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Theorems of Second Korovkin Type with respect to Triangular A-Statistical Convergence

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Abstract

This article is a continuation of our previous works. We mainly investigate a Korovkin type theorem for double sequences of positive linear operators defined in the space of all 2π-periodic and real valued continuous functions on the real two-dimensional space with help of the concept of triangular A-statistical convergence, which is a kind of statistical convergence for double real sequences. Also, we analyze the rate of convergence of double operators in this via modulus of continuity.

1. Introduction

Fast [1] (independently, Steinhaus [2]) introduced the concept of statistical convergence, which is an advantageous approach. This concept is studied in various fields and its generalization and properties are investigated. Bardaro et al. [3], introduced the concept of triangular A-statistical convergence which is a variant of statistical convergence in 2015. This new convergence offers another perspective as it is not comparable to statistical convergence. In addition, there are other studies in the literature [4–7].

The Korovkin type theorem has an important place in approximation theory as it enables us to check convergence with minimum calculations [8]. This theorem has been studied by many mathematicians in different spaces and with various types of convergence, with the aim of obtaining more general results [9–20].

Let $C^*\left(\mathbb{R}^2\right)$ stands for the space of all 2π-periodic and continuous functions on $\mathbb{R}^2$.

Our main aim in this study is to present a theorem of Korovkin type on $C^*\left(\mathbb{R}^2\right)$ in the light of the triangular A-statistical convergence given by Bardaro et al.

Before proceeding we recall some notation on the paper.

A double sequence $x=(x_{m,n})$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$, the set of all natural numbers, such that $|x_{m,n}-\iota|<\varepsilon$ whenever $m,n>N$, where ι is called the Pringsheim limit of x and denoted by $P-\lim x=\iota$ (see [21]). We shall call such an x, as P-convergent. By a bounded double sequence we mean there exists a $H>0$ such that $|x_{m,n}|\leq H$ for all $(m,n)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$. It is worthy of note that unlike the single sequences, the double sequence does not have to be bounded. Let $A=(a_{k,l,m,n})$ be a four-dimensional summability matrix. For a given double sequence $x=(x_{m,n})$, the A-transform of x, denoted by $Ax:=((Ax)_{k,l})$, is given by

$$(Ax)_{k,l}=\sum_{(m,n)\in\mathbb{N}^2}a_{k,l,m,n}x_{m,n}$$

provided the double series converges in Pringsheim's sense for every $(k,l)\in\mathbb{N}^2$.

If two dimensional matrix transformation of a given $x=(x_{m,n})$ sequence preserve $(Ax)_{k,l}$ limit, that is $P-\lim x=\iota$ whenever $P-\lim(Ax)_{k,l}=\iota$ then the matrix $A=(a_{k,l,m,n})$ is called a regular matrix.

Let's remember a four dimensional matrix $A=(a_{k,l,m,n})$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The well establish characterization of regularity for four-dimensional matrices is known as Robison-Hamilton conditions or RH-regularity (see, [22,23]) state that a four dimensional matrix $A=(a_{k,l,m,n})$ is RH-regular iff

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(i) $P-\lim_{k,l} a_{k,l,m,n}=0$ for each $(m,n)\in\mathbb{N}^2$,
(ii) $P-\lim_{k,l}\sum_{(m,n)\in\mathbb{N}^2} a_{k,l,m,n}=1$,
(iii) $P-\lim_{k,l}\sum_{m\in\mathbb{N}}\left|a_{k,l,m,n}\right|=0$ for each $n\in\mathbb{N}$,
(iv) $P-\lim_{k,l}\sum_{n\in\mathbb{N}}\left|a_{k,l,m,n}\right|=0$ for each $m\in\mathbb{N}$,
(v) $\sum_{(m,n)\in\mathbb{N}^2}\left|a_{k,l,m,n}\right|$ is P-convergent for each $(k,l)\in\mathbb{N}^2$,
(vi) there exist finite $A,B>0$ such that $\sum_{m,n>B}\left|a_{k,l,m,n}\right|<A$ holds for every $(k,l)\in\mathbb{N}^2$.

Firstly let $A=(a_{k,l,m,n})$ be a non-negative RH-regular summability matrix, and let $K\subset\mathbb{N}^2$. Then A-density of K is given as below

$$\delta_A^2(K):=P-\lim_{k,l}\sum_{(m,n)\in K} a_{k,l,m,n}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. Now recall the definiton of A-statistical convergence by considering the concept of A-density. A real double sequence $x=(x_{m,n})$ is said to be A-statistically convergent to a number L if, for every $\varepsilon>0$,

$$\delta_A^2\left(\{(m,n)\in\mathbb{N}^2:|x_{m,n}-\iota|\geq\varepsilon\}\right)=0.$$

At this state, we can show it as $st_A^2-\lim x=\iota$. Also, while $P-\lim x=\iota$, $st_A^2-\lim x=\iota$ is true but when $st_A^2-\lim x=\iota$ is not always $P-\lim x=\iota$. Furhermore, the double sequence does not require to be bounded when $st_A^2-\lim x=\iota$ is exist.
It is worth noting that now with the special choices of the A matrix in concept of A-statistical convergence for double sequences, the following relations are obtained. If one replaces the matrices A the double Cesáro matrix, then A-statistical convergence coincides to the statistical convergence i.e., $st^2_{C(1,1)}-\lim x=st^2-\lim x$ [24].

