Universal Journal of Mathematics and Applications, 5 (4) (2022) 145-155
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1207852

Binomial Transform for Quadra Fibona-Pell Sequence and Quadra Fibona-Pell Quaternion

Arzu Özkoç Öztürk[1*] and Eda Gündüz[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Türkiye*
[*]*Corresponding author*

Article Info

Keywords: *Binomial transform, Fibonacci sequence, Pell sequence, Quaternion.*
2010 AMS: *11B37, 11B65, 11B83, 11R52.*
Received: *21 November 2022*
Accepted: *26 December 2022*
Available online: *29 December 2022*

Abstract

The main object of the study is to consider the binomial transform for quadra Fibona-Pell sequence and quadra Fibona-Pell quaternion. In the paper, which consists of two parts in terms of the results found, the first step was taken for the sequence by defining the binomial transform for the quadra Fibona-Pell sequence in the first part and then finding the recurrence relation of this new binomial transform. Then, the Binet formula, generating function and various sum formulas of the sequence were found. In the second part, the binomial transform is applied for the quadra Fibona-Pell quaternion, which was discussed in a thesis before. Similar results in the first section are covered in the quaternion binomial transform.

1. Introduction

Number sequences such as Fibonacci, Pell, quadra Fibona-Pell, and their generalized versions are widely used in the literature. Fibonacci numbers form a sequence defined by the following recurrence relation: $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. The first Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots$. The characteristic equation of F_n is $x^2 - x - 1 = 0$ and hence the roots of it are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.It has become known as Binet's formula $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for $n \geq 0$. The Pell numbers are defined by the recurrence relation $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. The first few terms of the sequence are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378 \cdots$. See [1] for detailed information on Fibonacci and Pell number sequences.
In [2], the author examined sequence with fourth-order recurrence relation and discussed a new sequence of fourth-order formed by the roots of the characteristic equation of both Fibonacci and Pell number sequences. Later, different authors worked with similar integer sequences with the same logic, see [3,4]. In addition, we can come across many studies on integer sequences with fourth-order recurrence relation in the literature, see [5]. Quadra Fibona-Pell sequence as follows in [2]:

$$W_n = 3W_{n-1} - 3W_{n-3} + W_{n-4} \tag{1.1}$$

for $n \geq 4$, with initial values $W_0 = W_1 = 0, W_2 = 1, W_3 = 3$ and W_n is $n-$th quadra Fibona-Pell sequence. Note that here, the roots of the characteristic equation of W_n are the roots of the characteristic equations of both Fibonacci and Pell sequences, so $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ (α, β are the roots of the characteristic equation of Fibonacci numbers and γ, δ are the roots of the characteristic equation of Pell numbers). The Binet formula for the quadra Fibona-Pell sequence is given by

$$W_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} - \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for $n \geq 0$. Besides that generating function for the quadra Fibona-Pell sequence is

$$W(x) = \frac{x^2}{x^4 + 3x^3 - 3x + 1}.$$

Email addresses and ORCID numbers: **arzuozkoc@edu.tr, 0000-0002-2196-3725 (A. Özkoç Öztürk), eda.gunduz.97@gmail.com, 0000-0003-1985-6506 (E. Gündüz)**

Normed division algebra, nowadays which is so important topic consists of the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbf{H}$. Quaternions are non-commutative normed division algebra over the real numbers, even it looks like things are going to be done with quaternions. For $a_0, a_{1,} a_{2,} a_3 \in \mathbb{R}$, a quaternion is defined by

$$e = a_0 + a_1 i + a_2 j + a_3 k$$

where i, j and k are unit vectors which verifies the following rules

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.2}$$

From equation (1.2), we get

$$ij = -ji = k, \quad jk = -kj = i, \quad jk = -kj = i.$$

You can find detailed information about quaternions from [6–8].

2. Binomial Transform of Quadra Fibona-Pell Sequence

It is possible to find different articles in the literature on binomial transforms of sequences as [9]. Actually, one of these was studied by Chen [10] and later was studied by Falcon in [11]. In [12], given a sequence $A = \{a_1, a_2, \cdots\}$, its binomial transform B is the sequence $B(A) = \{b_n\}$ defined as follows:

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i. \tag{2.1}$$

Also, detailed information about binomial transform can be found in [13, 14]. Some authors considered special binomial sequences which are based on fourth-order recurrence relations, for example binomial transform of quadrapell sequences in [15].
In this part of the study, with similar logic, we apply the binomial transform of quadra Fibona-Pell sequence. When the binomial transform of the quadra Fibona-Pell sequence, which has a fourth-order recurrence relation, is made, some additional identities, especially the generating function, Binet formula and sum formulas, will be found for the new sequence obtained.

Definition 2.1. *Let W_n be the $n-$th Quadra Fibona-Pell sequence. Then the binomial transform of quadra Fibona-Pell sequence is*

$$b_n = \sum_{i=0}^{n} \binom{n}{i} W_i. \tag{2.2}$$

Lemma 2.2. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. Then*

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (W_i + W_i).$$

Proof. By the help of (2.2), we get

$$b_{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} W_i = \sum_{i=1}^{n+1} \binom{n+1}{i} W_i + W_0.$$

Also $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$ and $\binom{n}{n+1} = 0$, we get

$$b_{n+1} = \sum_{i=1}^{n+1} \left[\binom{n}{i-1} + \binom{n}{i} \right] W_i + W_0.$$

Thus we obtain that

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (W_i + W_{i+1}).$$

This completes the proof. □

From Lemma 2.2, we can give the following result for the binomial transform of quadra Fibona-Pell sequence.

Corollary 2.3. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. Then*

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} W_{i+1}.$$

The recurrence relation of the binomial transforms of quadra Fibona-Pell sequence is obtained below.

Theorem 2.4. *Let b_n is the binomial transform of quadra Fibona-Pell sequence. b_n states the following recurrence relation*

$$b_{n+3} = 7b_{n+2} - 15b_{n+1} + 10b_n - 2b_{n-1} \tag{2.3}$$

for $n \geq 4$, where $b_0 = b_1 = 0, b_2 = 1$ and $b_3 = 6$.

