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Existence and Uniqueness of Solutions in Inverse Sturm-Liouville Problems

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Abstract

In this study, a boundary value problem is investigated for the Sturm-Liouville equation defined in the interval [0, L]. The problem with [0, L] corresponds to the small vibrations of a fixed-end straight rope. In these problems, the necessary and sufficient conditions for the unique determination of the potential by only one spectrum at certain parameters of the boundary conditions are investigated. In the inverse problem, it is possible with a spectrum to describe the potential of the problem, hence the intensity of the array. Taking this into consideration and using the Leray-Schauder fixed point theorem in Banach space, the existence and uniqueness results of the problem are proved.

Keywords: Sturm-Liouville problem, boundary conditions, Leray–Schauder, eigenvalue, spectrum.

Ters Sturm-Liouville Problemlerinde Çözümlerin Varlık ve Teklik Durumları

Özet

Bu çalışmada, [0, L] aralığında tanımlanan Sturm-Liouville denklemi için bir sınır değer problemi incelenmiştir. [0, L] ile ilgili problem, sabit uçlu düz bir halatın küçük titreşimlerine karşılık gelir. Bu problemlerde, sınır koşullarının belirli parametrelerinde sadece bir spektrum tarafından potansiyelin benzersiz olarak belirlenmesi için gerekli ve yeterli koşullar araştırılır. Ters problemde, problemin potansiyelini, dolayısıyla dizinin yoğunluğunu tanımlamak bir spektrumla mümkündür. Bu durum dikkate alınarak Banach uzayında Leray-Schauder sabit nokta teoremi kullanılarak problemin varlığı ve tekliği sonuçları ispatlanmıştır.

Anahtar Kelimeler : Sturm-Liouville problemi, sınır koşulları, Leray–Schauder, özdeğer, spektrum

1. Introduction

The eigenvalues and structures of operators with a discrete spectrum in quantum mechanics are interesting. The spectral theory, especially for second-order operators, is called the Sturm-Liouville theory. One of the most important recent studies in this field belongs to Birkoff, examines the properties and eigenvalues of ordinary differential operators in a finite range depending on boundary conditions. The first spectral studies on singular operators were made by H. Weyl. Later studies in this field were carried out by F. Riesz, J. Von Neumann, the general spectral theory of symmetric and self-congruent operators belongs to Friedrichs and other scientists.

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Spectral theory of differential operators in different singular cases, asymptotics of eigenvalues, eigenfunctions and their completeness have been studied by researchers such as Courant, Salamyak, Birman, Maslov, and Keldish.

In general, a relationship is established between the solutions of two different Sturm-Liouville equations. Levitan is one of the first founders of the transformation operator in this field. The transformation operator for any Sturm-Liouville equation is worked out by Povzner. He used these operators in his inverse problem theory.

Boundary value problems for nonlinear equations have become very interesting in recent years. In these problems, many methods have been used, mainly topological crossover [1], Apart from these, upper solution method [2], The Lyapunov-Schmidt procedure for O-epi maps and the continum theory [3], are the main subjects studied. Marchenko [4] and Carlson [5] obtained two spectra with potential and boundary conditions certain. Levitan [6], Gasymov [7] and Zhornitskaya [8] also performed similar studies, demonstrating the uniqueness of the solution with potential boundary conditions and two spectra. The results found were supported by Borg [9] and Hochstadt [10], and the existence and uniqueness of these problems and the availability of solutions have been further improved by different methods with recent studies [11]- [18]. According to the results obtained, the smallest of the eigenvalues from the spectra can be ignored. Given the boundary conditions and a single spectrum, it is possible to obtain the potential provided the function is in the middle of the range. In the theory of differential equations, it is an important issue to investigate under which conditions the equation has a solution and is unique without solving the equation. Existence and uniqueness theorems have been developed for such problems. If the existence and uniqueness of the solution of the equation are known, a solution can be revealed with appropriate solution methods. Therefore, knowing the existence and uniqueness of the solution of an equation is more important than solving the equation. While investigating the existence and uniqueness of the solutions of equations, a system of integral equations of a certain type emerges. The existence and uniqueness of the solution of this system is determined with the help of fixed point theorems is being examined. Therefore Leray-Schauder fixed point theorem in Banach space take an important place in practice. Now, in this study, we will take the last case and examine a supportive result in a different way. Although some of the results to be obtained are similar to others, existence has been proven with a newer technique using the method applied by Leray Schauder.

