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RESEARCH ARTICLE

NONIC B-SPLINE APPROACH FOR ADVECTION DIFFUSION EQUATION

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ABSTRACT

In this paper, a highly accurate method is introduced to achieve the numerical solution of the advection diffusion equation (ADE). This approach contains collocation technique based on nonic B-spline functions in the spatial-domain discretization and Adams Moulton scheme in the temporal-domain discretization. Two test problems are studied to validate effectiveness of the new presented method and efficiency of the approximate results are tested by calculating rate of temporal-convergence and error norm L_∞ for the suggested method. The obtained numerical results are compared in the tables by the other available studies in literature and it is observed that a better approximate solution is provided than the existing methods.

Keywords: Nonic B-spline, Advection diffusion equation, Collocation method

1. INTRODUCTION

ADE used to model a lot of real problems in physics and engineering is expressed in the following form:

$$w_t + \alpha w_x - \mu w_{xx} = 0, 0 \leq x \leq l \tag{1}$$

along with the boundary conditions (BCs)

$$\begin{matrix} w(0,t) = w(l,t) = 0 \\ w_x(0,t) = w_x(l,t) = 0 \end{matrix}, t \in [0,T] \tag{2}$$

and the initial condition (IC)

$$w(x,0) = \psi(x), 0 \leq x \leq l \tag{3}$$

where α and μ denote the steady uniform fluid velocity and the constant diffusion coefficient, respectively.

Numerous techniques have been implemented to ADE to solve it numerically so far including Finite difference method (FDM) [1-3], least-squares method [4], Taylor-Galerkin technique [5], cubic B-spline differential quadrature method (CBSDQM) [6], extended cubic B-spline collocation method (ECBSCM) [7], differential quadrature method based on quartic and quintic B-splines [8], extended cubic B-spline Galerkin method (ECBSGM) [9], Galerkin method [10] and collocation technique based on fourth-order cubic B-spline [11].

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The main purpose in this paper is to investigate the approximate solution of ADE by the new approach. In this approach, ADE is fully discretized by employing nonic B-spline collocation technique in spatial direction and Adams Moulton method in temporal direction. What is notable in this work is that the use of nonic B-spline functions that have not been utilized before to achieve the numerical solution of ADE. The rest structure of the paper is as follows. In section 2, the temporal and spatial discretizations of ADE are performed. In section 3, two test problems are examined to see the efficiency and accuracy of the present method. A brief summary about main findings of the suggested method is presented in section 4.

2. APPLICATION OF THE PROPOSED METHOD

In this work, the analytical solution of the unknown function at the grid points is represented by

$$w(x_r, t_n) = w_r^n, r = 0,1, \dots, M; \quad n = 0,1,2, \dots$$

where $x_r = rh$, $t_n = n\Delta t$ and the approximate value of w_r^n is denoted by W_r^n.

2.1. Time Discretization

Considering ADE of the form

$$w_t = \mu w_{xx} - \alpha w_x \tag{4}$$

and employing the following two-step method

$$w^{n+1} = w^n + \Delta t(\theta_1 w_t^{n+1} + \theta_2 w_t^n + \theta_3 w_t^{n-1}) \tag{5}$$

we have the temporal discretization of the Eq. (4). Choosing the coefficients in (5) as

$$\theta_1 = \frac{1}{2}, \theta_2 = \frac{1}{2}, \quad \theta_3 = 0$$

gives Crank-Nicolson (CN) method having order two in time and then choosing the coefficients in (5) as

$$\theta_1 = \frac{5}{12}, \theta_2 = \frac{2}{3}, \quad \theta_3 = -\frac{1}{12}$$

yields the third-order implicit Adams Moulton method which is going to be used to discretizate the temporal domain. Using Eq. (5), the temporal discretization of the Equation (4) is obtained as

$$\begin{aligned} w^{n+1} - \theta_1 \Delta t(\mu w_{xx}^{n+1} - \alpha w_x^{n+1}) \\ = w^n + \theta_2 \Delta t(\mu w_{xx}^n - \alpha w_x^n) + \theta_3 \Delta t(\mu w_{xx}^{n-1} - \alpha w_x^{n-1}) \end{aligned} \tag{6}$$