Proof. Using Lemma 2.2 we get,

$$b_{n+3} = K_1 b_{n+2} + L_1 b_{n+1} + M_1 b_n + N_1 b_{n-1}.$$

Then, if we solve the system of equations

$$\begin{aligned} n=1 &\Rightarrow b_4 = K_1 b_3 + L_1 b_2 + M_1 b_1 + N_1 b_0, \\ n=2 &\Rightarrow b_5 = K_1 b_4 + L_1 b_3 + M_1 b_2 + N_1 b_1, \\ n=3 &\Rightarrow b_6 = K_1 b_5 + L_1 b_4 + M_1 b_3 + N_1 b_2, \\ n=4 &\Rightarrow b_7 = K_1 b_6 + L_1 b_5 + M_1 b_4 + N_1 b_3, \end{aligned}$$

by considering Definition 2.1, we deduce

$$K_1 = 7, L_1 = -15, M_1 = 10, N_1 = -2.$$

which is completed the proof. □

The generating function of the new binomial transform is found below.

Theorem 2.5. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. The generating function of the related binomial transform is*

$$b(x) = \frac{x^2 - x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4},$$

where $b_0 = b_1 = 0, b_2 = 1$ and $b_3 = 6$.

Proof. Assume that

$$b(x) = \sum_{i=0}^{\infty} b_i x^i$$

is the generating function of the binomial transform for W_n. Then

$$\begin{aligned} b(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ 7xb(x) &= 7b_0 x + 7b_1 x^2 + 7b_2 x^3 + 7b_3 x^4 + \cdots \\ 15x^2 b(x) &= 15b_0 x^2 + 15b_1 x^3 + 15b_2 x^4 + 15b_3 x^5 + \cdots \\ 10x^3 b(x) &= 10b_0 x^3 + 10b_1 x^4 + 10b_2 x^5 + 10b_3 x^6 + \cdots \\ 2x^4 b(x) &= 2b_0 x^4 + 2b_1 x^5 + 2b_2 x^6 + 2b_3 x^7 + \cdots. \end{aligned}$$

Since, from equation (2.3), we obtain

$$(1 - 7x + 15x^2 - 10x^3 + 2x^4) b(x) = x^2 - x^3$$

and hence, the generating function for the binomial transform of the b_n is

$$b(x) = \frac{x^2 - x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4}.$$

□

Another formula that is essential to find other results of the binomial transform is the Binet formula, which is provided below.

Theorem 2.6. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. The Binet formula for b_n is*

$$b_n = \frac{(\gamma+1)^n - (\delta+1)^n}{\gamma - \delta} - \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \tag{2.4}$$

for $n \geqslant 0$.

Proof. Note that the generating function is $W(x) = \frac{x^2 - x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4}$. It is easily seen that $1 - 7x + 15x^2 - 10x^3 + 2x^4 = (2x^2 - 4x + 1)(x^2 - 3x + 1)$. So we can rewrite $W(x)$ as

$$\begin{aligned} \frac{x^2 - x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4} &= \frac{x}{2x^2 - 4x + 1} - \frac{x}{x^2 - 3x + 1} \\ &= \sum_{n=0}^{\infty} \frac{(\gamma+1)^n - (\delta+1)^n}{\gamma - \delta} - \sum_{n=0}^{\infty} \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \\ &= \sum_{n=0}^{\infty} \left(\frac{(\gamma+1)^n - (\delta+1)^n}{\gamma - \delta} - \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \right) \end{aligned}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$. Using roots in quadra Fibona-Pell sequence, we get

$$b_n = \frac{(\gamma+1)^n - (\delta+1)^n}{\gamma - \delta} - \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}$$

the result. □

In the most general case, the following result was found for the series expansions.

Theorem 2.7. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. Then*

$$\sum_{n=0}^{\infty} b_{mn+s}x^n = b_s\left(\frac{1}{A_1}+\frac{1}{B_1}\right) - b_{s-m}\left(\frac{2^m}{A_1}+\frac{1}{B_1}\right)x + C_1$$

for all $n\in\mathbb{N}$ and $m,s\in\mathbb{N}$, $s>m$,

$$\begin{aligned}
A_1 =&(1-(\gamma+1)^m x)(1-(\delta+1)^m x),\\
B_1 =&(1-\alpha^{2m}x)(1-\beta^{2m}x),\\
C_1 =&\frac{1}{A_1}\left(\frac{\gamma^{2s}-\delta^{2s}}{\gamma-\delta}\right)-\frac{1}{B_1}\left(\frac{(\gamma+1)^s-(\delta+1)^s}{\gamma-\delta}\right)-\frac{2^m}{A_1}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{\alpha-\beta}\right)x+\frac{1}{B_1}\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{\gamma-\delta}\right)x.
\end{aligned}$$

Proof. Again from equation (2.4), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{mn+s}x^n =& \sum_{n=0}^{\infty}\left(\frac{(\gamma+1)^{mn+s}-(\delta+1)^{mn+s}}{\gamma-\delta}-\frac{\alpha^{2(mn+s)}-\beta^{2(mn+s)}}{\alpha-\beta}\right)x^n\\
=& \frac{(\gamma+1)^s}{\gamma-\delta}\sum_{n=0}^{\infty}((\gamma+1)^m x)^n-\frac{(\delta+1)^s}{\gamma-\delta}\sum_{n=0}^{\infty}((\delta+1)^m x)^n-\frac{\alpha^{2s}}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^{2m}x)^n+\frac{\beta^{2s}}{\alpha-\beta}\sum_{n=0}^{\infty}(\beta^{2m}x)^n
\end{aligned}$$