2. Triangular Statistical Convergence

Let $x=(x_{m,n})$ be a double sequence and suppose that $x=(x_{m,n})$ is neither A-statistical convergent nor convergent in the Pringsheim's sense. On the question of whether a different convergence is considered in such a case, Bardaro et al. introduced the notion of triangular A-statistical convergence in [3]. First, consider the regular matrix for double sequences [3].
The Silverman-Toeplitz conditions, which have an important place in the literature for the regular characterization of the two-dimensional matrix transformation, are as follows (see, for instance, [25]).

(i) $\|A\|=\sup_m\sum_{n=1}^{\infty}|a_{m,n}|<\infty$,
(ii) $\lim_m a_{m,n}=0$ for each $n\in\mathbb{N}$,
(iii) $\lim_m\sum_{n=1}^{\infty}a_{m,n}=1$.

Let $A=(a_{m,n})$ be a nonnegative regular summability matrix, K denotes the set $\{(m,n)\in\mathbb{N}^2:n\leq m\}$ and K_m is the m-section of K, i.e., the set of all $n\in\mathbb{N}$ such that $(m,n)\in K$, then we define triangular A-density of K by

$$\delta_A^T(K):=\lim_m\sum_{n\in K_m}a_{m,n}$$

provided that the limit on the right-hand side exists [3].
Also,

(i) $\delta_A^T(\mathbb{N}^2)=1$,
(ii) if $K\subset L$ then $\delta_A^T(K)\leq\delta_A^T(L)$,
(iii) if K has triangular A-density then $\delta_A^T(\mathbb{N}^2/K)=1-\delta_A^T(K)$,

triangular A-density has the above properties.

Definition 2.1 ([3]). *Let $A=(a_{m,n})$ be a nonnegative regular summability matrix. The number sequence $x=(x_{m,n})$ is triangular A-statistically convergent to ι provided that for every $\varepsilon>0$*

$$\lim_m\sum_{n\in K_m(\varepsilon)}a_{m,n}=0,$$

where $K_m(\varepsilon)=\{n\in\mathbb{N}:n\leq m,\ |x_{m,n}-\iota|\geq\varepsilon\}$ also written as $st_A^T-\lim_m x_{m,n}=\iota$.

The case in which $A=C_1$ the Cesaro matrix of order one reduces to the triangular statistical convergence i.e., $st_A^T-\lim x=st_{C_1}^T-\lim x$.
Triangular density $\delta^T(K)$ is given by

$$\delta^T(K)=\lim_m\frac{1}{m}|K_m|$$

or equivalently

$$\delta^{T}(K)=\lim_{m}\left(C_1\chi_{K_m}(n)\right)_m=\lim_{m}\sum_{n=1}^{\infty}c_{m,n}\chi_{K_m}(n)$$

if it exists. The number sequence $x=(x_{m,n})$ is triangular statistically convergent to ι provided that for every $\varepsilon>0$, the set $K:=K_m(\varepsilon):=\{n\in\mathbb{N}:n\leq m,\ |x_{m,n}-\iota|\geq\varepsilon\}$ if $\delta^{T}(K_m(\varepsilon))=0$; then we can write $st^{T}-\lim_{m}x_{m,n}=\iota$.

Let st_A^T be the set of all triangular A-statistically convergent sequences. As we mentioned before, triangular A-statistical convergence is a variant of statistical convergence. Here we give examples showing that these two convergences cannot be compared.

Example 2.2. *Let $A=C_1$ and*

$$x_{m,n}=\begin{cases}2, & m=n=j^2\\ \frac{j}{3(j+1)}, & m=2j,\ n=2j+1\\ \frac{2j}{3(j+2)}, & m=2j-1,\ n=2(j+1)\\ 0, & \text{otherwise}\end{cases},\ j\in\mathbb{N}.$$

$x=(x_{m,n})$ be given as above. For every $\varepsilon>0$,

$$\frac{1}{m}\left|\{n\in\mathbb{N}:n\leq m,\ |x_{m,n}-0|\geq\varepsilon\}\right|=\begin{cases}\frac{1}{j^2}, & m=j^2\\ 0, & \text{otherwise}\end{cases},\ j\in\mathbb{N}$$

clearly,

$$\lim_{m}\frac{1}{m}\left|\{\,n\in\mathbb{N}:n\leq m,\ |x_{m,n}-0|\geq\varepsilon\}\right|=0.$$

So, we obtain $st_{C_1}^{T}-\lim_{m}x_{m,n}=0$. Nevertheless, $x=(x_{m,n})$ is not Pringsheim's and $C(1,1)$-statistically convergent.

