Iğdır Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 13(1), 563-571, 2023
Journal of the Institute of Science and Technology, 13(1), 563-571, 2023

ISSN: 2146-0574, eISSN: 2536-4618

Mathematics **DOI: 10.21597/jist.1207287**

Research Article

Received: 19.11.2022 **Accepted: 19.12.2022**

To Cite: Akuzum, Y. (2023). The complex-type Narayana-Fibonacci Numbers. *Journal of the Institute of Science and Technology*, 13(1), 563-571.

The Complex-type Narayana-Fibonacci Numbers

Yeşim AKÜZÜM[1*]

Highlights:

- A new sequence called the complex-type Narayana-Fibonacci numbers is defined
- The generating matrix and Binet formula are produced for the defined numbers
- Some number theoretic properties are obtained for the defined numbers

Keywords:

- The Narayana-Fibonacci number
- matrix
- representation

ABSTRACT:

In this paper, the complex-type Narayana-Fibonacci numbers are defined. Additionally, we arrive at correlations between the complex-type Narayana-Fibonacci numbers and this generating matrix after deriving the generating matrix for these numbers. Eventually, we get their the Binet formula, the combinatorial, permanental, determinantal, exponential representations, and the sums by matrix methods are just a few examples of numerous features.

[1] Yeşim AKÜZÜM (**Orcid ID:** 0000-0001-7168-8429), Kafkas University, Faculty of Arts and Sciences, Department of Mathematics, Kars, Turkey

***Corresponding Author:** Yeşim AKÜZÜM, e-mail: yesim_036@hotmail.com

INTRODUCTION

As it is known, recurrence sequences are frequently encountered at the point of interdisciplinary relations. As an example of such studies, we can give the scientific outputs in (Mandelbaum, 1972; Adams and Shanks, 1982; Kirchoof and Rutishauser, 1990; Stein, 1993; Becker, 1994; Fraenkel and Klein, 1996; Spinadel, 2002; El Naschie, 2005). Algebraically reducing sequences, many of its features such as the generating matrix, generating function, Binet formula, exponential, permanental, and combinatorial representations have been studied and continue to be studied by many scientists. We can give (Shannon and Bernstein, 1973; Shannon and Horadam, 1994; Lee, 2000; Stakhov and Rozin, 2006; Gogin and Myllari, 2007; Ozkan, 2007; Yılmaz and Bozkurt, 2009; Tasci and Firengiz, 2010; Tuglu et al., 2011; Akuzum and Deveci, 2021; Halici and Deveci, 2021; Erdag et al., 2022) studies as an example of the current ones among these studies. In many of these studies, matrices corresponding to reduced sequences have been used to obtain various results. The authors constructed new sequences utilizing quaternions and complex numbers, particularly the complex-type k-Fibonacci numbers were defined by Deveci and Shannon in (Deveci and Shannon, 2021), after which they provided a variety of attributes and numerous applications for the sequences they had developed. The complex-type Narayana-Fibonacci numbers are defined in this paper. Then, we arrive at the Binet formula for the complex-type Narayana-Fibonacci numbers utilizing the roots of characteristic polynomials of these numbers. Additionally, using matrix methods, we derive their different features, including the combinatorial, permanental, determinantal, exponential representations, and sums.

MATERIALS AND METHODS

The homogeneous linear recurrence relation given below for $k \geq 0$ defines the Narayana-Fibonacci sequence (Akuzum and Deveci, 2022).

$$n_{k+5}^{f} = 2n_{k+4}^{f} - n_{k+1}^{f} - n_{k}^{f}$$

in which $n_0^f = n_1^f = n_2^f = n_3^f = 0$ and $n_4^f = 1$.

The complex Fibonacci sequence $\{F_n^*\}$ is given (Horadam, 1961) with the subsequent equation: for $n \geq 0$

$$F_n^* = F_n + iF_{n+1}$$

Such that $\sqrt{-1} = i$ and the n^{th} Fibonacci number is designated as F_n (Horadam, 1963; Berzsenyi, 1975).

As it is known, all of the reduction sequences defined by their unique reduction relations and initial values are special cases of the reduction sequences defined by the relation:

$$\aleph_{n+k} = \gamma_0 \aleph_n + \gamma_1 \aleph_{n+1} + \cdots + \gamma_{k-1} \aleph_{n+k-1}$$

Such that real constants $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ are present.