with the help of sum formula, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{mn+s}x^n =&\frac{(\gamma+1)^s}{\gamma-\delta}\left(\frac{1}{1-(\gamma+1)^m x}\right)-\frac{(\delta+1)^s}{\gamma-\delta}\left(\frac{1}{1-(\delta+1)^m x}\right)-\frac{\alpha^{2s}}{\alpha-\beta}\left(\frac{1}{1-\alpha^{2m}x}\right)+\frac{\beta^{2s}}{\alpha-\beta}\left(\frac{1}{1-\beta^{2m}x}\right)\\
=&\frac{\frac{(\gamma+1)^s-(\delta+1)^s}{\gamma-\delta}-((\gamma+1)(\delta+1))^m\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{\gamma-\delta}\right)x}{1-((\gamma+1)^m+(\delta+1)^m)x+((\gamma+1)(\delta+1))^m x^2}-\frac{\frac{\alpha^{2s}-\beta^{2s}}{\alpha-\beta}-(\alpha\beta)^{2m}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{\alpha-\beta}\right)x}{1-(\alpha^{2m}+\beta^{2m})x+(\alpha\beta)^{2m}x^2}\\
=&\frac{1}{A_1}\left(\frac{(\gamma+1)^s-(\delta+1)^s}{\gamma-\delta}\right)-\frac{((\gamma+1)(\delta+1))^m}{A_1}\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{\gamma-\delta}\right)x\\
&-\frac{1}{B_1}\left(\frac{\alpha^{2s}-\beta^{2s}}{\alpha-\beta}\right)+\frac{(\alpha\beta)^{2m}}{B_1}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{\alpha-\beta}\right)x,
\end{aligned}$$

if necessary arrangements are made, then we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_{mn+s}x^n =&\frac{(\alpha-\beta)}{A_1}\left(\frac{(\gamma+1)^s-(\delta+1)^s}{(\gamma-\delta)(\alpha-\beta)}\right)-\frac{(\gamma-\delta)}{B_1}\left(\frac{\alpha^{2s}-\beta^{2s}}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{((\gamma+1)(\delta+1))^m(\alpha-\beta)}{A_1}\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{(\gamma-\delta)(\alpha-\beta)}\right)x+\frac{(\alpha\beta)^{2m}(\gamma-\delta)}{B_1}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{(\alpha-\beta)(\gamma-\delta)}\right)x\\
=&\frac{1}{A_1}\left(\frac{b_s((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(\alpha^{2s}-\beta^{2s})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{1}{B_1}\left(\frac{(\alpha-\beta)((\gamma+1)^s-(\delta+1)^s)-b_s((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{((\gamma+1)(\delta+1))^m}{A_1}\left(\frac{b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(\alpha^{2(s-m)}-\beta^{2(s-m)})}{(\alpha-\beta)(\gamma-\delta)}\right)x\\
&+\frac{(\alpha\beta)^{2m}}{B_1}\left(\frac{(\alpha-\beta)((\gamma+1)^{s-m}-(\delta+1)^{s-m})-b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)x\\
=&b_s\left(\frac{1}{A_1}+\frac{1}{B_1}\right)-b_{s-m}\left(\frac{2^m}{A_1}+\frac{1}{B_1}\right)x+C_1.
\end{aligned}$$

Hence the result is obvious. □

Now let's get a grand total formula that includes all the sum results that deal with the sum formulas of $mk+s$ terms.

Theorem 2.8. *Let b_n be the binomial transform of quadra Fibona-Pell sequence. Then*

$$\sum_{k=0}^{n} b_{mk+s} = \frac{2^m}{A_2}(b_{mn+s}-b_{s-m})+\frac{1}{B_2}(b_{mn+s}+b_{s-m}+b_{mn+m+s}-b_s)+\frac{1}{A_2}(b_s-b_{mn+m+s})+C_2$$

for all $n \in \mathbb{N}$ and $m, s \in \mathbb{Z}, s > m$, where

$$\begin{aligned}
A_2 =& (\gamma+1)^m(\delta+1)^m - ((\gamma+1)^m + (\delta+1)^m) + 1 \\
B_2 =& \alpha^{2m}\beta^{2m} - \left(\alpha^{2m} + \beta^{2m}\right) + 1 \\
C_2 =& \frac{2^m}{A_2}\left(\frac{\alpha^{2(mn+s)} - \beta^{2(mn+s)}}{\alpha-\beta}\right) - \frac{1}{B_2}\left(\frac{(\gamma+1)^{mn+s} - (\delta+1)^{mn+s}}{\gamma-\delta}\right) - \frac{2^m}{A_2}\left(\frac{\alpha^{2(s-m)} - \beta^{2(s-m)}}{\alpha-\beta}\right) \\
&+ \frac{1}{B_2}\left(\frac{(\gamma+1)^{s-m} - (\delta+1)^{s-m}}{\gamma-\delta}\right) - \frac{1}{A_2}\left(\frac{\alpha^{2(mn+m+s)} - \beta^{2(mn+m+s)}}{\alpha-\beta}\right) + \frac{1}{B_2}\left(\frac{(\gamma+1)^{mn+m+s} - (\delta+1)^{mn+m+s}}{\gamma-\delta}\right) \\
&+ \frac{1}{A_2}\left(\frac{\alpha^{2s} - \beta^{2s}}{\alpha-\beta}\right) - \frac{1}{B_2}\left(\frac{(\gamma+1)^s - (\delta+1)^s}{\gamma-\delta}\right).
\end{aligned}$$

