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Brinkman-Forchheimer denklemlerinin Darcy Katsayısına sürekli bağımlılığı üzerine

Orkun Yeşilova [1] ve Ülkü Dinlemez [2]

[1] *Milli Eğitim Bakanlığı, Çubuk Anadolu Öğretmen Lisesi, Ankara,*

[2] *Gazi Üniversitesi, Fen Fakültesi, Matematik Bölümü, Ankara,*

orkunyesilova@hotmail.com

Özet

Bu çalışmada; Brinkman-Forchheimer Denklemlerinin çözümlerinin H^1 normunda a Darcy Katsayısına sürekli bağımlılığı gösterilmiştir.

Anahtar Kelimeler : Brinkman-Forchheimer Denklemleri, sürekli bağımlılığı, Darcy Katsayısı.

On continuous dependence on Darcy coefficient of the Brinkman-Forchheimer equations

Abstract

In this paper the continuous dependence of solutions of the Brinkman-Forchheimer equations on the Darcy coefficient in H^1 norm is proved.

Keywords : Brinkman-Forchheimer equations, continuous dependence on the coefficient, Darcy coefficient.

1. Introduction

In this paper we consider the following initial-boundary value problem for the Brinkman- Forchheimer equations:

$$u_t = \gamma\Delta u - au - b|u|^{\alpha} u - \nabla p, \qquad \nabla . u = 0, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u = 0, \qquad x \in \partial\Omega, \quad t > 0 \tag{1.3}$$

Here $u = (u_1, u_2, u_3)$ is the fluid velocity vector, $\gamma > 0$ is the Brinkman coefficient, $a > 0$ is the Darcy coefficient, $b > 0$ is the Forchheimer coefficient, p is the pressure, $\alpha \in [1,2]$ is a given number, Ω is a bounded domain of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is assumed to be of class C^2. We study the problem of continuous dependence of solutions to the problem $(1.1)-(1.3)$ on coefficient a.

This type works can be found the articles of Ames ve Straughan $[1]$, Ames ve Payne $[2]$. They investigated structural stability in flows of fluid in porous media represented by the Darcy and Brinkman systems. Çelebi, Kalantarov and Uğurlu [3-4] proved continuous dependence of solutions of the Brinkman- Forchheimer coefficients in H^1 norm and Dinlemez [5] worked the structural stability for a class of nonlinear wave equations. Continuous dependence of solutions on coefficients of equations reflects the effect of small changes in coefficients of equations on the solutions. Our purpose is to present continuous dependence on Darcy coefficient in H^1 norm. Throughout this paper we will use the function spaces $\widetilde{H}_0^1(\Omega,\mathbb{R}^3)=\{\, u\in H_0^1(\Omega,\mathbb{R}^3):\ \nabla.u=0 \,\}$ and $\widetilde{L}^2(\Omega,\mathbb{R}^3)$, where $\widetilde{L}^2(\Omega,\mathbb{R}^3)=\overline{\widetilde{H}_0^1(\Omega,\mathbb{R}^3)}\subset L^2(\Omega,\mathbb{R}^3)$.

For convenience we will write $\widetilde{L}^2(\Omega,\mathbb{R}^3)=\widetilde{L}^2(\Omega)$ and $\widetilde{H}_0^1(\Omega,\mathbb{R}^3)=\widetilde{H}_0^1(\Omega)$. We define $\|u\|_p=\left(\int_\Omega u^p(x)\,dx\right)^{\frac{1}{p}}$ for the norm in $L^p(\Omega)$ where $1<p<\infty$.

2. Contınuous Dependence On The Darcy Coefficient

In this section we are going to prove that the solution of the problem (1,1)-(1,3) depends continuous on the Darcy coefficient a in $H^1(\Omega)$ norm. The following existence and uniqueness theorem for the problem (1,1)-(1,3) can be found in [4].

Theorem 2.1.

Assume that $1\le\alpha\le 2$. Then for any $u_0\in\widetilde{H}_0^1(\Omega)$, there exists a unique solution $u\in C\left([0,T];\ \widetilde{H}_0^1(\Omega)\right)$ of the problem (1.1) – (1.3). Furthermore, we have

$$\sup_{0\le t\le T}\|\nabla u(t)\|\le D \qquad \text{and} \qquad \int_0^T \|u_t(t)\|^2\,dt\le D \tag{E_1}$$

for any $T>0$, where D is a generic positive constant depending on the initial data and the parameters of (1,1).