In this sense, along with the relation of the Narayana-Fibonacci sequence, new reduction sequences will be defined with the help of appropriate reduction relations and initial values, taking into account their structural properties.

Think about the given above sequence $\{\aleph_n\}$. By using the companion matrix method, Kalman constructed the following closed-form formulas in (Kalman, 1982):

Suppose that M is defined as

$$M=\left[M_{i,j}\right]_{k\times k}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ 0 & 0 & 0 & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ \gamma_0 & \gamma_1 & \gamma_2 & & \gamma_{k-2} & \gamma_{k-1}\end{bmatrix},$$

Then

$$M^n\begin{bmatrix}\aleph_0\\ \aleph_1\\ \vdots\\ \aleph_{k-1}\end{bmatrix}=\begin{bmatrix}\aleph_n\\ \aleph_{n+1}\\ \vdots\\ \aleph_{n+k-1}\end{bmatrix}$$

for $n \geqslant 0$.

RESULTS AND DISCUSSION

We next define the complex-type Narayana-Fibonacci numbers by integer constants $n_0^{f,*}=\cdots=n_3^{f,*}=0$ and $n_4^{f,*}=1$ and the recurrence relation:

$$n_{k+5}^{f,*}=2i.n_{k+4}^{f,*}-n_{k+1}^{f,*}-i.n_k^{f,*} \tag{1}$$

for $k \geq 0$.

Using relation (1), the generating matrix for the complex-type Narayana-Fibonacci numbers can be written as below:

$$N^c=\begin{bmatrix}2i & 0 & 0 & -1 & -i\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{bmatrix}_{(5)\times(5)}.$$

The companion matrix N^c is called a complex-type Narayana-Fibonacci matrix.

An inductive argument can be used to prove that for $\alpha \geq 4$,

$$(N^c)^\alpha=\begin{bmatrix}n_{\alpha+4}^{f,*} & -n_{\alpha+1}^{f,*}-i.n_{\alpha}^{f,*} & -n_{\alpha+2}^{f,*}-i.n_{\alpha+1}^{f,*} & -n_{\alpha+3}^{f,*}-i.n_{\alpha+2}^{f,*} & -i.n_{\alpha+3}^{f,*}\\ n_{\alpha+3}^{f,*} & -n_{\alpha}^{f,*}-i.n_{\alpha-1}^{f,*} & -n_{\alpha+1}^{f,*}-i.n_{\alpha}^{f,*} & -n_{\alpha+2}^{f,*}-i.n_{\alpha+1}^{f,*} & -i.n_{\alpha+2}^{f,*}\\ n_{\alpha+2}^{f,*} & -n_{\alpha-1}^{f,*}-i.n_{\alpha-2}^{f,*} & -n_{\alpha}^{f,*}-i.n_{\alpha-1}^{f,*} & -n_{\alpha+1}^{f,*}-i.n_{\alpha}^{f,*} & -i.n_{\alpha+1}^{f,*}\\ n_{\alpha+1}^{f,*} & n_{\alpha-2}^{f,*}-i.n_{\alpha-3}^{f,*} & -n_{\alpha-1}^{f,*}-i.n_{\alpha-2}^{f,*} & -n_{\alpha}^{f,*}-i.n_{\alpha-1}^{f,*} & -i.n_{\alpha}^{f,*}\\ n_{\alpha}^{f,*} & -n_{\alpha-3}^{f,*}-i.n_{\alpha-4}^{f,*} & n_{\alpha-2}^{f,*}-i.n_{\alpha-3}^{f,*} & -n_{\alpha-1}^{f,*}-i.n_{\alpha-2}^{f,*} & -i.n_{\alpha-1}^{f,*}\end{bmatrix}.$$

It is significant to remember that $\det N^c=-i$.

It is clear that each of the eigenvalues of the matrix N^c are distinct. Let $\{\eta_1,\eta_2,\eta_3,\eta_4,\eta_5\}$ be the sets of the eigenvalues of the matrix N^c and let W^c be 5×5 Vandermonde matrices as below: $W^c=$

$$\begin{bmatrix}(\eta_1)^4 & (\eta_2)^4 & (\eta_3)^4 & (\eta_4)^4 & (\eta_5)^4\\ (\eta_1)^3 & (\eta_2)^3 & (\eta_3)^3 & (\eta_4)^3 & (\eta_5)^3\\ (\eta_1)^2 & (\eta_2)^2 & (\eta_3)^2 & (\eta_4)^2 & (\eta_5)^2\\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5\\ 1 & 1 & 1 & 1 & 1\end{bmatrix}.$$

