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Zaman-Kesirli Mertebeden Burgers Denklemi İçin Optimal Bir Parametre ile Homotopi Analiz Yönteminin Geliştirilmesi

Improving Homotopy Analysis Method with An Optimal Parameter for Time-Fractional Burgers Equation

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Özet. Çalışmanın amacı h keyfi parametresinin seçimi ile ilgili artık hata fonksiyonunu kullanarak bu parametrenin optimal değerini belirleyerek mutlak hatayı azaltmaktır. Bazı sayısal örnekler çözülmüş ve mevcut sonuçlarla karşılaştırılmıştır. Homotopi analiz yöntemi, seri çözümler elde etmek için Burgers denklemine başarıyla uygulanmıştır. Gerekli denklemler için elde edilen çözümlere dayanarak, bu yöntemin zaman-kesirli kısmi diferansiyel denklemlere uygulanabileceği gösterilmiştir.

Anahtar Kelimeler: Homotopi analiz metodu, Burgers denklemi, Caputo kesirli türevi

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Abstract. The aim of the study is to reduce the absolute error by determining the optimal value of this arbitrary parameter using the residual error function related to the selection of the arbitrary parameter h. Some numerical examples are solved and compared to existing results. The homotopy analysis method has been successfully implemented to Burgers equation to obtain serial solutions. On the base of the solutions obtained for the required equations, it has been shown that this method is applicable to time-fractional partial differential equations.

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Key words: Homotopy analysis method, Burgers equation, Caputo fractional derivative

1. Introduction

Analytical solution of nonlinear problems is incredibly challenging. The region of convergence of the series solution in quasi-analytical methods is frequently determined by physical parameters. When non-linearity is strong, quasi-analytical approaches frequently fail. Homotopy analysis method (HAM) provides practical results to remove these unsuccessful results. HAM is a semi-analytical general technique to produce series solutions to numerous nonlinear equations. This method is used to find algebraic, ordinary differential, and partial differential equation solutions. This method first provides a continuous transform that takes the original approximation to the exact soluiton in order to solve the considered problem. An auxiliary linear operator is selected to generate this type of continuous transform. An auxiliary parameter is utilized to verify convergence of the resulting series solution. This method allows the choice of any initial approximation and linear auxiliary operators. A complex nonlinear problem is reduced into an enourmous number of easier linear subproblems with the help of HAM.

Liao (1992) proposed HAM for the first time in his phd thesis [11]. In the years that followed, HAM was used to a variety of differential problems [12-14]. With HAM, convergence control was also supplied in an analytical series solution technique. Using this technique, numerous researchers have effectively solved a variety of physical and engineering problems. Sun (2004) applied HAM to the solution of nonlinear traveling waves modeled by Klein-Gordon equation [26]. HAM established for the integer order differential equation by Song and Zhang (2007) was firstly applied to fractional KdV-Burgers-Kuramoto problem [25]. Abbasbandy (2008) solved the generalized Benjamin-Bona-Mahony (BBM) problem by HAM [1]. Abdulaziz et al. (2008) solved a number of FPDEs via HAM, such as the fractional wave-like, hyperbolic, and Fisher equations [2].

Dehghan et al. (2009) solved nonlinear FPDEs by utilizing HAM [5]. In their study, Yusufoglu and Selam (2010) determined the range of convergence for the h convergence-control parameter by employing HAM to analyze the modified equal-width wave equation, and they verified the method's efficiency by determining the appropriate h value [31]. In their study (Elsaid, 2011), partial differential equations (PDEs) with spatial and temporal fractional derivatives were solved using homotopy analysis in terms of Riesz and Caputo [6]. Using HAM, Arafa et al. (2012) obtained effective numerical solutions for a system consisting of two fractional reaction-diffusion equations (fractional Schnakenberg systems) that model morphogen systems in developmental biology [3, 6]. Vishal et al. (2012) utilized HAM to get approximate solution to time-fractional Swift-Hohenberg equation [30]. In their study, Sakar and Erdogan (2013) utilized HAM to numerically solve the nonlinear Fornberg-Whitham equation, compared the results acquired with Adomian decomposition method (ADM), and determined the optimal values for the convergence-control parameter in the examined ranges [22]. In the study of Shaiq et al. (2013), the nonlinear time-fractional wave-like equation was solved by using HAM [24]. The method of multi-step homotopy analysis was utilized by Freihat et al. (2013) to create a modified epidemiological model for computer viruses [8].