Example 2.3. *Take $A=C(1,1)$ and*

$$x_{m,n}=\begin{cases}\sqrt{mn}, & m=n=j^2\\ \frac{3}{mn}, & \text{otherwise}\end{cases},\ j\in\mathbb{N}.$$

$x=(x_{m,n})$ be given as above. Obviously, $st_{C(1,1)}^{2}-\lim_{m,n}x_{m,n}=0$ but x is not Pringsheim's and triangular statistically convergent.

Example 2.4. *Let $A=C_1$ and*

$$x_{m,n}=\begin{cases}-2, & m=n=j^2\\ 0, & \text{otherwise}\end{cases},\ j\in\mathbb{N}.$$

$x=(x_{m,n})$ be given as above. Similarly, $st_{C_1}^{T}-\lim_{m}x_{m,n}=0$ and $st_{C(1,1)}^{2}-\lim_{m,n}x_{m,n}=0$.

Example 2.5. *Let $A=C_1$ and*

$$x_{m,n}=\begin{cases}1, & m=n=j^2\\ \frac{j}{2j+1}, & m=2j+1,\ n=2j-1\\ \frac{j}{4j+2}, & m=2j,\ n=2(j+1)\\ k, & m=j^2,\ n=j^2+1\\ 0, & \text{otherwise}\end{cases},\ j\in\mathbb{N}.$$

$x=(x_{m,n})$ be given as above. So, we can easily see that $st_{C_1}^{T}-\lim_{m}x_{m,n}=0$. Neither $x=(x_{m,n})$ is Pringsheim's and $C(1,1)$-statistically convergent nor bounded.

Remark 2.6. *(i) Triangular statistical convergence and statistical convergence are incompatible; i.e., $st_A^T\nsubseteq st_A^2$ and $st_A^2\nsubseteq st_A^T$.*
(ii) A P-convergent double sequence is A-statistically convergent and triangular A-statistically convergent to the same value but the inverse implications are not true, i.e., $st_A^2\nsubseteq c^2$ and $st_A^T\nsubseteq c^2$.

3. A Korovkin-Type Approximation Theorem

In this section using the concept of triangular A-statistical convergence for double sequence and test function $1, sins, coss, sint, cost$, we provide a Korovkin type theorem for positive linear operators on the space $C^{*}\left(\mathbb{R}^2\right)$.

If a function f on $\mathbb{R}^2$ has a 2π-period, then, for all $(s,t)\in\mathbb{R}^2$,

$$f(s,t)=f(s+2k\pi,t)=f(s,t+2k\pi)$$

holds for $k=0,\pm1,\pm2,\ldots$. This space is equipped with the supremum norm

$$\|f\|_{C^{*}(\mathbb{R}^2)}=\sup_{(s,t)\in\mathbb{R}^2}|f(s,t)|,\ \left(f\in C^{*}\left(\mathbb{R}^2\right)\right).$$

Theorem 3.1 ([26]). *Let $A=(a_{k,l,m,n})$ be a non-negative RH-regular summability matrix and let $(L_{m,n})$ be a double sequence of positive linear operators acting from $C^*\left(\mathbb{R}^2\right)$ into $C^*\left(\mathbb{R}^2\right)$. Then, for all $f\in C^*\left(\mathbb{R}^2\right)$*

$$st_A^2-\lim\|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}=0$$

iff the following statements hold:

$$st_A^2-\lim\|L_{m,n}(f_r)-f_r\|_{C^*(\mathbb{R}^2)}=0,\ r=0,1,2,3,4,$$

where $f_0(s,t)=1,\ f_1(s,t)=\sin s,\ f_2(s,t)=\sin t,\ f_3(s,t)=\cos s$ and $f_4(s,t)=\cos t$.

Theorem 3.2. *Let $A=(a_{m,n})$ be a nonnegative regular summability matrix and $(L_{m,n})$ be a double sequence of positive linear operators from $C^*\left(\mathbb{R}^2\right)$ into $C^*\left(\mathbb{R}^2\right)$. Then, for all $f\in C^*\left(\mathbb{R}^2\right)$*

$$st_A^T-\lim_m\|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}=0 \tag{3.1}$$

iff the following statements hold:

$$st_A^T-\lim_m\|L_{m,n}(f_r)-f_r\|_{C^*(\mathbb{R}^2)}=0,\ \textit{for every } r=0,1,2,3,4 \tag{3.2}$$

where $f_0(s,t)=1,\ f_1(s,t)=\sin s,\ f_2(s,t)=\sin t,\ f_3(s,t)=\cos s$ and $f_4(s,t)=\cos t$.