In spectral theory, inverse problems appear as follows, eigenvalue sequences and norming constants or spectrum are used to calculate the potential. It has been shown that when a single spectrum is taken, necessary and sufficient conditions are created for it, thereby uniquely determining the potential and thus the intensity of the array.

Now let's take the problem,

$$-y'' + (q(t) - \lambda)y = f(t, y(t)), \qquad 0 \le \mathrm{t} \le L$$

$$y(0) = y(L),\ y'(0) = y'(L).$$

$q(t)$, let be a continuous real function satisfying the condition $q(L-t) = q(t)$.

Let $L > 0$ and $\varepsilon = \{y: y \in C^1(0,L) \cap C^2(0,L),\ y(t) > 0\ for\ t \in (0,L)\}$. Here $\lambda \ge 0$ is a constant and $f(t, y(t)) \ge 0$ on a suitable subset of $(0,L) \times (0,\infty) \times R$.

$f\colon (0,L)xR \to R$ is a $\phi \in L^1$-Caratheodory function, for every $R > 0$ there are $\phi\ L^1\ (0,L)$ and

$$|\ f(t,x)\ | \le \phi(t)\ ,$$

for $t \in (0, L)$ and all $x \epsilon R$, with $|x| \leq R$.

Here, $C(0, L)$ denotes the continuous subspace of $AC(0, L)$ absolute continuous functions.

The norm of $y \in C(0, L)$ is

$$\|y\|_0 = sup|y(t)|.$$

If we consider the point partial ordering defined in the $C(0, L)$ space, we determine the interval as follows

$$[y, w] = \{y \in C(0, L): \mathrm{v} \leq \mathrm{y} \leq \mathrm{w}\}.$$

Let $f(t)$ be a vector function with period L. Using the definition of Euclidean norms for vectors and matrices, let's define two norms for this function as follows,

$$\|f\|_q = \left[\frac{1}{L}\int_0^L \|f(t)\|^2\, dt\right]^{\frac{1}{2}} \text{ and } \|f\|_n = max\|f(t)\| .$$

Let's take $y \in AC(0, L)$ as a solution of (1), with $y: C(0, L) \to R$.

2. Materials and Methods

In this study, the existence and uniqueness of the solutions in the case of a single spectrum of the Sturm-Liouville equation were examined using the Leray-Schauder method.

2.1. Statement of the Main Results

Basic Existence Theory

Let $\lambda \in R$, $F: (0, L)xR \to R$ a L^1-Caratheodory function and let's deal with the problem

$$-y'' + (q(t) - \lambda)y = F(t, y(t)), \qquad 0 \leq t \leq L$$

$$y(0) = y(L), \quad y'(0) = y'(L), \tag{1}$$

and $q(L - t) = q(t)$ that evidently if $F(t, y) = f(t, y) + \lambda y$ and y is a solution to (1) then y is a solution to problem. It is also $y \in C(0, L)$ for $y = Ay$.

$A = C(0, L) \to C(0, L)$, the following equation is written

$$(Ay)(t) = \int_0^L G(t, s)F\big(s, y(s)\big)ds, \tag{2}$$

where G is the Green's function

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T-t+s)}}{e^{\lambda T}+1}, & 0 \leq s \leq t \leq L, \\ \frac{e^{\lambda(-t+s)}}{e^{\lambda T}+1}, & 0 \leq t \leq s \leq L . \end{cases}$$

Problem (1) is linear for each $\lambda \in R$ and has a solution, and the y function satisfies the equation (2). Now we construct two nonlinear existence principles for (1) while applying the Leray-Schauder alternative.

Let us explain the principle of the Leray-Schauder alternative with the that theorem.

Theorem 2.1. Let C be the locally convex and fully convex subset of the topological space E and U

the open subset of C with $p \in U$. Also, let $F: \overline{U} \to C$ be a continuous, compact transformation, in which case either
A1) F contains a fixed point at U^{-}, or,
A2) With $y = \mu F(y) + (1-\mu)p$, there is a a $y \in \partial U$ and $\mu \epsilon (0, L)$.