In their study, Lu and Liu (2014) utilized HAM to numerically solve the variable coefficient KdV Burgers equation numerically [16]. In the study [4], solutions to nonlinear wave-like equations were obtained using HAM. In their paper, Odibat and Bataineh (2015) presented a novel approach to HAM for nonlinear problems [19]. This proposed approach can easily overcome the difficulty of calculating complex integrals. In addition, homotopy polynomials are proposed that decompose the nonlinear term of the problem into a set of polynomials, and a computational algorithm is developed for such polynomials that make the solution procedure simpler and more effective. An effective approach was developed for determining the optimal convergence-control parameters employed in HAM's analysis of the convergence region [27]. In their paper, Pandey and Mishra (2017) developed a hybrid technique for solving third-order fractional dispersive wave equations by combining HAM and Sumudu transform [21]. PDEs in engineering were solved by using HAM [9]. Jia et al. (2017) implemented optimal homotopy analysis method (OHAM) to optimal control problems [10]. Odibat (2018) proposed a new approach for the optimal choice of linear operator and initial approach

[20]. Van Gorder (2019) developed a novel method for OHAM and error control for nonlinear ordinary differential equations (ODEs) [29].

The aim of the study is to get numerical solutions of Burgers equation, including the arbitrary parameter h, by using HAM. For this reason, the residual error method is utilized to determine the optimal parameter values of the h parameter.

The organization of this article is as follows: In the second chapter, information is provided concerning about HAM and its development. Chapter 3 presents the numerical solution to the time-fractional Burgers equation by using HAM. Numerical results and discussion is given in Chapter 4. In the conclusion chapter, the most significant outcomes of the research are emphasized.

2. Homotopy Analysis Method

HAM is a semi-analytical technique developed by Liao (1992) for the solution of nonlinear ODEs and PDEs [11]. It is based on homotopy and Taylor formula, which is an essential idea in topology. The flexibility with which the initial approach and auxiliary linear operators can be chosen is one of the most crucial components of this method. HAM for PDEs will now be introduced.

2.1. Homotopy Analysis Method for Partial Differential Equations

Examine the following nonlinear equation to explain the fundamental concept underlying the technique utilized in this investigation:

$$\mathcal{N}[y(x,t)] = 0, \tag{1}$$

where $\mathcal{N}$ is a nonlinear operator, $\boldsymbol{x}$ and $\boldsymbol{t}$ are independent spatial and temporal variables, $y(x,t)$ is unknown function dependent on these variables.

Liao (2003) developed the zero-order deformation equation by utilizing the classical homotopy concept [13]. Let $q \in [0,1]$ be embedding parameter, $h \neq 0$ be nonzero convergence-control parameter, $H(x,t)$ be an auxiliary function, M be auxiliary linear operator and $y_0(x,t)$, be an initial estimate of $y(x,t)$. Therefore, the zero-order deformation equation for the solution series $\varphi(x,t;q)$ is given by

$$(1-q)M[\varphi(x,t;q)-y_0(x,t)] = qhH(x,t)\mathcal{N}[\varphi(x,t;q)]. \quad (2)$$

Homotopy analysis process method provides arbitrary choices of the convergence-control parameter h, the auxiliary function $H(x,t)$, and the auxiliary linear operator M. $\varphi(x,t;q)$ which is the solution of the equation is dependent not only on $y_0(x,t)$, M, h, and $H(x,t)$, but also on the embedding parameter $q \in [0,1]$.

When $q = 0$ and $q = 1$ in Eq. (2), the following expressions are obtained, respectively:

$$\varphi(x,t;0) = y_0(x,t), \varphi(x,t;1) = y(x,t). \quad (3)$$

When the homotopy parameter q increases from 0 to 1, $\varphi(x,t;q)$ continuously converges from the initial approximation $y_0(x,t)$ to the exact solution $y(x,t)$. In homotopy, this continuous change is referred to as deformation.