Now assume that (u,p) is the solution of the problem

$$u_t=\gamma\Delta u-a_1u-b|u|^\alpha u-\nabla p,\qquad \nabla.u=0,\quad x\in\Omega,\quad t>0, \tag{2.1}$$

$$u(x,0)=u_0(x),\qquad x\in\Omega, \tag{2.2}$$

$$u=0,\qquad x\in\partial\Omega,\ t>0, \tag{2.3}$$

and (v,q) is the solution of the problem

$$v_t=\gamma\Delta v-a_2v-b|v|^\alpha v-\nabla q,\qquad \nabla.u=0,\quad x\in\Omega,\quad t>0, \tag{2.4}$$

$$v(x,0)=u_0(x),\qquad x\in\Omega, \tag{2.5}$$

$$v=0,\qquad x\in\partial\Omega,\ t>0. \tag{2.6}$$

Let $w=u-v$, $\pi=p-q$ and $\hat{a}=a_1-a_2$. Then (w,π) is a solution of the problem,

$$w_t=\gamma\Delta w-a_1w-\hat{a}v-b\left(|u|^\alpha u-|v|^\alpha v\right)-\nabla\pi,\qquad \nabla w=0,\quad x\in\Omega,\quad t>0 \tag{2.7}$$

$$w(x,0)=0,\qquad x\in\Omega, \tag{2.8}$$

$$w = 0 \quad , \quad x \in \partial\Omega , \ t > 0 \tag{2.9}$$

The main result of this paper is the following theorem:

Theorem 2.2.

Let w be the solution of the problem $(2.7)-(2.9)$ and $\frac{2\gamma}{a_1} > 1$. Then w satisfies the estimate,

$$\| \nabla w(t) \|^2 + \| w(t) \|^2 \leq L(a_1 - a_2)^2 \quad , \quad \forall t > 0 , \tag{2.10}$$

where L is a positive constant depending on the parameters of (1.1).

Proof :

Multiplying (2.7) by w in $L^2(\Omega)$ we get

$$\frac{1}{2}\frac{d}{dt} \| w(t) \|^2 + \gamma \| \nabla w(t) \|^2 + a_1 \| w(t) \|^2 = -\hat{a}\left(v(t) , w(t)\right) - b\left(| u(t) |^\alpha u(t) - | v(t) |^\alpha v(t) , w(t)\right). \tag{2.11}$$

Since the operator $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(u) = | u |^\alpha u$ is monotone, we have $\left(| u(t) |^\alpha u(t) - | v(t) |^\alpha v(t) , w(t)\right) \geq 0$.

Hence, from (2.11) we get

$$\Rightarrow \frac{1}{2}\frac{d}{dt} \| w(t) \|^2 + \gamma \| \nabla w(t) \|^2 + a_1 \| w(t) \|^2 \leq \left| \hat{a} \left(v(t) , w(t)\right) \right| ". \tag{2.12}$$

Using Young's inequality in (2.12) we have

$$\frac{1}{2}\frac{d}{dt} \| w(t) \|^2 + \gamma \| \nabla w(t) \|^2 + a_1 \| w(t) \|^2 \leq \frac{\hat{a}^2}{a_1} \| u(t) \|^2 + \frac{a_1}{2} \| w(t) \|^2 . \tag{2.13}$$

By Poincare-Friedrichs inequality,

$$\Rightarrow \frac{d}{dt} \| w(t) \|^2 + 2\gamma \| \nabla w \|^2 + a_1 \| w \|^2 \leq \frac{2\hat{a}^2}{a_1} b_0^2 \| \nabla u \|^2 \tag{2.14}$$

where b_0 is a positive constant.

Multiplication of (2.7) with w_t in $L^2(\Omega)$ we obtain

$$\frac{1}{2}\frac{d}{dt} \left\{ \gamma \| w_x(t) \|^2 + a_1 \| w(t) \|^2 \right\} + \| w_t(t) \|^2 = -\hat{a} \ (v(t) , w_t(t)) - b\left(| u(t) |^\alpha u(t) - | v(t) |^\alpha v(t) , w_t(t)\right) \tag{2.15}$$

Now we will estimate the following inequaity. Using the mean value theorem and Hölder's inequality respectively we obtain

$$\left| b \left(| u |^\alpha u - | v |^\alpha v , w_t\right) \right| \leq | b | 3\alpha \left(\| u \|_{3\alpha}^\alpha + \| v \|_{3\alpha}^\alpha \right) . \| w \|_6 . \| w_t \| . \tag{2.16}$$