Proof. From (2.4), we get

$$\begin{aligned}
\sum_{k=0}^{n} b_{mk+s} =& \sum_{k=0}^{n}\left(\frac{(\gamma+1)^{mk+s} - (\delta+1)^{mk+s}}{\gamma-\delta} - \frac{\alpha^{2(mk+s)} - \beta^{2(mk+s)}}{\alpha-\beta}\right) \\
=& \frac{(\gamma+1)^s}{\gamma-\delta}\left(\frac{\left(((\gamma+1)^m)^{n+1} - 1\right)((\delta+1)^m - 1)}{((\gamma+1)^m - 1)((\delta+1)^m - 1)}\right) - \frac{(\delta+1)^s}{\gamma-\delta}\left(\frac{\left(((\delta+1)^m)^{n+1} - 1\right)((\gamma+1)^m - 1)}{((\delta+1)^m - 1)((\gamma+1)^m - 1)}\right) \\
&- \frac{\alpha^{2s}}{\alpha-\beta}\left(\frac{\left((\alpha^{2m})^{n+1} - 1\right)(\beta^{2m} - 1)}{(\alpha^{2m} - 1)(\beta^{2m} - 1)}\right) + \frac{\beta^{2s}}{\alpha-\beta}\left(\frac{\left((\beta^{2m})^{n+1} - 1\right)(\alpha^{2m} - 1)}{(\beta^{2m} - 1)(\alpha^{2m} - 1)}\right) \\
=& \frac{\left\{\begin{array}{c} (\gamma+1)^m(\delta+1)^m\left(\frac{(\gamma+1)^{mn+s} - (\delta+1)^{mn+s}}{\gamma-\delta}\right) - (\gamma+1)^m(\delta+1)^m\left(\frac{(\gamma+1)^{s-m} - (\delta+1)^{s-m}}{\gamma-\delta}\right) \\ - \left(\frac{(\gamma+1)^{mn+m+s} - (\delta+1)^{mn+m+s}}{\gamma-\delta}\right) + \left(\frac{(\gamma+1)^s - (\delta+1)^s}{\gamma-\delta}\right) \end{array}\right\}}{(\gamma+1)^m(\delta+1)^m - ((\gamma+1)^m + (\delta+1)^m) + 1} \\
&- \frac{\left\{\begin{array}{c} (\alpha\beta)^{2m}\left(\frac{\alpha^{2(mn+s)} - \beta^{2(mn+s)}}{\alpha-\beta}\right) + (\alpha\beta)^{2m}\left(\frac{\alpha^{2(s-m)} - \beta^{2(s-m)}}{\alpha-\beta}\right) \\ + \left(\frac{\alpha^{2(mn+m+s)} - \beta^{2(mn+m+s)}}{\alpha-\beta}\right) - \left(\frac{\alpha^{2s} - \beta^{2s}}{\alpha-\beta}\right) \end{array}\right\}}{\alpha^{2m}\beta^{2m} - (\alpha^{2m} + \beta^{2m}) + 1}.
\end{aligned}$$

As a result of calculations, we obtained

$$\begin{aligned}
\sum_{k=0}^{n} b_{mk+s} =& \frac{(\gamma+1)^m(\delta+1)^m}{A_2}\left(\frac{b_{mn+s}((\gamma+1) - (\delta+1))(\alpha-\beta) + ((\gamma+1) - (\delta+1))(\alpha^{2(mn+s)} - \beta^{2(mn+s)})}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&- \frac{(\alpha\beta)^{2m}}{B_2}\left(\frac{(\alpha-\beta)((\gamma+1)^{mn+s} - (\delta+1)^{mn+s}) - b_{mn+s}((\gamma+1) - (\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&- \frac{(\gamma+1)^m(\delta+1)^m}{A_2}\left(\frac{b_{s-m}((\gamma+1) - (\delta+1))(\alpha-\beta) + ((\gamma+1) - (\delta+1))(\alpha^{2(s-m)} - \beta^{2(s-m)})}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&+ \frac{(\alpha\beta)^{2m}}{B_2}\left(\frac{(\alpha-\beta)((\gamma+1)^{s-m} - (\delta+1)^{s-m}) - b_{s-m}((\gamma+1) - (\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&- \frac{1}{A_2}\left(\frac{b_{mn+m+s}((\gamma+1) - (\delta+1))(\alpha-\beta) + ((\gamma+1) - (\delta+1))(\alpha^{2(mn+m+s)} - \beta^{2(mn+m+s)})}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&+ \frac{1}{B_2}\left(\frac{(\alpha-\beta)((\gamma+1)^{mn+m+s} - (\delta+1)^{mn+m+s}) - b_{mn+m+s}((\gamma+1) - (\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&+ \frac{1}{A_2}\left(\frac{b_s((\gamma+1) - (\delta+1))(\alpha-\beta) + ((\gamma+1) - (\delta+1))(\alpha^{2s} - \beta^{2s})}{(\alpha-\beta)(\gamma-\delta)}\right).
\end{aligned}$$

Note that if we substitute the roots of the characteristic equation,

$$\begin{aligned}\sum_{k=0}^{n} b_{mk+s} =& \frac{2^m}{A_2} b_{mn+s} + \frac{1}{B_2} b_{mn+s} + \frac{2^m}{A_2}\left(\frac{\alpha^{2(mn+s)}-\beta^{2(mn+s)}}{\alpha-\beta}\right) - \frac{1}{B_2}\left(\frac{(\gamma+1)^{mn+s}-(\delta+1)^{mn+s}}{\gamma-\delta}\right) - \frac{2^m}{A_2} b_{s-m} + \frac{1}{B_2} b_{s-m} \\ &+ \frac{2^m}{A_2}\left(\frac{\alpha^{2(s-m)}-\beta^{2(s-m)}}{\alpha-\beta}\right) + \frac{1}{B_2}\left(\frac{(\gamma+1)^{s-m}-(\delta+1)^{s-m}}{\gamma-\delta}\right) - \frac{1}{A_2} b_{mn+m+s} + \frac{1}{B_2} b_{mn+m+s} \\ &- \frac{1}{A_2}\left(\frac{\alpha^{2(mn+m+s)}-\beta^{2(mn+m+s)}}{\alpha-\beta}\right) + \frac{1}{B_2}\left(\frac{(\gamma+1)^{mn+m+s}-(\delta+1)^{mn+m+s}}{\gamma-\delta}\right) + \frac{1}{A_2} b_s - \frac{1}{B_2} b_s \\ &+ \frac{1}{A_2}\left(\frac{\alpha^{2s}-\beta^{2s}}{\alpha-\beta}\right) - \frac{1}{B_2}\left(\frac{(\gamma+1)^{s}-(\delta+1)^{s}}{\gamma-\delta}\right) \\ =& \frac{2^m}{A_2}\left(b_{mn+s}-b_{s-m}\right) + \frac{1}{B_2}\left(b_{mn+s}+b_{s-m}+b_{mn+m+s}-b_s\right) + \frac{1}{A_2}\left(b_s - b_{mn+m+s}\right) + C_2.\end{aligned}$$