The derivatives of the deformation equation of the m-th order is defined by:

$$y_0^{(m)}(x,t) = \left.\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\right|_{q=0}. \quad (4)$$

With the aid of Taylor's theorem, the expansion of $\varphi(x,t;q)$ to the power series of q is obtained as:

$$\varphi(x,t;q) = y_0(x,t) + \sum_{m=1}^{+\infty} y_m(x,t)q^m. \quad (5)$$

where,

$$y_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \quad (6)$$

And also, the power series of $\varphi(x,t;q)$ is

$$\varphi(x,t;q) = y_0(x,t) + \sum_{m=1}^{+\infty} y_m(x,t)q^m. \quad (7)$$

If auxiliary linear operator M, initial approximation $y_0(x,t)$, convergence-control parameter h, and auxiliary function $H(x,t)$ are appropriately chosen, the power series $\varphi(x,t;q)$ converges at $q = 1$ and it is obtained as

$$y(x,t) = y_0(x,t) + \sum_{m=1}^{\infty} y_m(x,t). \quad (8)$$

Liao (2005) showed in the literature that if one of the solutions to the nonlinear equation given at the beginning is $h = -1$ and $H(x,t) = 1$, the equation turns into the following form, which is utilized in the homotopy perturbation method [14].

$$(1-q)M[\varphi(x,t;q) - y_0(x,t)] + q\mathcal{N}[\varphi(x,t;q)] = 0. \quad (9)$$

If the vector $\vec{y}_n$ is defined as

$$\vec{y}_n = \{y_0(x,t), y_1(x,t), \dots, y_n(x,t)\}. \quad (10)$$

According to Eq. (6), the equation of $y_m(x,t)$ is obtained from the zero-order deformation equation.

Using the $\mathcal{X}_m$ function defined by

$$\mathcal{X}_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases} \quad (11)$$

it is obtained as

$$M[y_m(x,t) - \mathcal{X}_m y_{m-1}(x,t)] = hH(x,t)R_m(\vec{y}_{m-1}, x, t), \quad (12)$$

where $R_m(\vec{y}_{m-1}, x, t)$ is given as follows:

$$R_m(\vec{y}_{m-1}, x, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\varphi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}. \tag{13}$$

The m-th order deformation equation is found using Eq. (13). By using Eqs. (5)-(13), it is found as,

$$R_m(\vec{y}_{m-1}, x, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \mathcal{N}\left[\sum_{n=0}^{+\infty} y_n(x,t) q^n\right]\Bigg|_{q=0}. \tag{14}$$

m-th order solution approach is obtained as

$$y(x,t) = \sum_{k=0}^{m} y_k(x,t), \tag{15}$$

where the solution $y(x,t)$ includes the convergence-control parameter h. The calculation of the parameter h will be explained in detail in a subsequent section.

Theorem 2.1. [13] If the homotopy series in Eq. (8) is convergent, the result is as follows:

$$\sum_{n=1}^{\infty} R_n(\vec{y}_{n-1}, x, t) = 0. \tag{16}$$

Proof. For Theorem 2.1.'s proof, it can be seen [13].

Theorem 2.2. [13] If the homotopy series in Eq. (8) converges, then this series must be a solution to the original nonlinear Eq. (1).

Proof. For Theorem 2.2.'s proof, it can be seen [13].

2.2. Optimal Homotopy Analysis Method

Drawing h curves determines the range of the h convergence-control parameter in the classical HAM. The optimal parameter of the known method by Liao (2010) was investigated [15]. It was utilized a variety of methods to calculate this h parameter [7, 10, 18, 23, 27-29].

The exact square residual error in ordinary differential equations for the m-th order approximation is defined by [15]

$$\Delta_m = \int_{\Omega} \left(N\left(\sum_{i=0}^{m} s_i\,(w) \right) \right)^2 dw, \tag{17}$$

where the expression Δ_m includes an unknown h convergence-control parameter. For the m-th approximation, the optimal value of the convergence-control parameter h is the minimum value of Δ_m. That is

$$\frac{d\Delta_m}{dh} = 0. \tag{18}$$

However, it has been shown that when calculating Δ_m given by Liao with the formula (17), even if the approximation order is low, a significant amount of CPU time is required. In order to reduce CPU time, the average quadratic residual error ($\sqrt{E_m}$) is defined as follows [15]:

$$E_m = \frac{1}{n+1} \sum_{j=0}^{n} \left(N\left(\sum_{i=0}^{m} s_i\left(\frac{j}{n}, h\right) \right) \right)^2. \tag{19}$$

For Burgers equation discussed in this study, the valid version of the Eq. (18) was utilized.