2.2. Nonic B-spline Collocation Method

Let the spatial domain $[0, l]$ be splitted into uniformly M finite elements at the knots

$$0 = x_0 < x_1 < \cdots < x_M = l$$

where $h = x_r - x_{r-1},\ \ r = 1, \dots, M$. On this partition, the nonic B-splines $\varphi_r,\ \ r = -4, \dots, M+4,$ have the following form:

$$\varphi_r(x) = \frac{1}{h^9}\begin{cases} \phi_1, & x_{r-5} \le x < x_{r-4}, \\ \phi_2, & x_{r-4} \le x < x_{r-3}, \\ \phi_3, & x_{r-3} \le x < x_{r-2}, \\ \phi_4, & x_{r-2} \le x < x_{r-1}, \\ \phi_5, & x_{r-1} \le x < x_r, \\ \phi_6, & x_r \le x < x_{r+1}, \\ \phi_7, & x_{r+1} \le x < x_{r+2}, \\ \phi_8, & x_{r+2} \le x < x_{r+3}, \\ \phi_9, & x_{r+3} \le x < x_{r+4}, \\ \phi_{10}, & x_{r+4} \le x < x_{r+5}, \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where

$$\phi_1 = (x - x_{r-5})^9,$$
$$\phi_2 = h^9 + 9h^8(x - x_{r-4}) + 36h^7(x - x_{r-4})^2 + 84h^6(x - x_{r-4})^3 + 126h^5(x - x_{r-4})^4 + 126h^4(x - x_{r-4})^5 + 84h^3(x - x_{r-4})^6 + 36h^2(x - x_{r-4})^7 + 9h(x - x_{r-4})^8 - 9(x - x_{r-4})^9,$$
$$\phi_3 = 502h^9 + 2214h^8(x - x_{r-3}) + 4248h^7(x - x_{r-3})^2 + 4536h^6(x - x_{r-3})^3 + 2772h^5(x - x_{r-3})^4 + 756h^4(x - x_{r-3})^5 - 168h^3(x - x_{r-3})^6 - 216h^2(x - x_{r-3})^7 - 72h(x - x_{r-3})^8 + 36(x - x_{r-3})^9,$$
$$\phi_4 = 14608h^9 + 36414h^8(x - x_{r-2}) + 34272h^7(x - x_{r-2})^2 + 11256h^6(x - x_{r-2})^3 - 4032h^5(x - x_{r-2})^4 - 4284h^4(x - x_{r-2})^5 - 672h^3(x - x_{r-2})^6 + 504h^2(x - x_{r-2})^7 + 252h(x - x_{r-2})^8 - 84(x - x_{r-2})^9,$$
$$\phi_5 = 88234h^9 + 101934h^8(x - x_{r-1}) + 5544h^7(x - x_{r-1})^2 - 36456h^6(x - x_{r-1})^3 - 10836h^5(x - x_{r-1})^4 + 5796h^4(x - x_{r-1})^5 + 2856h^3(x - x_{r-1})^6 - 504h^2(x - x_{r-1})^7 - 504h(x - x_{r-1})^8 + 126(x - x_{r-1})^9,$$
$$\phi_6 = 156190h^9 - 88200h^7(x - x_r)^2 + 23940h^5(x - x_{r-1})^4 - 4200h^3(x - x_r)^6 + 603h(x - x_r)^8 - 126(x - x_r)^9,$$
$$\phi_7 = 88234h^9 - 101934h^8(x - x_{r+1}) + 5544h^7(x - x_{r+1})^2 + 36456h^6(x - x_{r+1})^3 - 10836h^5(x - x_{r+1})^4 - 5796h^4(x - x_{r+1})^5 + 2856h^3(x - x_{r+1})^6 + 504h^2(x - x_{r+1})^7 - 504h(x - x_{r+1})^8 + 84(x - x_{r+1})^9,$$
$$\phi_8 = 14608h^9 - 36414h^8(x - x_{r+2}) + 34272h^7(x - x_{r+2})^2 - 11256h^6(x - x_{r+2})^3 - 4032h^5(x - x_{r+2})^4 + 4284h^4(x - x_{r+2})^5 - 672h^3(x - x_{r+2})^6 - 504h^2(x - x_{r+2})^7 + 252h(x - x_{r+2})^8 - 36(x - x_{r+2})^9,$$
$$\phi_9 = 502h^9 - 2214h^8(x - x_{r+3}) + 4248h^7(x - x_{r+3})^2 - 4536h^6(x - x_{r+3})^3 + 2772h^5(x - x_{r+3})^4 - 756h^4(x - x_{r+3})^5 - 168h^3(x - x_{r+3})^6 + 216h^2(x - x_{r+3})^7 - 72h(x - x_{r+3})^8 + 9(x - x_{r+3})^9,$$
$$\phi_{10} = [h - (x - x_{r+4})]^9.$$