We get the result. □

3. Binomial Transform of Quadra Fibona-Pell Quaternions

In this section, we give the binomial transform of quadra Fibona-Pell quaternion sequence and obtain some certain identities related to this binomial transform. In [16], quaternion state of the Fibonacci-Pell sequence was investigated and here, Binet Formula, generating function, sum formulas are obtained. In [17], the results for the new sequence obtained by applying binomial transform to the quaternion sequence of the Horadam sequence, which is an integer sequence with a quadratic recurrence relation, are found. In [18], both quaternion and binomial transform are examined simultaneously for the first time.
In [16], let W_n be the quadra Fibona-Pell sequence, then

$$QW_n = W_n + W_{n+1} i + W_{n+2} j + W_{n+3} k$$

is called a quadra Fibona-Pell quaternion, containing the initial values of

$$\begin{aligned} QW_0 &= j + 3k, \\ QW_1 &= i + 3j + 9k, \\ QW_2 &= 1 + 3i + 9j + 24k, \\ QW_3 &= 3 + 9i + 24j + 62k. \end{aligned}$$

Let QW_n be the $n-th$ quadra Fibona-Pell quaternion. Then the binomial transform of quadra Fibona-Pell sequence is

$$bq_n = \sum_{i=0}^{n} \binom{n}{i} QW_i. \tag{3.1}$$

Let us give a Lemma as a first step to find the recurrence relation of the binomial transform.

Lemma 3.1. *Let b_n be the binomial transform of quadra Fibona-Pell quaternion. Then*

$$bq_{n+1} = \sum_{i=0}^{n} \binom{n}{i} \left(QW_i + QW_{i+1}\right).$$

Proof. Notice that the equation (2.1)

$$\begin{aligned} bq_n &= \sum_{i=0}^{n} \binom{n}{i} QW_i, \\ bq_{n+1} &= \sum_{i=0}^{n+1} \binom{n+1}{i} QW_i, \\ bq_{n+1} &= \sum_{i=1}^{n+1} \binom{n+1}{i} QW_i + QW_0. \end{aligned}$$

Since $\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$ and $\binom{n}{n+1} = 0$, we get

$$\begin{aligned} bq_{n+1} &= \sum_{i=1}^{n+1} \left[\binom{n}{i} + \binom{n}{i-1}\right] QW_i + QW_0 \\ &= \sum_{i=1}^{n+1} \binom{n}{i} QW_i + \sum_{i=1}^{n+1} \binom{n}{i-1} QW_i + QW_0 \\ &= \sum_{i=0}^{n} \binom{n}{i} QW_i + \sum_{i=0}^{n} \binom{n}{i} QW_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} \left(QW_i + QW_{i+1}\right). \end{aligned}$$

□

Theorem 3.2. *The binomial transform of quadra Fibona-Pell quaternion states following recurrence relation*

$$bq_{n+3} = 7bq_{n+2} - 15bq_{n+1} + 10bq_n - 2bq_{n-1} \tag{3.2}$$

for $n \geq 4$, *where*

$$\begin{aligned} bq_0 &= j+3k, \\ bq_1 &= i+4j+12, \\ bq_2 &= 1+5i+16j+45k, \end{aligned}$$

and

$$bq_3 = 6+21i+61j+164k.$$

Proof. Using Lemma 3.1 we get,

$$bq_{n+3} = K_2 bq_{n+2} + L_2 bq_{n+1} + M_2 bq_n + N_2 bq_{n-1}.$$

If we take $n = 1,2,3,4$ we take the system,

$$\begin{aligned} n=1 &\Rightarrow bq_4 = K_2bq_3 + L_2bq_2 + M_2bq_1 + N_2bq_0, \\ n=2 &\Rightarrow bq_5 = K_2bq_4 + L_2bq_3 + M_2bq_2 + N_2bq_1, \\ n=3 &\Rightarrow bq_6 = K_2bq_5 + L_2bq_4 + M_2bq_3 + N_2bq_2, \\ n=4 &\Rightarrow bq_7 = K_2bq_6 + L_2bq_5 + M_2bq_4 + N_2bq_3. \end{aligned}$$

By considering Cramer's rule for the system, we obtain

$$K_2 = 7,\ L_2 = -15,\ M_2 = 10,\ N_2 = -2$$

which is completed the proof. □

Theorem 3.3. *Let* bq_n *be the binomial transform of quadra Fibona-Pell quaternion sequences. The generating function of the related binomial transform is*

$$bq(x) = \frac{bq_0 + (bq_1 - 7bq_0)x + (bq_2 - 7bq_1 + 15bq_0)x^2 + (bq_3 - 7bq_2 + 15bq_1 - 10bq_0)x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4}$$

where $bq_0 = j+3k$, $bq_1 = i+4j+12$, $bq_2 = 1+5i+16j+45k$ *and* $bq_3 = 6+21i+61j+164k$.

Proof. Assume that

$$bq(x) = \sum_{i=0}^{\infty} bq_i x^i$$

is the generating function of the binomial transform for QW_n.Then

$$\begin{aligned} bq(x) &= bq_0 + bq_1x + bq_2x^2 + bq_3x^3 + \cdots \\ 7xbq(x) &= 7bq_0x + 7bq_1x^2 + 7bq_2x^3 + 7bq_3x^4 + \cdots \\ 15x^2bq(x) &= 15bq_0x^2 + 15bq_1x^3 + 15bq_2x^4 + 15bq_3x^5 + \cdots \\ 10x^3bq(x) &= 10bq_0x^3 + 10bq_1x^4 + 10bq_2x^5 + 10bq_3x^6 + \cdots \\ 2x^4bq(x) &= 2bq_0x^4 + 2bq_1x^5 + 2bq_2x^6 + 2bq_3x^7 + \cdots \end{aligned}$$

Since from equation (3.2), we obtain

$$(1 - 7x + 15x^2 - 10x^3 + 2x^4)bq(x) = j + 3k + (i - 3j - 9k)x + (1 - 2i + 3j + 6k)x^2 + (-1 + \hat{\imath} - j - k)x^3$$

and hence the generating function for the binomial transform of the bq_n is

$$bq(x) = \frac{bq_0 + (bq_1 - 7bq_0)x + (bq_2 - 7bq_1 + 15bq_0)x^2 + (bq_3 - 7bq_2 + 15bq_1 - 10bq_0)x^3}{1 - 7x + 15x^2 - 10x^3 + 2x^4}.$$