3. Burgers Equation

In this part, Burgers equation with a variable coefficient in one dimensional involving fractional derivative in the sense of Caputo is discussed [17]:

$$D_t^{\alpha} y(x,t) + k_1(x,t) y_x(x,t) + k_2(x,t) y(x,t) + k_3(x,t) y_x(x,t) +$$

$$k_4(x,t) y(x,t) y_x(x,t) = f(x,t), \quad 0 \leq x, t \leq 1\,, 0 < \alpha \leq 1, \tag{20}$$

where, $D_t^{\alpha}$is the Caputo fractional derivative operator of order α with respect to the time variable t. Also, $k_1(x,t)$, $k_2(x,t)$, $k_3(x,t)$, $k_4(x,t)$ and $f(x,t)$ are continuous functions. The initial and boundary conditions of the equation utilized are taken as follows:

$$y(x,0) = 0,$$

$$y(0,t) = y(1,t) = 0. \tag{21}$$

If $k_1(x,t) = -1$, $k_2(x,t) = 0$, $k_3(x,t) = 0$, $k_4(x,t) = 1$ ve $f(x,t) = \frac{2t^{2-\alpha}e^x}{\Gamma(3-\alpha)} + t^4 e^{2x} - t^2 e^x$ in Eq. (20), then Burgers equation becomes

$$D_t^{\alpha} y(x,t) - y_{xx}(x,t) + y(x,t).\, y_x(x,t) = \frac{2t^{2-\alpha}e^x}{\Gamma(3-\alpha)} + t^4 e^{2x} - t^2 e^x. \tag{22}$$

The exact solution of this problem is $y(x,t) = e^x t^2$. Now, the efficiency of the method is investigated by applying HAM to Eq. (22). Obtaining the optimal values for the arbitrary parameter h that appears in the solution will be shown.

Firstly, the linear operator M is chosen as follows to provide the property $M[c] = 0$:

$$M[\varphi(x,t;q)] = \frac{\partial^{\alpha} \varphi(x,t;q)}{\partial t^{\alpha}}, \tag{23}$$

where c is constant. Now, the nonlinear operator is chosen as

$$N[\varphi(x,t;q)] = \frac{\partial^{\alpha}\varphi(x,t;q)}{\partial t^{\alpha}} - \frac{\partial^{2}\varphi(x,t;q)}{\partial t^{2}} + \varphi(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x} - \frac{2t^{2-\alpha}e^{x}}{\Gamma(3-\alpha)}$$

$$-t^{4}e^{2x} + t^{2}e^{x}. \tag{24}$$

The zero-order deformation equation is formulated as

$$(1-q)M[\varphi(x,t;q) - y_0(x,t)] = qhN[\varphi(x,t;q)]. \tag{25}$$

By substituting $q = 0$ and $q = 1$ in Eq. (25), the following expressions can written:

$$\begin{cases} \varphi(x,t;0) = y_0(x,t), \\ \varphi(x,t;1) = y(x,t). \end{cases} \tag{26}$$

Thus, the m-th order deformation equation is written as

$$M[y_m(x,t) - \chi_m y_{m-1}(x,t)] = hR_m\left(\vec{y}_{m-1}(x,t)\right), \tag{27}$$

where,

$$R_m\left(\vec{y}_{m-1}(x,t)\right) = \frac{\partial^{\alpha}\varphi_{m-1}(x,t;q)}{\partial t^{\alpha}} - \frac{\partial^{2}\varphi_{m-1}(x,t;q)}{\partial x^{2}} + \varphi_{m-1}(x,t;q)\frac{\partial\varphi_{m-1}(x,t;q)}{\partial x}$$

$$-\left(\frac{2t^{2-\alpha}e^{x}}{\Gamma(3-\alpha)} + t^{4}e^{2x} - t^{2}e^{x}\right)(1-\chi_m). \tag{28}$$