The set of nonic B-spline functions

$$\{\varphi_{-4}(x), \varphi_{-3}(x), \ldots, \varphi_{M+3}(x), \varphi_{M+4}(x)\}$$

generates a basis over the spatial domain $[0, l]$. The global approximate solution $W(x,t)$ corresponding to the analytical solution $w(x,t)$ is expressed as a combination of separable solution of the nonic B-spline spatial terms $\varphi_j(x)$ and temporal terms $\delta_j(t)$ as

$$W(x,t) = \sum_{j=-4}^{M+4} \delta_j \varphi_j \tag{8}$$

where temporal term $\delta_j(t)$ is going to be determined by means of the collocation procedure. Since each subinterval $[x_{r-1}, x_r]$ is covered by ten B-splines, the approximate solution and its first two derivatives at the knots x_r are calculated in terms of temporal terms $\delta_j(t)$ using (7) and (8) as

$$\begin{aligned} W_r &= \delta_{r-4} + 502\delta_{r-3} + 14608\delta_{r-2} + 88234\delta_{r-1} + 156190\delta_r + 88234\delta_{r+1} + 14608\delta_{r+2} + 502\delta_{r+3} + \delta_{r+4}, \\ W_r' &= \frac{9}{h}(-\delta_{r-4} - 246\delta_{r-3} - 4046\delta_{r-2} - 11326\delta_{r-1} + 11326\delta_{r+1} + 4046\delta_{r+2} + 246\delta_{r+3} + \delta_{r+4}), \\ W_r'' &= \frac{72}{h^2}(\delta_{r-4} + 118\delta_{r-3} + 952\delta_{r-2} + 154\delta_{r-1} - 2450\delta_r + 154\delta_{r+1} + 952\delta_{r+2} + 118\delta_{r+3} + \delta_{r+4}) . \end{aligned} \tag{9}$$

Substituting (9) into (6), the fully-discretization of ADE is obtained as

$$\begin{aligned} &\delta_{r-4}^{n+1}(1 + a_1 - a_2) + \delta_{r-3}^{n+1}(502 + 118a_1 - 246a_2) + \delta_{r-2}^{n+1}(14608 + 952a_1 - 4046a_2) + \\ &\delta_{r-1}^{n+1}(88234 + 154a_1 - 11326a_2) + \delta_r^{n+1}(156190 - 2450a_1) + \delta_{r+1}^{n+1}(88234 + 154a_1 + 11326a_2) + \\ &\delta_{r+2}^{n+1}(14608 + 952a_1 + 4046a_2) + \delta_{r+3}^{n+1}(502 + 118a_1 + 246a_2) + \delta_{r+4}^{n+1}(1 + a_1 + a_2) \\ &= W_r^n + a_3(W'')_r^n + a_4(W')_r^n + a_5(W'')_r^{n-1} + a_6(W')_r^{n-1}, \quad 0 \le r \le M \end{aligned} \tag{10}$$

where

$$a_1 = -\theta_1 \Delta t \mu \frac{72}{h^2}, \quad a_2 = \theta_1 \Delta t \alpha \frac{9}{h}, \quad a_3 = \theta_2 \Delta t \mu \frac{72}{h^2},$$

$$a_4 = -\theta_2 \Delta t \alpha \frac{9}{h}, \quad a_5 = \theta_3 \Delta t \mu \frac{72}{h^2}, \quad a_6 = -\theta_3 \Delta t \alpha \frac{9}{h}.$$