Theorem 2.2. Let M be a constant, $\|u\|_0 \neq M$, independent of μ and for any solution function $y \in AC(0, L)$ to

$$\begin{gathered} -y'' + (q(t) - \lambda)y = \mu F(t, y(t)), \qquad t \in (0, L) \\ y(0) = y(L), \qquad y'(0) = y'(L). \end{gathered} \tag{3}$$

In this case, problem (3) has at least one solution in $AC(0, L)$. for every $\mu\, \epsilon (0, L)$.

Proof. A function $y \in C(0, L)$ is a solution to (3) if and only if $y = \mu A y$ where A is defined in (2),

$$(Ay)(t) = e^{-\lambda t} \int_0^L e^{\lambda s} F\big(s, y(s)\big) ds + \frac{e^{-\lambda L}}{1 + e^{-\lambda L}} e^{-\lambda t} \int_0^L e^{\lambda s} F\big(s, y(s)\big) ds$$

Since F is L^1-Caratheodory the continuity of A is easily shown.
Let's take $U = \{y \in C(0, L): \|y\|_0 < M\}$, $C = E = C(0, L)$, $p = 0$ and apply Theorem 2.1.
Theorem 2.3. Let $\varphi: [0, \infty) \to (0, \infty)$ be a continuous and undiminished function and $q \in L^1(0, L)$ are provided for $t \in C(0, L)$ and all $y \in R$,

$$\big|F(t, y)\big| \leq q(t)\varphi(|y|).$$

In addition suppose that

$$sup \frac{c}{\varphi(c)} > k_0\,, \tag{4}$$

with

$$k_0 = sup \int_0^L |g(t, s)|\, q(s) ds.$$

So (1) has at least one solution at $AC(0, L)$.
Proof. From (4) there exists $M > 0$ with

$$\frac{M}{\varphi(M)} > k_0\,, \tag{5}$$

for $\mu \in$ (0,1) . Let y be a solution of (3). Then for $t \in C(0, L)$,

$$y(t) = \mu \int_0^L g(t, s) F\big(s, y(s)\big) ds,$$

and so

$$\begin{aligned} |y(t)| &\leq \mu \int_0^L g(t, s) F\big(s, y(s)\big) ds \leq \int_0^L |g(t, s)| q(s) \varphi(|y(s)|) ds, \\ &\leq \varphi(\|y\|_0) \int_0^L |g(t, s)| q(s) ds. \end{aligned}$$

Consequently, $\|y\|_0 \leq k_0\, is\, \varphi(\|y\|_0)$ and $\|y\|_0 \neq M$ from (5). We will now show using Theorem 2 that (1) has a solution in $AC(0, L)$.
Lemma 2.4. Let us be the product of two real or complex solutions of the equation $y'' + \lambda y = qy$ with q in H^1.

$$\langle u, v\rangle = \int_0^T u(t)v(t)dt \text{ and let } D = \frac{d}{dt}.$$

Further, let

$$H_0^m = \{u \in H^m: [u] = 0\}.$$

Let $q \in H^1$ and let f and g be two solitions of $y'' + \lambda y = qy$ the product of these solutions is either L-periodic, or disappears at L. Thus,

$$2\lambda\langle fg,h\rangle = \langle fg,Ph\rangle,$$

$$P = -\frac{1}{2}D^2 + 2q + q'I \text{ being } h \in H_0^1, \text{ where } Ih = \int_0^1 h(t)dt.$$

Remark. The following equation also applies to $h \in H_0^2$,

$$\langle fg,Ph\rangle = \frac{1}{2}\langle (fg)',h'\rangle + \langle fg, 2qh + q'Ih\rangle.$$