Also where, Adomian polynomial for the nonlinear term is written as:

$$\sum_{n=0}^{m-1} \varphi_n\,(x,t;q)\frac{\partial\varphi_{m-1}(x,t;q)}{\partial x}. \tag{29}$$

Hence, Burgers equation becomes

$$R_m\left(\vec{y}_{m-1}(x,t)\right) = \frac{\partial^{\alpha}\varphi_{m-1}(x,t;q)}{\partial t^{\alpha}} - \frac{\partial^{2}\varphi_{m-1}(x,t;q)}{\partial t^{2}}$$

$$+\sum_{n=0}^{m-1} \varphi_n\,(x,t;q)\frac{\partial\varphi_{m-1}(x,t;q)}{\partial x} - \left(\frac{2t^{2-\alpha}e^{x}}{\Gamma(3-\alpha)} + t^{4}e^{2x} - t^{2}e^{x}\right)(1-\chi_m). \tag{30}$$

The solution to the m-th order deformation equations for $m \geq 1$ is obtained as

$$y_m(x,t) = \chi_m y_{m-1}(x,t) + hM^{-1}\left[R_m\left(\vec{y}_{m-1}(x,t)\right)\right]. \tag{31}$$

To begin the iteration, the initial approximation $y_0(x,t)$ must be known. Initial approach can be chosen arbitrarily to satisfy initial and boundary-value conditions.

If $y_0(x,t) = 0$ in Eq. (31), the first few iteration terms are as follows:

$$y_0(x,t) = 0, \tag{32}$$

$$y_1(x,t) = \frac{he^x(-\Gamma(\alpha+5)t^2 - 24e^x t^{\alpha+4} + 2\alpha^2 t^{\alpha+2} + 4\alpha t^{\alpha+2} + 24t^{\alpha+2})}{\Gamma(\alpha+5)}, \tag{33}$$

$$\vdots$$

4. Numerical Results and Discussion

The approximate solution is calculated as $u_m(x,t) = \sum_{k=0}^{m} y_m(x,t)$. In the HAM, h parameter is the convergence-control parameter. For the optimal value of this parameter, the squared residual error method will be used. The region will be taken as $(x,t) \in \left[0,\frac{1}{2}\right] \times \left[0,\frac{1}{2}\right]$. Let's define the residual function as follows:

$$r_3(x,t,h) = D_t^{\alpha} u_3(x,t,h) - \frac{\partial^2 u_3(x,t,h)}{\partial x^2} - u_3(x,t,h)\frac{\partial u_3(x,t,h)}{\partial x}$$
$$-\left(\frac{2t^{2-\alpha}e^x}{\Gamma(3-\alpha)} + t^4 e^{2x} - t^2 e^x\right). \tag{34}$$

Taking the 2nd norm of this residual function yields the expression

$$e_3(h) = \left(\frac{1}{\frac{1}{2}} \cdot \frac{1}{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |r_3(x,t,h)|^2 dt dx\right)^{\frac{1}{2}}. \tag{35}$$

In calculating the optimal values of the h parameter, the minimum of $e_3(h)$ according to norm 2 will be chosen.

The optimal h parameters for $\alpha = 1, 0.9, 0.8, 0.7$ are shown in the following table 1.

The numerical computations are shown in Tables 2-5 and Fig. 1.

Table 1. The optimal h parameters for $\alpha = 1, 0.9, 0.8, 0.7$.

m	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$
3	0.7671576951	-0.5298577453	-0.4075433436	-0.3242486092

Table 2. Comparison of exact solution and HAM solution for $\boldsymbol{\alpha = 1}$ in Eq. (22).

x	t	HAM	Exact Solution	Absolute Error (AE)
0.20	0.10	0.0120078877	0.0122140275	2.06E-4
0.20	0.20	0.0478689819	0.0488561103	9.87E-4
0.20	0.30	0.1076578753	0.1099262482	2.26E-3
0.20	0.40	0.1923444856	0.1954244413	3.07E-3
0.20	0.50	0.3045316031	0.3053506895	8.19E-4
0.50	0.10	0.0162104277	0.0164872127	2.76E-4
0.50	0.20	0.0646663837	0.0659488508	1.28E-3
0.50	0.30	0.1457253815	0.1483849144	2.65E-3
0.50	0.40	0.2614341670	0.2637954034	2.36E-3
0.50	0.50	0.4168778270	0.4121803178	4.69E-3