Hence we use Sobolev inequality , we get

$$\left| b \left(| u |^\alpha u - | v |^\alpha v , w_t\right) \right| \leq | b | 3\alpha \, d_0^{\alpha+1} \left(\| \nabla u \|^\alpha + \| \nabla v \|^\alpha \right) . \| \nabla w \| . \| w_t \| ,$$

where d_0 is the constant in the Sobolev inequality

$$\| v \|_p \leq d_0 \| \nabla v \| , \qquad 1 \leq p \leq 6 \tag{E_2}$$

which is valid for each $v \in H_0^1\,(\Omega,\mathbb{R}^3)$. Then from *Theorem 2.1* and using Young inequality we get

$$\Rightarrow \left| b\left(|u|^{\alpha} u - |v|^{\alpha} v \,,\, w_t \right) \right| \le \frac{1}{2} \| w_t \|^2 + 18 b^2 \alpha^2 d_0^{2\alpha+2} D^{2\alpha} \| \nabla w \|^2 . \tag{2.17}$$

Let use Cauchy-Schwarz and Young inequalities in (2.4) we obtain

$$\Rightarrow \left| -\hat{a}\left(v \,,\, w_t \right) \right| \le \frac{1}{2} \| w_t \|^2 + \frac{3\hat{a}^2}{a_2^2} \left[\| v_t \|^2 + \gamma^2 \| \Delta v \|^2 + b^2 \| v \|_{2\alpha+2}^{2\alpha+2} \right]. \tag{2.18}$$

Then by using (E_2) and from *Theorem 2.1* in (2.18) we have

$$\left| -\hat{a}\left(v \,,\, w_t \right) \right| \le \frac{1}{2} \| w_t \|^2 + \frac{3\hat{a}^2}{a_2^2} \| v_t \|^2 + \frac{3\hat{a}^2}{a_2^2} \left[\left(d_0\, \gamma D \right)^2 + b^2 \left(d_0\, D \right)^{2\alpha+2} \right]. \tag{2.19}$$

Therefore using (2.17) and (2.19) in (2.15) we get

$$\Rightarrow \frac{d}{dt} \left[\gamma \| \nabla w \|^2 + a \| w \|^2 \right] \le K_4\, \hat{a}^2 + \frac{6\hat{a}^2}{a_2^2} \| v_t \|^2 + L_0 \| \nabla w \|^2 \tag{2.20}$$

where $K_4 = \frac{6}{a_2^2} \left[\left(d_0\, \gamma D \right)^2 + b^2 \left(d_0\, D \right)^{2\alpha+2} \right]$ and $L_0 = 18 b^2 \alpha^2 d_0^{2\alpha+2} D^{2\alpha}$.

If we multiply (2.14) by $\tau = \frac{L_0}{a_1}$ we have

$$\tau \frac{d}{dt} \| w \|^2 + 2\gamma\tau \| \nabla w \|^2 + L_0 \| w \|^2 \le \tau \frac{2\hat{a}^2}{a_1} b_0^2 \| \nabla u \|^2 . \tag{2.21}$$

Adding (2.20) and (2.21) we obtain

$$\frac{d}{dt} \left[\gamma \| \nabla w \|^2 + (\tau + a) \| w \|^2 \right] + (2\gamma\tau - L_0) \| \nabla w \|^2 + L_0 \| w \|^2 \le 2L_0 \frac{\hat{a}^2}{a_1^2} b_0^2 \| \nabla u \|^2 + K_4 \hat{a}^2 + \frac{6\hat{a}^2}{a_2^2} \| v_t \|^2 . \tag{2.22}$$

Inequality of (2.22) implies

$$Z'(t) + k_0\, Z(t) \le \hat{a}^2 \left[\frac{2L_0}{a_1^2} b_0^2 \| \nabla u \|^2 + \frac{6}{a_2^2} \| v_t \|^2 + K_4 \right] \tag{2.23}$$

where $k_0 = \min\left\{ \frac{1}{2\tau} \,,\, \frac{(\tau + a)}{L_0} \right\}$ and $Z(t) = \gamma \| \nabla w \|^2 + (\tau + a) \| w \|^2$.

Solving the differential inequality (2.23) and using *Theorem 2.1* we arrive at

$$Z(t) \le \hat{a}^2 \left(\frac{2L_0}{a_1^2} b_0^2 D + \frac{6}{a_2^2} D + \frac{K_4}{d_0} \right).$$

Therefore we have completed the proof of the theorem we have

$$\| \nabla w(t) \| \to 0 \quad \text{as} \quad \hat{a} \to 0 \,, \quad t > 0 .$$

3. References

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