Proof. The two solutions of $y'' + \lambda y = qy, f$ and g give the following equation

$$L(fg) = 2\lambda D(fg),$$

where are $H = -\frac{1}{2}D^3 + qD + Dq$. Hence,

$$2\lambda fg = IH(fg) + c,$$

where $Iu = \int_0^1 h(t)dt.$ By matching this expression with $h \in H_0^1$, we get

$$2\lambda\langle fg,h\rangle = \langle IH(fg),h\rangle,$$

the term $\langle c,h\rangle = c[h]$ reads absent here.
If we integrate it piecewise, with $Ih|_0 = 0$ and $Ih|_1 = [h]= 0$,

$$\langle IH(fg),h\rangle = -\langle H(fg),Ih\rangle,$$

$$= \frac{1}{2}((fg)h' - (fg)'h)|_0^L + \langle fg, HIh\rangle.$$

Since fg is L-periodic, h is also L-periodic, if the boundary terms disappear and g disappears at the lower and upper bounds, then

$$((fg)'h - (fg)h')|_0^L = fg'h|_0^L = (fg' - f'g)h|_0^L,$$

$fg' - f'g$ since the Wronskian value of g is constant, the last term also disappears. So either way,

$$\langle IH(fg),h\rangle = \langle fg, HIh\rangle,$$

it turns out that $HI = P$.

Since γ_n must not disappear to be differentiable,

$$\delta_{\gamma_n} = \begin{cases} \partial\gamma_n & \text{when } \gamma_n \neq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Theorem 2.5. Let's take

$$F(\alpha) = 0\,. \qquad (6)$$

Let α and $F(\alpha)$ be vectors of the same size, and $F(\alpha)$, Ω form a system of equations corresponding to a continuously differentiable function of α. Let's take a solution of (6) $\alpha = \bar{\alpha}$ so that the determinant of the $J(\alpha)$ Jacobian matrix of F(α) does not disappear at $\alpha = \bar{\alpha}$, δ being a positive constant and $\kappa < 1$a non-negative constant,

(i) $\Omega_\delta = \{\alpha | \|\alpha - \bar{\alpha}\| \leqq \delta\} \subset \Omega$,

(ii) $\|J(\alpha) - J(\bar{\alpha})\| \leqq \frac{\kappa}{M'}$ for any $\alpha \in \Omega_\delta$, (7)

(iii) $\frac{M'r}{1-\kappa} \leqq \delta$,

where r and $M'(> 0)$ are numbers such that

$$\|F(\hat{\alpha})\| \leqq r \text{ and } \|J^{-1}(\hat{\alpha})\| \leqq M'. \qquad (8)$$

So system (6) has only one solution $\alpha = \bar{\alpha}$ in the Ω_δ region, and

$$\|\alpha - \bar{\alpha}\| \leqq \frac{M'r}{1-\kappa}. \qquad (9)$$

Proof. Let's apply Newton's sequential method, taking $A = J^{-1}(\hat{\alpha})$,

$$\alpha_{n+1} = \alpha_n - A\,F(\alpha_n) \quad (n = 0,1,2,\ldots.) \qquad (10)$$

where $\alpha_0 = \hat{\alpha}$.

In the first place, this sequential process can be continued indefinitely and

$$\| \ \alpha_{n+1} - \alpha_n \| \leqq \ \kappa^n \| \ \alpha_1 - \alpha_0 \|, \tag{11}$$

$$\alpha_{n+1} \in \Omega_\delta, \qquad (n = 0,1,2, \dots.) \tag{12}$$

for $n = 0$, (11) is evident. For α_1, we have succesively

$$\| \ \alpha_1 - \alpha_0 \| = \ \|A \, F(\alpha_0)\| \leqq \ M'r,$$
$$\leqq (1-\kappa)\delta < \delta, \tag{13}$$

and consequently $\alpha_1 \in \Omega_\delta$. This proves (12) for $n = 0$.

Let us assume that (11) and (12) hold up to $n - 1$. Then form (11) we have

$$\alpha_{n+1} - \alpha_n = (\alpha_n - \ \alpha_{n-1}) - A[F(\alpha_n) - F(\alpha_{n-1})],$$
$$= A \int_0^1 \{J(\alpha_0) - J[\alpha_{n-1} + \vartheta(\alpha_n - \ \alpha_{n-1})]\}.(\alpha_n - \ \alpha_{n-1})d\vartheta.$$