Table 3. Comparison of exact solution and HAM solution for $\boldsymbol{\alpha = 0.9}$ in Eq. (22).

x	t	HAM	Exact Solution	AE
0.20	0.10	0.0107364327	0.0122140275	1.47E-3
0.20	0.20	0.0422684704	0.0488561103	6.58E-3
0.20	0.30	0.0940180048	0.1099262482	1.59E-2
0.20	0.40	0.1664160774	0.1954244413	2.90E-2
0.20	0.50	0.2615618637	0.3053506895	4.37E-2
0.50	0.10	0.0144946722	0.0164872127	1.99E-3
0.50	0.20	0.0571195568	0.0659488508	8.82E-3
0.50	0.30	0.1273944830	0.1483849144	2.09E-2
0.50	0.40	0.2267136196	0.2637954034	3.70E-2
0.50	0.50	0.3595539798	0.4121803178	5.26E-2

Table 4. Comparison of exact solution and HAM solution for $\boldsymbol{\alpha = 0.8}$ in Eq. (22).

x	t	HAM	Exact Solution	AE
0.20	0.10	0.0093120640	0.0122140275	2.90E-3
0.20	0.20	0.0362293866	0.0488561103	1.26E-2
0.20	0.30	0.0798831342	0.1099262482	3.00E-2
0.20	0.40	0.1405973462	0.1954244413	5.48E-2
0.20	0.50	0.2204604068	0.3053506895	8.48E-2
0.50	0.10	0.0125726342	0.0164872127	3.91E-3
0.50	0.20	0.0489826902	0.0659488508	1.69E-2
0.50	0.30	0.1084038925	0.1483849144	3.99E-2
0.50	0.40	0.1921648849	0.2637954034	7.16E-2
0.50	0.50	0.3048139294	0.4121803178	1.07E-1

Table 5. Comparison of exact solution and HAM solution for $\boldsymbol{\alpha = 0.7}$ in Eq. (22).

x	t	HAM	Exact Solution	AE
0.20	0.10	0.0079127016	0.0122140275	4.30E-3
0.20	0.20	0.0303901811	0.0488561103	1.84E-2
0.20	0.30	0.0664724198	0.1099262482	4.34E-2
0.20	0.40	0.1165825275	0.1954244413	7.88E-2
0.20	0.50	0.1829901398	0.3053506895	1.22E-1
0.50	0.10	0.0106845311	0.0164872127	5.80E-3
0.50	0.20	0.0411183005	0.0659488508	2.48E-2
0.50	0.30	0.0904015854	0.1483849144	5.79E-2
0.50	0.40	0.1600752973	0.2637954034	1.03E-1
0.50	0.50	0.2550116282	0.4121803178	1.57E-1

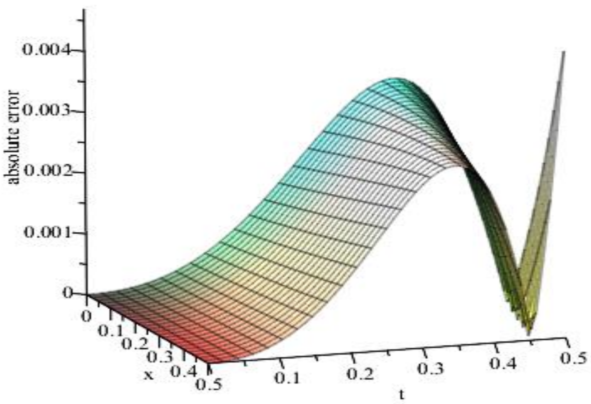

Figure 1. Absolute error graph for $\boldsymbol{\alpha = 1}$in Eq. (22).

5. Conclusion

In this study, the optimal parameter used for ordinary differential equations was successfully applied to the fractional partial Burgers equation, resulting in serial solutions. By narrowing the studied region or increasing the number of iterations, it is possible to lower the absolute error. In light of the solutions obtained for Burgers equation discussed in this study, it is thought that this method can be used to solve linear and nonlinear FPDEs.

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