Assume that $W_{i,j}^c$ is a 5×5 matrix created from the Vandermonde matrix W^c by exchanging the jth column of W^c by U^c, where, U^c is a 5×1 matrix as below:

$$U^c = \begin{bmatrix} (\eta_1)^{\alpha+5-l} \\ (\eta_2)^{\alpha+5-l} \\ (\eta_3)^{\alpha+5-l} \\ (\eta_4)^{\alpha+5-l} \\ (\eta_5)^{\alpha+5-l} \end{bmatrix}.$$

Theorem 1. *For* $\alpha \geq 4$,

$$n_{l,j}^{c,\alpha} = \frac{\det W_{l,j}^c}{\det W^c},$$

where $(N^c)^\alpha = [n_{l,j}^{c,\alpha}]$.

Proof. The matrix N^c may be diagonalized since $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ are distinct. Let $K_5 = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$, then we easily see that $N^c W^c = W^c K_5$. Since $\det W^c \neq 0$, the matrix W^c is invertible. Thus we may write $(W^c)^{-1} N^c W^c = K_5$. Consequently, the matrix N^c is similar to K_5; and we find $(N^c)^\alpha W^c = W^c (K_5)^\alpha$ for $\alpha \geq 4$. The resulting linear system of equations is as follows:

$$\begin{cases} n_{l,1}^{c,\alpha}(\eta_1)^4 + n_{l,2}^{c,\alpha}(\eta_1)^3 + \cdots + n_{l,5}^{c,\alpha} = (\eta_1)^{\alpha+5-l} \\ n_{l,1}^{c,\alpha}(\eta_2)^4 + n_{l,2}^{c,\alpha}(\eta_2)^3 + \cdots + n_{l,5}^{c,\alpha} = (\eta_2)^{\alpha+5-l} \\ \vdots \\ n_{l,1}^{c,\alpha}(\eta_5)^4 + n_{l,2}^{c,\alpha}(\eta_5)^3 + \cdots + n_{l,5}^{c,\alpha} = (\eta_5)^{\alpha+5-l} \end{cases}$$

As a result, for each $l, j = 1,2, \ldots ,5$, we get $n_{l,j}^{c,\alpha}$ as below

$$n_{l,j}^{c,\alpha} = \frac{\det W_{l,j}^c}{\det W^c}.$$

Corollary 1. Let $n_\alpha^{f,*}$ be the αth the complex-type Narayana-Fibonacci number for $\alpha \geq 4$. Then

$$n_\alpha^{f,*} = \frac{\det W_{5,1}^c}{\det W^c}$$

and

$$n_\alpha^{f,*} = -\frac{\det W_{4,5}^c}{i.\det W^c}.$$

Assume that $C(c_1, c_2, \ldots, c_v)$ is a $v \times v$ companion matrix as below:

$$C(c_1, c_2, \ldots, c_v) = \begin{bmatrix} c_1 & c_2 & \cdots & c_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2. (Chen and Louck, 1996). The following formula can be used to determine the (l, j) entry $c_{l,j}^{(\alpha)}(c_1, c_2, \ldots, c_v)$ in the matrix $C^\alpha(c_1, c_2, \ldots, c_v)$:

$$c_{l,j}^{(\alpha)}(c_1, c_2, \ldots, c_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} c_1^{t_1} \cdots c_v^{t_v} \quad (2)$$

where $\binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}$ represents a multinomial coefficient, the coefficients in equation (2) are defined as being 1 if $\alpha = l - j$, and the summing is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = \alpha - l + j$.

In this section, we look into a combinatorial representation of the complex-type Narayana-Fibonacci numbers.

Corollary 2. Suppose that $n_\alpha^{f,*}$ is the αth the complex-type Narayana-Fibonacci number. Then

i.

$$n_\alpha^{f,*} = \sum_{(t_1,t_2\ldots,t_5)} \binom{t_1+\cdots+t_5}{t_1,\ldots,t_5} (2i)^{t_1}(-1)^{t_4}(-i)^{t_5}$$

where $t_1+2t_2+\cdots+5t_5 = \alpha-4$ is the sum of non-negative numbers.

ii.

$$n_\alpha^{f,*} = -\frac{1}{i}\sum_{(t_1,t_2\ldots,t_5)} \frac{t_5}{t_1+t_2+\cdots+t_5} \times \binom{t_1+\cdots+t_5}{t_1,\ldots,t_5} (2i)^{t_1}(-1)^{t_4}(-i)^{t_5}$$

where $t_1+2t_2+\cdots+5t_5 = \alpha+1$ is the sum of non-negative numbers.