Finally, we can give the following result. □

Now let's find the Binet formula, which we will use for the identities. For this, let the following equations be given

$$\begin{aligned} A &= -10bq_0 + 15bq_1 - 7bq_2 + bq_3, \\ B &= -3bq_0 + 10bq_1 - 6bq_2 + bq_3, \\ C &= -10bq_0 + 24bq_1 - 13bq_2 + 2bq_3, \\ D &= 4bq_0 - 10bq_1 + 6bq_2 - bq_3. \end{aligned} \tag{3.3}$$

Theorem 3.4. *Let bq_n be the binomial transform of quadra Fibona-Pell quaternion sequences. Binet formula for the related binomial transform is*

$$bq_n = \frac{P(\gamma+1)^n - R(\delta+1)^n}{\gamma-\delta} + \frac{S\alpha^{2n} - T\beta^{2n}}{\alpha-\beta} \tag{3.4}$$

for $n \geqslant 0$, where

$$\begin{aligned} A+B(\gamma+1) &= P, \\ A+B(\delta+1) &= R, \\ C+D\left(\alpha^2\right) &= S, \\ C+D\left(\beta^2\right) &= T. \end{aligned}$$

Proof. Assume that, from the previous theorem

$$bq(x) = \frac{Ax+B}{2x^2-4x+1} + \frac{Cx+D}{x^2-3x+1}.$$

When the denominator is equal

$$\begin{aligned} B+D &= bq_0, \\ A+2C &= bq_3 - 7bq_2 + 15bq_1 - 10bq_0, \\ B-3A-4C+2D &= bq_2 - 7bq_1 + 15bq_0, \\ A-3B+C-4D &= bq_1 - 7bq_0. \end{aligned}$$

the equation is obtained (3.3). When the values are replaced, we get

$$\begin{aligned} A &= -10(j+3k) + 15(i+4j+12k) - 7(1+5i+16j+45k) + 6 + 21i + 61j + 164k \\ &= i - j - k - 1, \\ B &= -3(j+3k) + 10(i+4j+12k) - 6(1+5i+16j+45k) + 6 + 21i + 61j + 164k \\ &= i + 2j + 5k, \\ C &= -10(j+3k) + 24(i+4j+12k) - 13(1+5i+16j+45k) + 2(6+21i+61j+164k) \\ &= -1 + i + k, \\ D &= 4(j+3k) - 10(i+4j+12k) + 6(1+5i+16j+45k) - (6+21i+61j+164k) \\ &= -i - j - 2k. \end{aligned}$$

Finally, when necessary calculations are taken

$$\begin{aligned} bq(x) &= \frac{Ax+B}{2x^2-4x+1} + \frac{Cx+D}{x^2-3x+1} \\ &= \frac{Ax}{2x^2-4x+1} + \frac{B}{2x^2-4x+1} + \frac{Cx}{x^2-3x+1} + \frac{D}{x^2-3x+1} \\ &= \frac{A(\gamma+1)^n - A(\delta+1)^n + B(\gamma+1)^{n+1} - B(\delta+1)^{n+1}}{\gamma-\delta} + \frac{C\alpha^{2n} - C\beta^{2n} + D\alpha^{2n+2} - D\beta^{2n+2}}{\alpha-\beta} \\ &= \frac{(\gamma+1)^n (A+B(\gamma+1)) - (\delta+1)^n (A+B(\delta+1))}{\gamma-\delta} + \frac{\alpha^{2n}\left(C+D\alpha^2\right) - \beta^{2n}\left(C+D\beta^2\right)}{\alpha-\beta} \end{aligned}$$

we find the result

$$bq_n = \frac{P(\gamma+1)^n - R(\delta+1)^n}{\gamma-\delta} + \frac{S\alpha^{2n} - T\beta^{2n}}{\alpha-\beta}$$

where

$$\begin{aligned} A+B(\gamma+1) &= P, \\ A+B(\delta+1) &= R, \\ C+D\alpha^2 &= S, \\ C+D\beta^2 &= T. \end{aligned}$$

□

Now, we can give the following result.

Theorem 3.5. *Let bq_n be the binomial transform of quadra Fibona-Pell quaternion sequences. Then*

$$\sum_{n=0}^{\infty} bq_{mn+s}x^n = b_s\left(\frac{1}{E_1}-\frac{1}{F_1}\right) - b_{s-m}\left(\frac{2^m}{E_1}-\frac{1}{G_1}\right)x + H_1$$

for all $n \in \mathbb{N}$ and $m,s \in \mathbb{N}, s>m$,

$$E_1 = (1-(\gamma+1)^m x)(1-(\delta+1)^m x),$$
$$G_1 = (1-\alpha^{2m}x)(1-\beta^{2m}x),$$
$$H_1 = \frac{1}{E_1}\left(\frac{S\gamma^{2s}-T\delta^{2s}}{\gamma-\delta}\right) + \frac{1}{G_1}\left(\frac{P(\gamma+1)^s-R(\delta+1)^s}{\gamma-\delta}\right) - \frac{2^m}{E_1}\left(\frac{S\alpha^{2(s-m)}-T\beta^{2(s-m)}}{\alpha-\beta}\right)x - \frac{4^m}{G_1}\left(\frac{P(\gamma+1)^{s-m}-R(\delta+1)^{s-m}}{\gamma-\delta}\right)x.$$