Proof. Under the hypotheses, since 1, *sins*, *coss*, *sint* and *cost* belong to $C^*\left(\mathbb{R}^2\right)$, the necessity is clear. Suppose that (3.2) hold and let $f\in C^*\left(\mathbb{R}^2\right)$ and D,F be closed subinterval of length 2π of $\mathbb{R}$. Fix $(s,t)\in D\times F$. As in the proof of Theorem 2.1 in [17], it follows from the continuity of f that

$$|f(u,v)-f(s,t)|<\varepsilon+\frac{2M_f}{\sin^2\frac{\delta}{2}}\varphi(u,v)$$

which gives,

$$\begin{aligned}|L_{m,n}(f;s,t)-f(s,t)|\le&L_{m,n}(|f(u,v)-f(s,t)|;s,t)+|f(s,t)|\,|L_{m,n}(f_0;s)-f_0(s,t)|\\ \le&\left|L_{m,n}\left(\varepsilon+\frac{2M_f}{\sin^2\frac{\delta}{2}}\varphi(u,v);s,t\right)\right|+M_f\,|L_{m,n}(f_0;s)-f_0(s,t)|\\ \le&\left(\varepsilon+M_f\right)|L_{m,n}(f_0;s)-f_0(s,t)|+\frac{M_f}{\sin^2\frac{\delta}{2}}\{2\,|L_{m,n}(f_0;s)-f_0(s,t)|\\ &+|\sin x|\,|L_{m,n}(f_1;s,t)-f_1(s,t)|+|\sin y|\,|L_{m,n}(f_2;s,t)-f_2(s,t)|\\ &+|\cos x|\,|L_{m,n}(f_3;s,t)-f_3(s,t)|++|\cos t|\,|L_{m,n}(f_4;s,t)-f_4(s,t)|\}+\varepsilon\\ <&\varepsilon+N\sum_{r=0}^{4}|L_{m,n}(f_r;s)-f_r(s,t)|\end{aligned}$$

where $M_f=\|f\|_{C^*(\mathbb{R}^2)}$, $\varphi(u,v)=\sin^2\frac{u-s}{2}+\sin^2\frac{v-t}{2}$ and $N:=\varepsilon+M_f+\frac{2M_f}{\sin^2\frac{\delta}{2}}$. Then, taking supremum over $(s,t)\in\mathbb{R}^2$, we obtain

$$\|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}<\varepsilon+N\sum_{r=0}^{4}\|L_{m,n}(f_r)-f_r\|_{C^*(\mathbb{R}^2)}. \tag{3.3}$$

Now given $\varepsilon^{'}>0$, choose $\varepsilon>0$ such that $\varepsilon<\varepsilon^{'}$, and define

$$\begin{aligned}D_m&:=\left\{n\in\mathbb{N}:n\le m,\ \|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}\ge\varepsilon^{'}\right\},\\ D_m^r&:=\left\{n\in\mathbb{N}:n\le m,\ \|L_{m,n}(f_r)-f_r\|_{C^*(\mathbb{R}^2)}\ge\frac{\varepsilon^{'}-\varepsilon}{5N}\right\},\ r=0,1,2,3,4.\end{aligned}$$

It is easy see that from (3.3)

$$D_m\subseteq\bigcup_{r=0}^{4}D_m^r.$$

Hence, we may write

$$\sum_{n\in D_m}a_{m,n}\le\sum_{m=0}^{4}\sum_{n\in D_m^r}a_{m,n}.$$

Now taking the limit $m\to\infty$, (3.2) yield the result. □

Example 3.3. *We consider the following the double sequence of Fejer operators on* $C^{*}\left(\mathbb{R}^2\right)$

$$\sigma_{m,n}\left(f;s,t\right)=\frac{1}{(m\pi)(n\pi)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f\left(u,v\right)F_m\left(u\right)F_n\left(v\right)dudv \tag{3.4}$$

where $F_m\left(u\right)=\frac{\sin^2\frac{m(u-s)}{2}}{2\sin^2\frac{u-s}{2}}$ *and* $\frac{1}{\pi}\int_{-\pi}^{\pi}F_m\left(u\right)du=1$. *Analyze this*

$$\begin{aligned}&\sigma_{m,n}\left(f_0;s,t\right)=f_0(s,t),\ \sigma_{m,n}\left(f_1;s,t\right)=\frac{m-1}{m}f_1(s,t),\\&\sigma_{m,n}\left(f_2;s,t\right)=\frac{n-1}{n}f_2(s,t),\ \sigma_{m,n}\left(f_3;s,t\right)=\frac{m-1}{m}f_3(s,t),\\&\sigma_{m,n}\left(f_4;s,t\right)=\frac{n-1}{n}f_4(s,t).\end{aligned} \tag{3.5}$$