Hence, we get a linear system (10) consisting of $M+1$ algebraic equations in $M+9$ unknowns $(\delta_{-4}^{n+1}, \ldots, \delta_{N+4}^{n+1})$. Using BCs (2) and the following additional BCs

$$\begin{aligned} w_{xx}(0,t) = 0 \quad & w_{xx}(l,t) = 0 \\ w_{xxx}(0,t) = 0 \quad & w_{xxx}(l,t) = 0, \end{aligned}$$

the variables

$$\delta_{-4}^{n+1}, \delta_{-3}^{n+1}, \delta_{-2}^{n+1}, \delta_{-1}^{n+1}, \delta_{M+1}^{n+1}, \delta_{M+2}^{n+1}, \delta_{M+3}^{n+1} \text{ and } \delta_{M+4}^{n+1}$$

are eliminated from the above system. Thus, the system is reduced to solvable matrix system of $(M+1) \times (M+1)$ dimension. In order to commence iterative computation, the initial vector

$$\delta^0 = (\delta_{-4}^0, \ldots, \delta_{M+4}^0)^T$$

is first computed using ICs, BCs and additional BCs. After getting the initial vector δ^0 , the unknown vector

$$\delta^1 = (\delta^1_{-4}, \ldots, \delta^1_{M+4})$$

is obtained by making use of CN method. Thus, the unknown vectors

$$\delta^{n+1} = (\delta^{n+1}_{-4}, \ldots, \delta^{n+1}_{M+4})^T (n = 1,2, \ldots)$$

at any desired time level can be computed repeatedly by solving the recurrence relation two precious δ^n and δ^{n-1}.

3. NUMERICAL EXPERIMENTS

In this section, two test problem are examined to illustrate the efficiency and applicability of the suggested method. Accuracy of solution is tested by measuring error norm L_∞

$$L_\infty = \max_m |w_m - W_m| \tag{11}$$

and the order of temporal convergence is calculated by the formula

$$\text{order}=\frac{log\left|\frac{(L_\infty)_{\Delta t_i}}{(L_\infty)_{\Delta t_{i+1}}}\right|}{log\left|\frac{\Delta t_i}{\Delta t_{i+1}}\right|} \tag{12}$$

where $(L_\infty)_{\Delta t_i}$ is the error norm L_∞ for time step Δt_i.

3.1. Problem 1

Consider pure advection problem obtained by taking $\mu = 0$. The analytical solution of this problem is given by

$$w(x,t) = 10\, exp\left(-\frac{(x - \tilde{x}_0 - \alpha t)^2}{2\rho^2}\right). \tag{13}$$

The numerical computation is performed by choosing flow velocity $\alpha = 0.5 m/s$, initial peak location $\tilde{x}_0 = 2km$, $\rho = 264$ in the spatial domain $[0,9000]$ until the terminating time $t = 9600s$. In this case, the wave which is initially located at $\tilde{x}_0 = 2km$ with its peak moves to the right along a channel maintaining its initial shape and size by the time $t = 9600s$. The suggested program is running until the time $t = 9600s$ and the figures of initial solution and waves at various time levels are presented in Figure 1 with $h = 60, \Delta t = 1$. It can be observed from Figure 1 that wave preserves its initial state while moving to the right. Thus, the initial wave moves $4.8 \ km$ from the initial position and the peak of the wave remains stable 10 in progress of time. The graph of absolute error at $t = 9600s$ is also given in Figure 2 which shows that the influence of BCs can be neglected.

