\log^{[p_1+1]}\mu(r,f)} \ge \frac{A_lB_l}{A}$;

(c) if $\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)<0$, *then* $\lim_{r\to\infty}\frac{\log^{[p_1+1]}\mu(r,f_{n,g})}{\log^{[p_1+1]}\mu(r,f)}=\infty$.

Proof. From Definition 1.6, for chosen ε $(0<4\varepsilon<A_l)$, there exists a positive number r_0 such that for all $r\ge r_0$, we have

$$\exp^{[p_1+1]}\{(A_l-\varepsilon)\log^{[q_1]}r\} \le \mu(r,f) \le \exp^{[p_1+1]}\{(A+\varepsilon)\log^{[q_1]}r\}.$$

Case (I). When n is odd.

Now since $\mu(r,f)\ge\frac{1}{2}M(\frac{r}{2},f)$, for chosen ε $(0<4\varepsilon<\min\{A_l,B_l\})$, from (3.13) we get

$$\mu(r,f_{n,g}) \ge \tfrac{1}{2}\exp^{[p_1+1]}[(A_l-2\varepsilon)\exp^{[\frac{n-1}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-1)]}\{(A_l-3\varepsilon)\log^{[q_1]}\tfrac{r}{2}\}]$$

$$\ge \exp^{[p_1+1]}[(A_l-3\varepsilon)\exp^{[\frac{n-1}{2}(p_1-q_2)+\frac{n-1}{2}(p_2-q_1)+(n-1)]}\{(A_l-4\varepsilon)\log^{[q_1]}r\}].$$

Case (II). When n is even.

From (3.18), we get

$$\mu(r,f_{n,g}) \ge \exp^{[p_1+1]}[(A_l-3\varepsilon)\exp^{[\frac{n-2}{2}(p_1-q_2)+\frac{n}{2}(p_2-q_1)+(n-1)]}\{(B_l-4\varepsilon)\log^{[q_2]}r\}].$$

Using the similar reasoning as Theorem 3.3, we get the required results. $\square$

Remark 3.4. A series of results can be obtained for $\mu(r,f_{n,g})$ and $\mu(r,g)$ using similar arguments.

Note 3.1. When $\alpha=1$ and $n=2$, all the results are identical to H. Y .Xu et. al [16]

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