Let $A=C_1$ *and define a double sequence* $(u_{m,n})$ *by*

$$u_{m,n}=\begin{cases}1, & m=n=k^2\\ \frac{k}{3(k+1)}, & m=2k+1,\ n=2k-1\\ \frac{k}{2(k+1)}, & m=2k,\ n=2\left(k+1\right)\\ 0, & otherwise\end{cases},\ k\in\mathbb{N}. \tag{3.6}$$

In this case, observe that

$$st_{C_1}^{T}-\lim_{m}u_{m,n}=0. \tag{3.7}$$

Nevertheless, the sequence $(u_{m,n})$ *is not statistically convergent. Also using (3.4) and (3.6), we define the following double positive linear operators on* $C^{*}\left(\mathbb{R}^2\right)$ *as follows:*

$$L_{m,n}\left(f;s,t\right)=\left(1+u_{m,n}\right)\sigma_{m,n}\left(f;s,t\right). \tag{3.8}$$

Then, observe that the double sequence of positive linear operators $(L_{m,n})$ *defined by (3.8) satisfy all hypotheses of Theorem 3.2. Therefore, by (3.5) and (3.7), we have, for all* $f\in C^{*}\left(\mathbb{R}^2\right)$,

$$st_{A}^{T}-\lim_{m}\|L_{m,n}\left(f\right)-f\|_{C^{*}(\mathbb{R}^2)}=0.$$

Since $(u_{m,n})$ *is not statistically convergent, the Theorem 3.1 does not work for our operators* $(L_{m,n})$ *defined by (3.8).*

Example 3.4. *Fejer operators be the same in Example 3.3. Now let* $A=C\left(1,1\right)$ *and define a double sequence* $(\beta_{m,n})$ *by*

$$\beta_{m,n}=\begin{cases}\sqrt{mn}, & m=n=k^2,\\ \frac{1}{mn} & otherwise.\end{cases} \tag{3.9}$$

Obviously

$$st_{C(1,1)}^{2}-\lim_{m,n}\beta_{m,n}=0. \tag{3.10}$$

Combing (3.4) and (3.9), we define the following positive linear operators on $C\left(\mathbb{R}^2\right)$ *as follows:*

$$L_{m,n}(f;s,t)=\left(1+\beta_{m,n}\right)\sigma_{m,n}\left(f;s,t\right). \tag{3.11}$$

So, by the Theorem 3.1 and (3.10), we are seeing this

$$st_{A}^{2}-\lim_{m,n}\|L_{m,n}\left(f\right)-f\ \|_{C^{*}(\mathbb{R}^2)}=0.$$

Also, since $(\beta_{m,n})$ *is not triangular statistical convergent, here we can explain that the Korovkin theorem in triangular statistical sense does not work for operators defined by (3.11).*

4. Rate of Triangular A-Statistical Convergence

Definition 4.1 ([3]). *Let* $A=(a_{m,n})$ *be a nonnegative regular summability matrix and let* (α_m) *be a positive non-increasing sequence. A double sequence* $x=(x_{m,n})$ *is triangular A-statistically convergent to a number* $\imath$ *with the rate of* $o(\alpha_m)$ *if for every* $\varepsilon>0$,

$$\lim_{m}\frac{1}{\alpha_m}\sum_{n\in K_m(\varepsilon)}a_{m,n}=0,$$

where

$$K_m(\varepsilon):=\left\{n\in\mathbb{N}:n\leq m,\ |x_{m,n}-\imath|\geq\varepsilon\right\}.$$

We may write

$$x_{m,n}-\imath=st_{A}^{T}-o(\alpha_m)\ \ as\ \ m\to\infty.$$

Definition 4.2 ([3]). *Let $A=(a_{m,n})$ and (α_m) be the same as in Definition 4.1. Then, a double sequence $x=(x_{m,n})$ is triangular A-statistically bounded with the rate of $O(\alpha_m)$ if for every $\varepsilon>0$,*

$$\sup_m \frac{1}{\alpha_m}\sum_{n\in L_m(\varepsilon)} a_{m,n}<\infty,$$

where

$$L_m(\varepsilon):=\{\,n\in\mathbb{N}:n\le m,\ |x_{m,n}|\ge\varepsilon\,\}.$$

In this case, we write $x_{m,n}=st_A^T-O(\alpha_m)$ as $m\to\infty$.

We now use the modulus of continuity $\omega(f;\delta)$, expressed as below:

$$\omega(f;\delta):=\sup\left\{|f(u,v)-f(s,t)|:(u,v),(s,t)\in\mathbb{R}^2,\sqrt{(u-s)^2+(v-t)^2}\le\delta\right\}$$

where $f\in C^*\left(\mathbb{R}^2\right)$ and $\delta>0$. We will use the fundamental inequality to obtain our main result, for all $f\in C^*\left(\mathbb{R}^2\right)$ and for $\lambda,\delta>0$,

$$\omega(f;\lambda\delta)\le(1+[\lambda])\,\omega(f;\delta) \tag{4.1}$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ.
To obtain our main result we require the following theorem.