Proof. Again from equation (3.4), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} bq_{mn+s}x^n =& \sum_{n=0}^{\infty}\left(\frac{P(\gamma+1)^{mn+s}-R(\delta+1)^{mn+s}}{\gamma-\delta} + \frac{S\alpha^{2(mn+s)}-T\beta^{2(mn+s)}}{\alpha-\beta}\right)x^n \\
=& \frac{P(\gamma+1)^s}{\gamma-\delta}\sum_{n=0}^{\infty}((\gamma+1)^m x)^n - \frac{R(\delta+1)^s}{\gamma-\delta}\sum_{n=0}^{\infty}((\delta+1)^m x)^n + \frac{S\alpha^{2s}}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^{2m}x)^n - \frac{T\beta^{2s}}{\alpha-\beta}\sum_{n=0}^{\infty}(\beta^{2m}x)^n \\
=& \frac{P(\gamma+1)^s}{\gamma-\delta}\left(\frac{1}{1-(\gamma+1)^m x}\right) - \frac{R(\delta+1)^s}{\gamma-\delta}\left(\frac{1}{1-(\delta+1)^m x}\right) + \frac{S\alpha^{2s}}{\alpha-\beta}\left(\frac{1}{1-\alpha^{2m}x}\right) - \frac{T\beta^{2s}}{\alpha-\beta}\left(\frac{1}{1-\beta^{2m}x}\right) \\
=& \frac{1}{\gamma-\delta}\,\frac{(P(\gamma+1)^s-R(\delta+1)^s)-(P(\gamma+1)^s(\delta+1)^m-R(\delta+1)^s(\gamma+1)^m)x}{1-((\gamma+1)^m+(\delta+1)^m)x+((\gamma+1)(\delta+1))^m x^2} \\
&+ \frac{1}{\alpha-\beta}\,\frac{(S\alpha^{2s}-T\beta^{2s})-(S\alpha^{2s}\beta^{2m}-T\beta^{2s}\alpha^{2m})x}{1-(\alpha^{2m}+\beta^{2m})x+(\alpha\beta)^{2m}x^2} \\
=& \frac{1\,(\alpha-\beta)}{E_1}\left(\frac{P(\gamma+1)^s-R(\delta+1)^s}{(\gamma-\delta)(\alpha-\beta)}\right) + \frac{1\,(\gamma-\delta)}{G_1}\left(\frac{S\alpha^{2s}-T\beta^{2s}}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&- \frac{((\gamma+1)(\delta+1))^m(\alpha-\beta)}{E_1}\left(\frac{P(\gamma+1)^{s-m}-R(\delta+1)^{s-m}}{(\gamma-\delta)(\alpha-\beta)}\right)x - \frac{(\alpha\beta)^{2m}(\gamma-\delta)}{G_1}\left(\frac{S\alpha^{2(s-m)}-T\beta^{2(s-m)}}{(\alpha-\beta)(\gamma-\delta)}\right)x.
\end{aligned}$$

If necessary arrangements are made, then we get

$$\begin{aligned}
\sum_{n=0}^{\infty} bq_{mn+s}x^n =& \frac{1}{E_1}\left(\frac{b_s((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2s}-T\beta^{2s})}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&+ \frac{1}{G_1}\left(\frac{(\alpha-\beta)(P(\gamma+1)^s-R(\delta+1)^s)-b_s((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right) \\
&- \frac{((\gamma+1)(\delta+1))^m}{E_1}\left(\frac{b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2(s-m)}-T\beta^{2(s-m)})}{(\alpha-\beta)(\gamma-\delta)}\right)x \\
&- \frac{(\alpha\beta)^{2m}}{G_1}\left(\frac{(\alpha-\beta)\left(P(\gamma+1)^{s-m}-R(\delta+1)^{s-m}\right)-b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)x \\
=& b_s\left(\frac{1}{E_1}-\frac{1}{G_1}\right) - b_{s-m}\left(\frac{2^m}{E_1}-\frac{1}{G_1}\right)x + H_1.
\end{aligned}$$

Hence the result is obvious. □

Theorem 3.6. *Let bq_n be the binomial transform of quadra Fibona-Pell quaternion sequences. Then*

$$\sum_{k=0}^{n} bq_{mk+s} = \frac{2^m}{E_2}(b_{mn+s}-b_{s-m}) - \frac{1}{G_2}(b_{mn+s}-b_{s-m}-b_{mn+m+s}+b_s) - \frac{1}{E_2}(b_{mn+m+s}+b_s) + H_2$$

for all $n \in \mathbb{N}$ and $m,s \in \mathbb{Z}$, $s>m$,

$$E_2 = (\gamma+1)^m(\delta+1)^m - ((\gamma+1)^m+(\delta+1)^m)+1,$$
$$G_2 = \alpha^{2m}\beta^{2m} - \left(\alpha^{2m}+\beta^{2m}\right)+1,$$
$$\begin{aligned}
H_2 =& \frac{2^m}{E_2}\left(\frac{S\alpha^{2(mn+s)}-T\beta^{2(mn+s)}}{\alpha-\beta}\right) + \frac{1}{G_2}\left(\frac{P(\gamma+1)^{mn+s}-R(\delta+1)^{mn+s}}{\gamma-\delta}\right) - \frac{2^m}{E_2}\left(\frac{S\alpha^{2(s-m)}-T\beta^{2(s-m)}}{\alpha-\beta}\right) \\
&- \frac{1}{G_2}\left(\frac{P(\gamma+1)^{s-m}-R(\delta+1)^{s-m}}{\gamma-\delta}\right) - \frac{1}{E_2}\left(\frac{S\alpha^{2(mn+m+s)}-T\beta^{2(mn+m+s)}}{\alpha-\beta}\right) - \frac{1}{G_2}\left(\frac{P(\gamma+1)^{mn+m+s}-T(\delta+1)^{mn+m+s}}{\gamma-\delta}\right) \\
&+ \frac{1}{E_2}\left(\frac{S\alpha^{2s}-T\beta^{2s}}{\alpha-\beta}\right) + \frac{1}{G_2}\left(\frac{P(\gamma+1)^s-R(\delta+1)^s}{\gamma-\delta}\right).
\end{aligned}$$

Proof. From (3.4), we get

$$
\begin{aligned}
\sum_{k=0}^{n} bq_{mk+s} =& \sum_{k=0}^{n}\left(\frac{P(\gamma+1)^{mk+s}-R(\delta+1)^{mk+s}}{\gamma-\delta}+\frac{S\alpha^{2(mk+s)}-T\beta^{2(mk+s)}}{\alpha-\beta}\right)\\
=&\frac{P(\gamma+1)^{mn+m+s}(\delta+1)^m-P(\gamma+1)^{mn+m+s}-P(\gamma+1)^s(\delta+1)^m+P(\gamma+1)^s}{(\gamma-\delta)((\gamma+1)^m(\delta+1)^m-((\gamma+1)^m+(\delta+1)^m)+1)}\\
&-\frac{R(\delta+1)^{mn+m+s}(\gamma+1)^m-R(\delta+1)^{mn+m+s}-R(\delta+1)^s(\gamma+1)^m+R(\delta+1)^s}{(\gamma-\delta)((\gamma+1)^m(\delta+1)^m-((\gamma+1)^m+(\delta+1)^m)+1)}\\
&+\frac{S\alpha^{2(mn+m+s)}\beta^{2m}-S\alpha^{2(mn+m+s)}-S\alpha^{2s}\beta^{2m}+S\alpha^{2s}}{(\alpha-\beta)(\alpha^{2m}\beta^{2m}-(\alpha^{2m}+\beta^{2m})+1)}\\
&-\frac{T\beta^{2(mn+m+s)}\alpha^{2m}-T\beta^{2(mn+m+s)}-T\beta^{2s}\alpha^{2m}+T\beta^{2s}}{(\alpha-\beta)(\alpha^{2m}\beta^{2m}-(\alpha^{2m}+\beta^{2m})+1)}.
\end{aligned}
$$