Proof. In Theorem 2, if we chose $l=5$, $j=1$, $c_1=2i$, $c_4=-1$ and $c_5=-i$ for the case i., $l=4$, $j=5$, $c_1=2i$, $c_4=-1$ and $c_5=-i$ for the case ii., so we can immediately view the outcomes from $(N^c)^\alpha$.

We are now focusing on the permanental representations of the complex-type Narayana-Fibonacci numbers.

Definition 1. If the k^{th} column (resp. row) of a $u\times v$ real matrix $A=[a_{i,j}]$ includes precisely two non-zero entries, the k^{th} column (resp. row) is said to be a contractible matrix.

According to Brualdi and Gibson's findings in citation (Brualdi and Gibson, 1977), if A is a real matrix of order $\alpha>1$ and B is a contraction of A, $per(A)=per(B)$.

Let $F_r=\left[f_{l,j}^{(r)}\right]$ be the $r\times r$ super-diagonal matrix, defined by

$$F_r = \begin{array}{c} \begin{array}{cccccccccc} &&&& (5)\text{th} &&&&& \\ &&&& \downarrow &&&&& \end{array} \\ \begin{bmatrix} 2i & 0 & 0 & -1 & -i & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2i & 0 & 0 & -1 & -i & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2i & 0 & 0 & -1 & -i & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2i & 0 & 0 & -1 & -i & 0 \\ 0 & 0 & \cdots & 0 & 1 & 2i & 0 & 0 & -1 & -i \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2i & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2i \end{bmatrix} \end{array}.$$

For the complex-type Narayana-Fibonacci numbers, we can then provide a permanental representation.

Theorem 3. *For* $r \geqslant 5$,

$$perF_r = n_{r+4}^{f,*}.$$

Proof. Assuming that the equation is valid for $r\geqslant 5$, we now demonstrate that it is also valid for $r+1$. When we expand the $perF_r$ by the Laplace expansion of the permanent with regard to the first row, we reach

$$perF_{r+1} = 2i.perF_r - perF_{r-3} - i.perF_{r-4}.$$

Since $perF_r = n_{r+4}^{f,*}$, $perF_{r-3} = n_{r+1}^{f,*}$ and $perF_{r-4} = n_r^{f,*}$, from definition of the complex-type Narayana-Fibonacci number $n_k^{f,*}$ It is obvious that

$$perF_{r+1} = n_{r+5}^{f,*}.$$

Thus, the evidence is conclusive.

Define the $r\times r$ matrix $G_r = \left[g_{l,j}^{(r)}\right]$ as shown:

$$g_{l,j}^{(r)} = \begin{cases} 2i & \text{if } l = j = \lambda \quad \text{for } 1 \leq \lambda \leq r, \\ -2i & \text{if } l = j = r-3, \\ -i & \text{if } l = \lambda + 4 \text{ and } j = \lambda \quad \text{for } 1 \leq \lambda \leq r-4, \\ -1 & \text{if } l = \lambda + 3 \text{ and } j = \lambda \quad \text{for } 1 \leq \lambda \leq r, \\ 1 & \text{if } l = \lambda - 1 \text{ and } j = \lambda \quad \text{for } 2 \leq \lambda \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that the $r \times r$ matrix $P_r = \left[p_{l,j}^{(r)}\right]$ is indicated by

$$P_r = \begin{array}{c} \begin{array}{ccccccc} & & (r-4)\text{th} & & & & \\ & & \downarrow & & & & \end{array} \\ \left[\begin{array}{c|cccccc} 1 & \cdots & 1 & 0 & & \cdots & 0 \\ \hline 1 & & & & & & \\ 0 & & & & & & \\ \vdots & & & G_{r-1} & & & \\ 0 & & & & & & \\ 0 & & & & & & \end{array}\right] \end{array},$$

where $r > 5$. The next intriguing findings are as follows.

Theorem 4. Let $n_r^{f,*}$ be the rth the complex-type Narayana-Fibonacci number. Then

(i). For $r \geqslant 5$,

$perG_r = n_r^{f,*}$.