Theorem 4.3. *Let $(L_{m,n})$ be a double sequence of positive linear operators acting from $C^*\left(\mathbb{R}^2\right)$ into itself and let $A=(a_{m,n})$ be a nonnegative regular summability matrix, and let (α_m) and (β_m) be positive non-increasing sequences. Then, for all $f\in C^*\left(\mathbb{R}^2\right)$,*

$$\|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}=st_A^T-o(\gamma_m),\ \text{as } m\to\infty,\ \text{with } \gamma_m:=\max\{\alpha_m,\beta_m\}\ \text{for each } m\in\mathbb{N}$$

provided that the following conditions hold:
(i) $\|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}=st_A^T-o(\alpha_m)$ as $m\to\infty$, with $f_0(u,v)=1$,
(ii) $\omega(f;\delta_{m,n})=st_A^T-o(\beta_m)$ as $m\to\infty$, where $\delta_{m,n}:=\sqrt{\|L_{m,n}(\Psi)\|_{C^*(\mathbb{R}^2)}}$ with $\Psi(u,v)=\sin^2\frac{u-s}{2}+\sin^2\frac{v-t}{2}$ for each $(s,t),(u,v)\in\mathbb{R}^2$.
Also, analogue results holds when the symbol "o" is replaced by "O".

Proof. To express it, we first assume that $(s,t)\in[-\pi,\pi]\times[-\pi,\pi]$ and $f\in C^*\left(\mathbb{R}^2\right)$ be fixed, and that (i) and (ii) hold. Let $\delta>0$. Also, it is as in the the proof Theorem 9 in [26]. Using the definition of modulus of continuity and the linearity and the positivity of the operators $L_{m,n}$ for all $(m,n)\in\mathbb{N}^2$, we get

$$\begin{aligned}|L_{m,n}(f;s,t)-f(s,t)|\le&L_{m,n}(|f(u,v)-f(s,t)|;s,t)+|f(s,t)|\,|L_{m,n}(f_0;s,t)-f_0(s,t)|\\ \le&\omega(f;\delta)L_{m,n}(f_0,s,t)+\pi^2\frac{\omega(f;\delta)}{\delta^2}L_{m,n}(\Psi;s,t)+|f(s,t)|\,|L_{m,n}(f_0,s,t)-f_0(s,t)|.\end{aligned}$$

Taking supremum over (s,t) on the both-sides of the above inequality and $\delta:=\delta_{m,n}:=\sqrt{\|L_{m,n}(\Psi)\|_{C^*(\mathbb{R}^2)}}$, then we get

$$\|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}\le\omega(f;\delta)\,\|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}+\left(1+\pi^2\right)\omega(f;\delta)+M\,\|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)} \tag{4.2}$$

where the quantity $M:=\|f\|_{C^*(\mathbb{R}^2)}$ is a finite number since $f\in C^*\left(\mathbb{R}^2\right)$. Then, given $\varepsilon>0$, define the following sets:

$$\begin{aligned}D_m:&=\left\{n\in\mathbb{N}:n\le m,\ \|L_{m,n}(f)-f\|_{C^*(\mathbb{R}^2)}\ge\varepsilon\right\},\\ D_m^1:&=\left\{n\in\mathbb{N}:n\le m,\ \omega(f;\delta)\,\|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}\ge\frac{\varepsilon}{3}\right\},\\ D_m^2:&=\left\{n\in\mathbb{N}:n\le m,\ \omega(f;\delta)\ge\frac{\varepsilon}{3\left(1+\pi^2\right)}\right\},\\ D_m^3:&=\left\{n\in\mathbb{N}:n\le m,\ \|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}\ge\frac{\varepsilon}{3M}\right\}.\end{aligned}$$

Then, thanks to (4.2) that $D_m\subset D_m^1\cup D_m^2\cup D_m^3$. Also, defining

$$\begin{aligned}D_m^4:&=\left\{n\in\mathbb{N}:n\le m,\ \omega(f;\delta)\ge\sqrt{\frac{\varepsilon}{3}}\right\},\\ D_m^5:&=\left\{n\in\mathbb{N}:n\le m,\ \|L_{m,n}(f_0)-f_0\|_{C^*(\mathbb{R}^2)}\ge\sqrt{\frac{\varepsilon}{3}}\right\},\end{aligned}$$

we have $D_m^1\subset D_m^4\cup D_m^5$, which yields

$$D_m\subseteq\bigcup_{r=2}^{5}D_m^r.$$

Therefore, since $\gamma_m := \max\{\alpha_m, \beta_m\}$, we get the result for all $m \in \mathbb{N}$,

$$\frac{1}{\gamma_m}\sum_{n\in D_m} a_{m,n} \le \frac{1}{\beta_m}\sum_{n\in D_m^2} a_{m,n} + \frac{1}{\alpha_m}\sum_{n\in D_m^3} a_{m,n} + \frac{1}{\beta_m}\sum_{n\in D_m^4} a_{m,n} + \frac{1}{\alpha_m}\sum_{n\in D_m^5} a_{m,n}. \tag{4.3}$$