As a result of calculations, we obtained

$$
\begin{aligned}
\sum_{k=0}^{n} bq_{mk+s} =& \frac{(\gamma+1)^m(\delta+1)^m}{E_2}\left(\frac{b_{mn+s}((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2(mn+s)}-T\beta^{2(mn+s)})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&+\frac{(\alpha\beta)^{2m}}{G_2}\left(\frac{(\alpha-\beta)(P(\gamma+1)^{mn+s}-R(\delta+1)^{mn+s})-b_{mn+s}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{(\gamma+1)^m(\delta+1)^m}{E_2}\left(\frac{b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2(s-m)}-T\beta^{2(s-m)})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{(\alpha\beta)^{2m}}{G_2}\left(\frac{(\alpha-\beta)(P(\gamma+1)^{s-m}-R(\delta+1)^{s-m})-b_{s-m}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{1}{E_2}\left(\frac{b_{mn+m+s}((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2(mn+m+s)}-T\beta^{2(mn+m+s)})}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&-\frac{1}{G_2}\left(\frac{(\alpha-\beta)(P(\gamma+1)^{mn+m+s}-R(\delta+1)^{mn+m+s})-b_{mn+m+s}((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&+\frac{1}{E_2}\left(\frac{b_s((\gamma+1)-(\delta+1))(\alpha-\beta)+((\gamma+1)-(\delta+1))(S\alpha^{2s}-T\beta^{2s}}{(\alpha-\beta)(\gamma-\delta)}\right)\\
&+\frac{1}{G_2}\left(\frac{(\alpha-\beta)(P(\gamma+1)^s-R(\delta+1)^s)-b_s((\gamma+1)-(\delta+1))(\alpha-\beta)}{(\alpha-\beta)(\gamma-\delta)}\right)\\
=&\frac{2^m}{E_2}(b_{mn+s}-b_{s-m})-\frac{1}{G_2}(b_{mn+s}-b_{s-m}-b_{mn+m+s}+b_s)-\frac{1}{E_2}(b_{mn+m+s}+b_s)+H_2.
\end{aligned}
$$

□

4. Conclusion

Our aim in this study is to study the binomial transform for quadra Fibona-Pell sequence and its binomial transform of quaternion sequence. In the article, the binomial transform of the sequence is found in the first part, and then the results related to this transform are mentioned. In the second part, similar results were obtained by binomial transform of quadra Fibona-Pell quaternion sequence, which was found before.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] P. Ribenboim. *My Numbers, My Friends*, Popular Lectures on Number Theory. Springer-Verlag, New York, Inc. 2000.

[2] A. Özkoç, *Some algebraic identities on quadra Fibona-Pell integer sequence*, Adv. Differ. Equ., **2015** (2015), Article Number: 148, 10 pages.

[3] C. Kızılateş, *On the quadra Lucas-Jacobsthal numbers*, Karaelmas Sci. Eng. J., **7**(2) (2017), 619-621.

[4] O. Dişkaya, H. Menken, *On the quadra Fibona-Pell and hexa-Fibona-Pell-Jacobsthal sequences*, Mathematical Sciences and Applications E-notes, **7**(2) (2019), 149-160.

[5] A. Tekcan, A. Özkoç, M. Engür, M. E. Özbek, *On algebraic identities on a new integer sequence with four parameters*, Ars Comb., **127** (2016), 225-238.

[6] W. R. Hamilton, *Elements of Quaternions*, London, England, Green Company, 1866.

[7] M. N. S. Swamy, *On generalized Fibonacci quaternions*, Fibonacci Q., **5** (1973), 547-550.

[8] A. F. Horadam, *Quaternion recurrence relations*, Ulam Q., **2**(2) (1993), 22-33.

[9] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison Wesley Publishing Company, 1998.

[10] K. W. Chen, *Identities from the Binomial Transform*, J. Number Theory, 124, 142-150, 2007.

[11] S. Falcon, A. Plaza, *Binomial Transforms of the* $k-$*Fibonacci Sequences*, Int. J. Nonlinear Sci. Numer. Simul., 10, 1305-1316, 2009.

[12] H. Prodinger, *Some Information about the Binomial Transform*, The Fibonacci Q., 32 (5), 412-415, 1994.

[13] H. W. Gould, *Series Transformations for Findings Recurrences for Sequences*, The Fibonacci Q., 28(2), 166-171, 1990.

[14] S. Falcon, *Binomial Transform of the Generalized* $k-$*Fibonacci Numbers*, Commun. Math. Anal., Vol. 10, No. 3, pp. 643–651, 2019

[15] C. Kızılateş, N. Tuğlu, B. Çekim, *Binomial transform of quadrapell sequences and quadrapell matrix sequences*, J. Sci. Arts, **1**(38) (2017), 69-80.

[16] T. Çetinalp, *Kuadra Fibona-Pell kuaterniyon dizileri ¨üzerine bazı cebirsel özdeşlikler*, MSc. Thesis, Karamanoğlu Mehmetbey University, 2017.

[17] F. Kaplan, A. Özkoç Öztürk, *On the binomial transforms of the Horadam quaternion sequences*, Math. Meth. Appl. Sci., **45** (2022), 12009-12022.

[18] E. Polatlı, On certain properties of quadrapell sequences, Karaelmas Sci. Eng. J., **8**(1) (2018), 305-308.