(ii). For $r > 5$,

$perP_r = \sum_{\beta=0}^{r-1} n_\beta^{f,*}$.

Proof. The induction method will be used to r.

(i). Suppose that $perG_r = n_r^{f,*}$ for $r \geqslant 5$. We examine the case $r+1$. When the matrix G_r is defined by expanding the $perG_r$ by the permanent Laplace expansion with regard to the first row, we are left with

$$perG_{r+1} = 2i.perG_r - perG_{r-3} - i.perG_{r-4} = 2i.n_r^{f,*} - n_{r-3}^{f,*} - i.n_{r-4}^{f,*}.$$

So the result holds.

(ii). If we expand the $perP_r$ by the permanent's Laplace expansion with regard to the first row, we get

$$perP_r = perP_{r-1} + perG_{r-1}.$$

The proof is evident by the conclusion of part (i) of Theorem 4.

If there is an $k \times k$ $(1, -1)$-matrix Q such that $perD = \det(D \circ Q)$, where $D \circ Q$ stands for the Hadamard product of D and Q, then the matrix D is said to be convertible.

Let $r > 5$, and let L be the $r \times r$ matrix, defined by

$$L = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 3. For $r > 5$,

$\det(F_r \circ L) = n_{r+4}^{f,*}$,

$\det(G_r \circ L) = n_r^{f,*}$

and

$$\det(P_r \circ L) = \sum_{\beta=0}^{r-1} n_\beta^{f,*}.$$

Proof. Since $perF_r = \det(F_r \circ L) = n_{r+4}^{f,*}$, $perG_r = \det(G_r \circ L) = n_r^{f,*}$ and $perP_r = \det(P_r \circ L) = \sum_{\beta=0}^{r-1} n_\beta^{f,*}$, the conclusions are clear from Theorems 3 and 4.

We can observe that the generating function of complex-type Narayana-Fibonacci numbers $n_k^{f,*}$ is

$$g_n^*(x) = \frac{x^4}{1-2i.x+x^4+i.x^5}.$$

Theorem 5. The exponential representation for the complex-type Narayana-Fibonacci numbers is as follows:

$$g_n^*(x) = x^4 \exp\left(\sum_{n=1}^{\infty} \frac{x^n}{n}(2i-x^3-x^4)^n\right).$$

Proof. It is obvious that

$$\ln\frac{g_n^*(x)}{x^4} = -\ln(1-2i.x+x^4+i.x^5).$$

By using the function $\ln x$, we can derive the relationship:

$$-\ln(1-2i.x+x^4+i.x^5) = -[-x(2i-x^3-x^4) - \frac{1}{2}x^2(2i-x^3-x^4)^2 - \cdots - \frac{1}{n}x^n(2i-x^3-x^4)^{\mathrm{n}}].$$

Therefore, we obtain

$$\ln\frac{g_n^*(x)}{x^4} = \exp\left(\sum_{n=1}^{\infty} \frac{x^n}{n}(2i-x^3-x^4)^n\right).$$

Thus, we have reached conclusion.

The sums of complex-type Narayana-Fibonacci numbers are now being considered.

Let

$$S_r = \sum_{k=0}^{r} n_k^{f,*}$$

for $r \geq 1$, let M_r be the $(6) \times (6)$ matrix as follows:

$$M_r = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & N^c & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Then it can be shown by induction that

$$(M_r)^\alpha = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{r+4} & & & \\ S_{r+3} & & (N^c)^\alpha & \\ \vdots & & & \\ S_{r-1} & & & \end{bmatrix}.$$

CONCLUSION

We defined the complex-type Narayana-Fibonacci numbers in this study and found their generating matrix. Then, for the complex-type Narayana-Fibonacci numbers, we came up with the Binet formula. Additionally, we obtained their numerous qualities, including their sums, combinatorial representation, permanental representation, determinantal representation, and exponential representation.

Conflict of Interest

The author declares that there is no conflict of interest.

Author's Contributions

I hereby declare that the planning, execution, and writing of the article were done by me as the sole author of the article.

REFERENCES

Adams, W. and Shanks, D. (1982). Strong Primality Tests that are not sufficient. *Mathematics of Computation*, 36(159), 255-300.

Akuzum, Y. and Deveci, O. (2021). The arrowhead-Pell sequences. *Ars Combinatoria*, 155, 231-246.