Letting $m \to \infty$ on both sides of (4.3), we get

$$\lim_{m\to\infty}\frac{1}{\gamma_m}\sum_{n\in D_m} a_{m,n} = 0.$$

Thus ends the proof. □

Now, having experienced from Theorem 4.3, we can introduce the ordinary rates of convergence of a sequence of positive linear operators defined on the space $C^*(\mathbb{R}^2)$. Firstly, we should point out if we choose $\alpha_m = \beta_m = 1$ for all $m \in \mathbb{N}$, then Theorem 3.2 is get from Theorem 4.3 at once. So our theorem gives us the rate of triangular A-statistical convergence in Theorem 3.2.

5. An Application to Theorem 4.3

Let $A = (a_{m,n})$ be a nonnegative regular summability matrix. Then, we consider the following operators defined by (3.8) on $C^*(\mathbb{R}^2)$:

$$L_{m,n}(f;s,t) = (1+u_{m,n})\,\sigma_{m,n}(f;s,t). \tag{5.1}$$

Then, we take $A = C_1 := (c_{m,n})$, the Cesáro matrix. Then, setting $(\alpha_m) = \left(\frac{1}{\sqrt{m}}\right)$, we get, for any $\varepsilon > 0$,

$$\frac{1}{\alpha_m}\sum_{n:|u_{i,j}|\ge\varepsilon} c_{m,n} = \sqrt{m}\sum_{n:|u_{m,n}|\ge\varepsilon}\frac{1}{m} \le \frac{2\sqrt{m}}{m} = \frac{2}{\sqrt{m}}. \tag{5.2}$$

Taking the limit as $m \to \infty$ in (5.2), we get, for any $\varepsilon > 0$,

$$\lim_m \frac{1}{\alpha_m}\sum_{n:|u_{m,n}|\ge\varepsilon} c_{m,n} = 0$$

which gives,

$$u_{m,n} = st_A^T - o\left(\frac{1}{\sqrt{m}}\right) \text{ as } m \to \infty. \tag{5.3}$$

Also, observe that

$$\begin{aligned}
L_{m,n}(f_0;s,t) &= (1+u_{m,n}),\\
L_{m,n}(f_1;s,t) &= (1+u_{m,n})\frac{m-1}{m}f_1(s,t),\\
L_{m,n}(f_2;s,t) &= (1+u_{m,n})\frac{n-1}{n}f_2(s,t),\\
L_{m,n}(f_3;s,t) &= (1+u_{m,n})\frac{m-1}{m}f_3(s,t),\\
L_{m,n}(f_4;s,t) &= (1+u_{m,n})\frac{n-1}{n}f_4(s,t),
\end{aligned}$$

where $f_0(s,t) = 1$, $f_1(s,t) = \sin s$, $f_2(s,t) = \sin t$, $f_3(s,t) = \cos s$ and $f_4(s,t) = \cos t$. Since $\|L_{m,n}(f_0) - f_0\|_{C^*(\mathbb{R}^2)} = u_{m,n}$, we obtain from (5.3)

$$\|L_{m,n}(f_0) - f_0\|_{C(\mathbb{R}^2)} = st_A^T - o(\alpha_m) \text{ as } m \to \infty. \tag{5.4}$$

Now, we calculate the quantity $L_{m,n}(\Psi;s,t)$, where $\Psi(u,v) = \sin^2\frac{u-s}{2} + \sin^2\frac{v-t}{2}$. After some calculations, we have

$$L_{m,n}(\Psi;s,t) = \frac{1+u_{m,n}}{2}\left(\frac{1}{m}+\frac{1}{n}\right).$$

So, we get $\delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|_{C^*(\mathbb{R}^2)}} = \sqrt{\frac{1+u_{m,n}}{2}\left(\frac{1}{m}+\frac{1}{n}\right)}$. In this case, setting $(\beta_m) = \left(\frac{1}{\sqrt[4]{m}}\right)$, we have, for any $\varepsilon > 0$,

$$\frac{1}{\beta_m}\sum_{n:|\delta_{m,n}|\ge\varepsilon} c_{k,l,m,n} = \sqrt[4]{m}\sum_{n:|\delta_{m,n}|\ge\varepsilon}\frac{1}{m} \le \frac{2\sqrt[4]{m}}{m} = \frac{2}{\sqrt[4]{m^3}}$$

which gives that

$$\lim_m \frac{1}{\beta_m}\sum_{n:|\delta_{m,n}|\ge\varepsilon} c_{k,l,m,n} = 0.$$