Here $\alpha_{n-1} + \vartheta(\alpha_n - \ \alpha_{n-1}) \in \ \Omega_\delta$ $(0 \leqq \vartheta \leqq 1)$ since $(\alpha_n, \ \alpha_{n-1}) \in \ \Omega_\delta$ by the assumption. The by (ii) of (7), we have

$$\| \ \alpha_{n+1} - \alpha_n \| \leqq M'.\frac{\kappa}{M'} \| \ \alpha_n - \alpha_{n-1} \| = \kappa \| \ \alpha_n - \alpha_{n-1} \|, \tag{14}$$

which proves (11) for n because

$$\| \ \alpha_n - \alpha_{n-1} \| \leqq \ \kappa^{n-1} \| \ \alpha_1 - \alpha_0 \|,$$

by the assumption. Since

$$\| \ \alpha_{n+1} - \alpha_0 \| \leqq \| \ \alpha_{n+1} - \alpha_n \| + \| \ \alpha_n - \alpha_{n-1} \| + \cdots + \| \ \alpha_1 - \alpha_0 \|,$$

it follows from (11) and (12) that

$$\| \ \alpha_{n+1} - \alpha_0 \| \leqq (\kappa^n + \ \kappa^{n-1} + \cdots \ \kappa + 1) \ \| \ \alpha_1 - \alpha_0 \|,$$
$$\leqq \ \frac{M'r}{1-\kappa} \leqq \ \delta, \tag{15}$$

which proves (12) for n.

It is seen that the steps can continue indefinitely in $\Omega_\delta \subset \Omega$ with the repetition rule in equation (10) and the use of (11) and (12).

Continuing the sequential operation results in an infinite and convergent sequence $\{\alpha_n\}$ in Ω_δ because $| \ \kappa| < 1$. Let

$$\bar{\alpha} = \lim_{n \to \infty} \ \alpha_n.$$

the resulting $\bar{\alpha}$ is a solution of obvious equation (6). The inequality (9) can also be easily obtained from (15). To complete the proof, let us show the uniqueness of the final solution. Let $\bar{\alpha}'$ be another solution of (6) in the region.Then,

$$\bar{\alpha} = \bar{\alpha} - AF(\bar{\alpha}),$$
$$\overline{\alpha'} = \bar{\alpha}' - AF\left(\overline{\alpha'}\right),$$

analogously to (14) we have

$$\left\|\bar{\alpha} - \overline{\alpha'}\right\| \leqq \kappa \|\bar{\alpha} - \overline{\alpha'}\|,$$

which implies

$$\|\bar{\alpha} - \overline{\alpha'}\| = 0, \tag{16}$$

$0 \leqq \ \kappa < 1$ and equality of (16) indicates that the solution is unique.

3. Results and Discussion

Firstly, the Hilbert space boundary value problem, which consists of the Sturm-Liouville equation and boundary conditions, the general properties of the operator suitable for the problem are introduced, the asymptotics of the solution functions and their characteristics are obtained by using

Green's function. The stability and uniqueness of the solutions are demonstrated by Leray-Schauder. The following problems are considered for differential operators; It is possible to define the operator according to which spectral data. For Sturm-Liouville operators, the transformative role of the operator in inverse problem theory is important. According to the given conditions, the smallest eigenvalue of the spectra may not be taken. Given the boundary conditions and a single spectrum, it is possible to obtain the potential provided the function is in the middle of the range. Considering this situation in the study and a supportive result is shown by a different method.
Although some of the results obtained are similar to others, the existence has been proven with a newer technique using the method applied by Leray Schauder. In inverse problem theory, it is shown that while the solution of the inverse problem is based on two spectrums, the solution potential is obtained with a single spectrum.

4. Conclusions

In general, it is known that in regular or singular problems, the potential function can be determined uniformly with the help of two eigenvalue sequences. The quasi-inverse spectral problem involves reconstructing the operator over the entire range in an operational situation where the spectrum and potential are known in the half range. If the function $q(x)$ is known in the given range, the function with only the eigenvalue sequence, the function $q(x)$ can be determined uniquely over the entire range. In spectral theory, the case for inverse problems of Sturm-Liouville problems given with boundary conditions is as follows, eigenvalue sequences to calculate the potential and norming constants or spectrum are used. It has been shown that when a single spectrum is taken, necessary and sufficient conditions are created for it, thereby uniquely determining the potential and thus the intensity of the array. For the inverse problem, it has been shown that a single spectrum is effective in determining the potential. In addition, necessary conditions are given for the Sturm-Liouville problem to have a spectrum with real value potential. The existence has been proven with a newer technique using the method implemented by Leray Schauder. The results obtained gave a different perspective to the evidence in the literature.

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