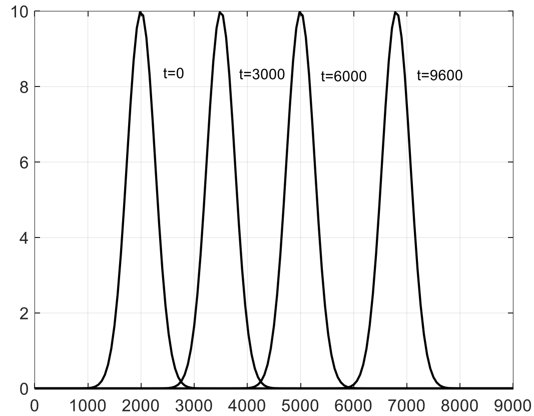

Figure 1. Waves at $t = 0,3000,6000,9600$.

The error norms L_∞ are listed in Table 1 to make a comparison with previous studies given in [4,7,9]. The obtain results confirm that the suggested method gives better results then the other methods. Also, the order of temporal convergence is calculated for $h = 100$ and different temporal steps Δt_i . The calculated order of convergence along with error norm is listed in Table 2. As expected from the theoretical results, the order of the temporal convergence is three.

Table 1. Comparison of L_∞ at time $t = 9600$

Method	h	Δt	L_∞
Proposed	200	50	4.56×10^{-2}
Proposed	100	50	1.93×10^{-2}
Proposed	50	50	1.94×10^{-2}
[9]	200	50	2.18×10^{-1}
[9]	100	50	1.90×10^{-1}
[9]	50	50	1.90×10^{-1}
[7]	200	50	1.29
[7]	100	50	3.25×10^{-1}
[7]	50	50	1.98×10^{-1}
[4]	200	50	5.18×10^{-1}
[4]	100	50	3.76×10^{-1}
[4]	50	50	3.73×10^{-1}

Table 2. The error norms and temporal order of convergence with $h = 100$ at $t = 9600$

Δt_i	Order	L_∞
100	-	1.63×10^{-1}
50	3.07	1.95×10^{-2}
25	3.01	2.41×10^{-3}
12.5	3.00	3.01×10^{-4}

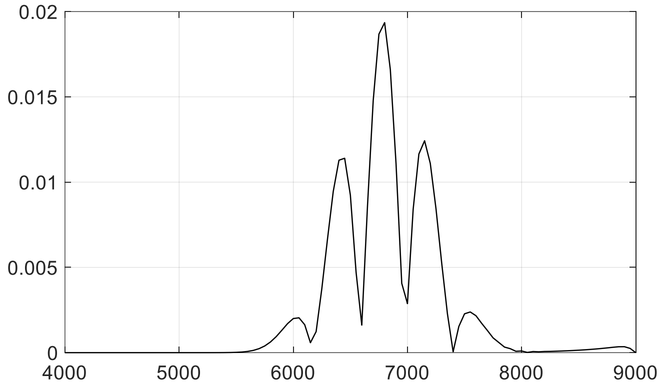

Figure 2. Absolute error for Problem 1 with $h=\Delta t = 50$.

3.2. Problem 2

The analytical solution of this problem, modeling fade out of an initial bell shaped concentration of height 1, is

$$w(x,t) = \frac{1}{\sqrt{4t+1}} exp\left(-\frac{(x-\tilde{x}_0-\alpha t)^2}{\mu(4t+1)}\right) \tag{13}$$

The initial wave forms the peak of which is initially located at $\tilde{x}_0$, moves to the right along x axis as time progresses.

The computation is carried out by taking the parameters $\alpha = 0.8 \; m/s$ and $\mu = 0.005 \; m^2/s$. Table 3 gives the comparison of L_∞ error norms with $\Delta t = 0.0125$. It can be seen from the Table 3 that our method produces better results than the methods given in Method I [6] and [9]. But the result of Method II [6] is a bit better than the result of the present method for $h = 0.025$. The order of convergence and L_∞ error norms are listed Table 4. It is seen from Table 4 that when time step size is reduced from 0.01 to 0.00125, the order of temporal convergence approaches to three.