Hence, we obtain $\delta_{m,n} = st^T_{C_1} - o(\frac{1}{\sqrt[4]{m}})$ as $m \to \infty$. By the uniform continuity of f on $\mathbb{R}^2$, we can write as follows:

$$\omega(f;\delta_{m,n}) = st^T_{C_1} - o(\frac{1}{\sqrt[4]{m}}) \text{ as } m \to \infty. \tag{5.5}$$

Then, the sequence of positive linear operators $(L_{m,n})$ satisfy all hypotheses of Theorem 4.3 from (5.4) and (5.5). So, we have, for all $f \in C^*(\mathbb{R}^2)$,

$$\|L_{m,n}(f) - f\|_{C^*(\mathbb{R}^2)} = st^T_{C_1} - o(\frac{1}{\sqrt[4]{m}}) \text{ as } m \to \infty.$$

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References

[1] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241-244.
[2] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymtotique*, Colloq. Math. **2** (1951), 73-74.
[3] C. Bardaro, A. Boccuto, K. Demirci, I. Mantellini, S. Orhan, *Triangular A-statistical approximation by double sequences of positive linear operators*, Results in Mathematics, **68** (2015), 271-291.
[4] C. Bardaro, A. Boccuto, K. Demirci, I. Mantellini, S. Orhan, *Korovkin-Type Theorems for Modular Ψ-A-Statistical Convergence*, Journal of Function Spaces, **2015** (2015), 1-11.
[5] K. Demirci, F. Dirik, P. Okçu, *Approximation in Triangular Statistical Sense to B-Continuous Functions by Positive Linear Operators*, Annals of the Alexandru Ioan Cuza University-Mathematics, **63**(3) (2017).
[6] S. Çınar, *Triangular A-statistical relative uniform convergence for double sequences of positive linear operator*, Facta Universitatis. Series: Mathematics and Informatics, (2021) 065-077.
[7] S. Çınar, S. Yıldız, K. Demirci, *Korovkin type approximation via triangular A-statistical convergence on an infinite interval*, Turkish Journal of Mathematics **45**(2) (2021), 929-942.
[8] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Co., Delhi, 1960.
[9] C. Bardaro, I. Mantellini, *Korovkin's theorem in modular spaces*, Commentationes Math. **47** (2007), 239-253.
[10] K. Demirci, A. Boccuto, S. Yıldız, F. Dirik, *Relative uniform convergence of a sequence of functions at a point and Korovkin-type approximation theorems*, Positivity, **24**(1) (2020), 1-11.
[11] K. Demirci, S. Orhan, *Statistically relatively uniform convergence of positive linear operators*, Results Math., **69** (2016), 359-367.
[12] K. Demirci, S. Orhan, B. Kolay, *Relative Hemen Hemen Yakınsaklık ve Yaklaşım Teoremleri*, Sinop Üniversitesi Fen Bilimleri Dergisi, **1**(2) (2016), 114-122.
[13] K. Demirci, S. Yıldız, F. Dirik, *Approximation via power series method in two-dimensional weighted spaces*, Bulletin of the Malaysian Mathematical Sciences Society, **43**(6) (2020), 3871-3883.
[14] K. Demirci, F. Dirik, *Approximation for periodic functions via statistical σ-convergence*, Mathematical Communications, **16**(1) (2011), 77-84.
[15] K. Demirci, F. Dirik, S. Yıldız, *Approximation via equi-statistical convergence in the sense of power series method*, Revista de la Real Academia de Ciencias Exactas, **116**(2) (2022), 1-13.
[16] O. Duman, *Statistical approximation for periodic functions*, Demons. Math., **36**(4) (2003), 873-878.
[17] O. Duman, E. Erkuş, *Approximation of continuous periodic functions via statistical convergence*, Comput. Math. Appl., **52** (2006) 967-974.
[18] O. Duman, M. K. Khan, C. Orhan, *A-statistical convergence of approximating operators*, Math. Inequal. Appl., **6** (2003) 689-699.
[19] A. D. Gadjiev, C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math., **32** (2002), 129-138.
[20] M. Ünver, C. Orhan, *Statistical convergence with respect to power series methods and applications to approximation theory*, Numerical Functional Analysis and Optimization, **40**(5) (2019), 535-547.
[21] A. Pringsheim, *Zur theorie der zweifach unendlichen zahlenfolgen*, Math. Ann., **53** (1900), 289-321.
[22] H.J. Hamilton, *Transformations of multiple sequences*, Duke Math. J., **2** (1936), 29-60.
[23] G.M. Robison, *Divergent double sequences and series*, Amer. Math. Soc. Transl., **28** (1926), 50-73.
[24] F. Moricz, *Statistical convergence of multiple sequences*, Arch. Math. (Basel), **81** (2004), 82-89.
[25] G.H. Hardy, *Divergent Series*, Oxford Univ. Press, London, 1949.
[26] K. Demirci, F. Dirik, *Four-dimensional matrix transformation and rate of A-statistical convergence of periodic functions*, Math. Comput. Modelling, **52** (2010), 1858-1866.