Akuzum, Y. and Deveci, O. (2022). *The Narayana-Fibonacci sequence and its Binet formulas*. 6th International Congress on Life, Social, and Health Sciences in a Changing World, İstanbul, July 02-03, pp:303-307.

Becker, P.G. (1994). *k*-Regular power series and Mahler-type functional equations. *Journal of Number Theory*, 49(3), 269-286.

Berzsenyi, G. (1975). Sums of products of generalized Fibonacci numbers. *The Fibonacci Quarterly*, 13(4), 43-344.

Brualdi, R.A. and Gibson, P.M. (1977). Convex polyhedra of doubly stochastic matrices I: applications of permanent function. *Journal of Combinatorial Theory, Series A*, 22 (2), 194-230.

Chen, W.Y.C. and Louck, J.D. (1996). The combinatorial power of the companion matrix. *Linear Algebra and its Applications*, 232, 261-278.

Deveci, O. and Shannon, A.G. (2021). The complex-type *k*-Fibonacci sequences and their applications. *Communications in Algebra*, 49(3), 1352-1367.

El Naschie, M.S. (2005). Deriving the Essential features of standard model from the general theory of relativity. *Chaos, Solitons & Fractals*, 26, 1-6.

Erdag, O., Halici, S. and Deveci, O. (2022). The complex-type Padovan-*p* sequences. *Mathematica Moravica*, 26, 77-88.

Fraenkel, A.S. and Klein, S.T. (1996). Robutst Universal complete codes for transmission and compression. *Discrete Applied Mathematics*, 64, 31-55.

Gogin, N. and Myllari, A.A. (2007). The Fibonacci-Padovan sequence and MacWilliams transform matrices. *Programming and Computer Software*, 33, 74-79.

Halici. S. and Deveci, O. (2021). On fibonacci quaternion matrix. *Notes on Number Theory and Discrete Mathematics*, 27(4), 236-244.

Horadam, A.F. (1961). A generalized Fibonacci sequence. *The American Mathematical Monthly*, 68(5), 455-459.

Horadam, A.F. (1963). Complex Fibonacci numbers and Fibonacci quaternions. *The American Mathematical Monthly*, 70(3), 289-291.

Kalman, D. (1982). Generalized Fibonacci numbers by matrix methods. *The Fibonacci Quarterly*, 20, 73-76.

Kirchoof, B.K. and Rutishauser, R. (1990). The Phyllotaxy of Costus (Costaceae). *Botanical Gazette*, 51(1), 88-105.

Lee, G-Y. (2000). *k*-Lucas numbers and associated bipartite graphs. *Linear Algebra and its Applications*, 320, 51-61.

Mandelbaum, D.M. (1972). Synchronization of codes by means of Kautz's Fibonacci encoding. *IEEE Transactions on Information Theory*, 18(2), 281-285.

Ozkan, E. (2007). On truncated Fibonacci sequences. *Indian Journal of Pure and Applied Mathematics*, 38(4), 241-251.

Shannon, A.G. and Bernstein, L. (1973). The Jacobi-Perron algorithm and the algebra of recursive sequences. *Bulletin of the Australian Mathematical Society*, 8, 261-277.

Shannon, A.G. and Horadam, A.F. (1994). Arrowhead curves in a tree of Pythagorean triples. *Journal of Mathematical Education in Science and Technology*, 25, 255-261.

Spinadel, V.W. (2002). The Metallic Means Family and Forbidden Symmetries. *International Journal of Mathematics*, 2(3), 279-288.

Stakhov, A. and Rozin, B. (2006). Theory of Binet formulas for Fibonacci and Lucas *p*-numbers. *Chaos, Solitons & Fractals*, 27, 1162-1177.

Stein, W. (1993). Modelling The Evolution of Stelar Architecture in Vascular Plants. *International Journal of Plant Sciences*, 154(2), 229-263.

Tasci, D. and Firengiz, M.C. (2010). Incomplete Fibonacci and Lucas *p*-numbers. *Mathematical and Computer Modelling*, 52(9-10), 1763-1770.

Tuglu, N., Kocer, E.G. and Stakhov, A. (2011). Bivariate Fibonacci like *p*-polynomials. *Applied Mathematics and Computation*, 217, 10239-10246.

Yılmaz, F. and Bozkurt, D. (2009). The generalized order-*k* Jacobsthal numbers. *International Journal of Contemporary Mathematical Sciences*, 4, 1685-1694.