Table 3. Comparison of L_∞ at time $t = 5$ with $0 \le x \le 9$

h	Proposed	[9]	Method I [6]	Method II [6]
0.2	1.36×10^{-1}	1.33×10^{-1}	1.25×10^{-1}	1.36×10^{-1}
0.1	4.01×10^{-3}	4.23×10^{-3}	6.96×10^{-3}	1.46×10^{-2}
0.05	3.94×10^{-5}	8.43×10^{-4}	1.21×10^{-4}	2.89×10^{-4}
0.025	3.98×10^{-5}	8.43×10^{-4}	3.07×10^{-4}	1.81×10^{-5}

Table 4. The error norms and temporal order of convergence with $h = 0.01$ at $t = 5$.

Δt_i	Order	L_∞
0.01	-	2.04×10^{-5}
0.005	3.00	2.55×10^{-6}
0.0025	3.00	3.19×10^{-7}
0.00125	3.00	3.98×10^{-8}

Figure 3 shows the behaviour of numerical solutions up to time $t = 5$ over the spatial interval [0,9] with $h = \Delta t = 0.001$. Figure 4 gives the graph of the absolute error with $h = 0.01, \Delta t = 0.00125$ at time $t = 5$. It can be seen from Figure 4 that maximum error appears at about the peak of the final wave. Thus, it can be said that the there is no problem in applying BCs.

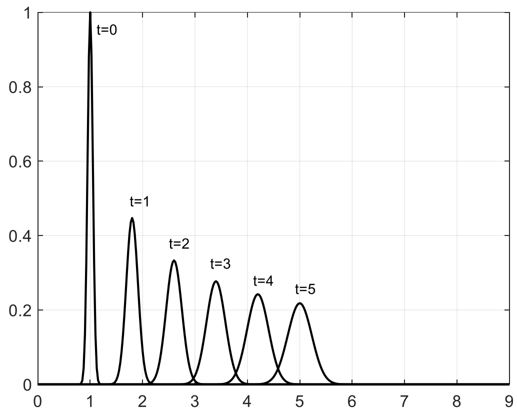

Figure 3. Waves at $t = 0,1,2,3,4,5$.

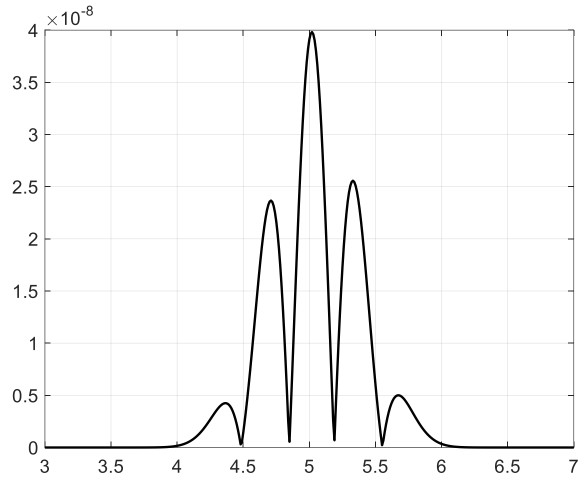

Figure 4. Absolute error for Problem 2 with $h = 0.01, \Delta t = 0.00125$ at $t = 5$.

4. CONCLUSION

In this paper, nonic B-spline collocation technique in collaboration with Adams Moulton method has been proposed to get approximate solution of ADE. To show the effectiveness of the present method, two test problems are used by computing error norms L_∞ and compared with the results existing in the literature. The results obtained by the present method is found to be better than the existed studies given in [4,6,7,9]. The order of temporal convergence is calculated numerically, which agrees with theoretical rate. Consequently, nonic B-spline functions can be applied to obtain approximate solution of the high order nonlinear partial differential equations.

CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

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