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On $(p,q)-$Fibonacci Polynomials Connected with Finite Operators

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Abstract

The aim of this study is to obtain some properties of the $(p,q)-$Fibonacci finite operator polynomials by implementing the finite operator to the $(p,q)-$Fibonacci polynomials. Firstly, we obtain the Binet formula, generating function, exponential generating function, Poisson generating function, and binomial sum of $(p,q)-$Fibonacci finite operator polynomials. After that we give determinantal expressions for these finite operator polynomials. Lastly, we regain, in a different way, recurrence relation for these finite operator polynomials.

Keywords: *Binet-like formula; finite operator; generating function; $(p,q)-$Fibonacci polynomials; tridiagonal determinant.*
2010 Mathematics Subject Classification: *11B39; 11C08; 11C20; 11Y55.*

1. Introduction and Preliminaries

Recently, in [26], Şimşek introduced an operator as follows:

$$\mathbb{Y}_{\lambda,\beta}\left[f;a,b\right](x) = \lambda E^{a}\left[f\right](x) + \beta E^{b}\left[f\right](x),$$

where a,b are real parameters, λ,β are real or complex parameters. We note that $E^{a}\left[f\right](x) = f(x+a)$. Using this new operator, Şimşek defined two new classes of special polynomials and numbers. He also gave many relations between some known polynomials and number sequences. Let a and b be integers, λ and β be real parameters. For any polynomial sequence $f_n(x)$ and $i \geq 1$, ith finite operator $\mathbb{Y}^{(i)}_{\lambda,\beta}\left[f_n;a,b\right](x)$ (or briefly $f_n^{(i)}(x)$) is defined by the following relation:

$$\mathbb{Y}^{(i)}_{\lambda,\beta}\left[f_n;a,b\right](x) = f_n^{(i)}(x) = \mathbb{Y}_{\lambda,\beta}\left[f;a,b\right](x)\left(\mathbb{Y}^{(i-1)}_{\lambda,\beta}\left[f_n;a,b\right](x)\right), \tag{1.1}$$

where $\mathbb{Y}^{(1)}_{\lambda,\beta}\left[f_n;a,b\right](x) = f_n^{(1)}(x) = \lambda f_n(x+a) + \beta f_n(x+b)$.
We note that this operator generalizes some well known operators such as the identitiy operator, the forward difference operator, the backward difference operator, the means operator, and the Gould operator. These special cases of the finite operator are given respectively as in Table 1.

Table 1: Some particular cases of new finite operator

λ	β	a	b	Operator
1	0	0	0	$\mathbb{Y}_{1,0}\left[f;0,0\right](x) = I(f(x)) = f(x)$
1	-1	1	0	$\mathbb{Y}_{1,-1}\left[f;1,0\right](x) = \Delta(f(x)) = f(x+1) - f(x)$
1	-1	0	-1	$\mathbb{Y}_{1,-1}\left[f;0,-1\right](x) = \bigtriangledown(f(x)) = f(x) - f(x-1)$
$\frac{1}{2}$	$-\frac{1}{2}$	1	0	$\mathbb{Y}_{\frac{1}{2},-\frac{1}{2}}\left[f;1,0\right](x) = M(f(x)) = \frac{1}{2}\left(f(x+1) - f(x)\right)$
1	-1	$a \to a+b$	$b \to a$	$\mathbb{Y}_{1,-1}\left[f;a+b,a\right](x) = G_{ab}(f(x)) = f(x+a+b) - f(x+a)$

We remark that these operators have lots of applications in engineering, physics, and applied mathematics. In addition, many researchers in different fields frequently use finite operators in their calculations. Please see [26, 27] for more detail.

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Table 2: Some special cases of $(p,q)-$Fibonacci polynomials

$\phi_0(x)$	$\phi_1(x)$	$p(x)$	$q(x)$	**Polynomial**
0	1	x	1	Fibonacci polynomial, $F_n(x)$
0	1	$2x$	1	Pell polynomial, $P_n(x)$
0	1	$3x$	-2	Fermat polynomial, $\Phi_n(x)$
0	1	$2x$	-1	Chebyshev second kind polynomial, $U_n(x)$
0	1	1	$2x$	Jacobsthal polynomial, $J_n(x)$
0	1	$x+2$	-1	Morgan-Voyce polynomial, $\mathbb{B}_n(x)$
0	1	x	-1	Vieta polynomial, $V_n(x)$
0	1	$6x$	-1	Balancing polynomial, $B_n(x)$

In [18], Lee and Aşçı defined and examined a generalization for the Fibonacci polynomials named $(p,q)-$Fibonacci polynomials. The authors defined these polynomials as follows:

$$F_{p,q,n+1}(x)=p(x)F_{p,q,n}(x)+q(x)F_{p,q,n-1}(x), \quad (n\geq 1)$$

where $F_{p,q,0}(x)=0$, $F_{p,q,1}(x)=1$. Here $p(x)$ and $q(x)$ are the polynomials with real coefficients. After that Wang [29] derived some interesting combinatorial properties of these polynomials by using elementary methods and techniques. For further information connected to Fibonacci polynomials and their generalizations, please see [12, 13, 17, 20, 21, 24, 25]. Some special cases of $(p,q)-$Fibonacci polynomials are given as in Table 2.

Recently, in [28], Terzioğlu et al. have defined a new family of quaternions whose components are the Fibonacci finite operator numbers and have provided some properties of these types of quaternions.

We indicate that we will use the symbol $\phi_n(x)$ instead of $F_{p,q,n}(x)$ throughout this paper for simplicity.

In this study, inspired by the above papers, we obtain some properties of the $(p,q)-$ Fibonacci finite operator polynomials. Firstly, we give the Binet formula, generating function, exponential generating function, Poisson generating function, and binomial sum of $(p,q)-$Fibonacci finite operator polynomials. After that we give determinantal expressions of these finite operator polynomials. Lastly, we regain, by taking a different tack, recurrence relation for these finite operator polynomials given in Theorem 2.1.

2. Main Results

In this section, we apply the finite operator to $(p,q)-$Fibonacci polynomials and named these polynomials as $(p,q)-$Fibonacci finite operator polynomials.

Now we start to apply the finite operator to $(p,q)-$Fibonacci polynomials. Using the eq. (1.1), we obtain

$$\mathbb{Y}^{(1)}_{\lambda,\beta}\left[\phi_n;a,b\right](x)=\phi_n^{(1)}(x)=\lambda\phi_n(x+a)+\beta\phi_n(x+b),$$

where $\phi_n^{(1)}(x)$ is the first finite operator of $\phi_n(x)$. At his point one can obtain the special cases of the finite operator $\phi_n^{(1)}(x)$ by taking the special values in Table 1 such as $(p,q)-$Fibonacci identity operator polynomial, $(p,q)-$Fibonacci forward difference polynomial, $(p,q)-$Fibonacci backward difference polynomial, $(p,q)-$Fibonacci means operator polynomial, $(p,q)-$Fibonacci-Horadam operator polynomial respectively.

If we consider the eq. (1.1) again, the second finite operator of $\phi_n(x)$ is given by

$$\mathbb{Y}^{(2)}_{\lambda,\beta}\left[\phi_n;a,b\right](x)=\phi_n^{(2)}(x)=\lambda^2\phi_n(x+2a)+2\lambda\beta\phi_n(x+a+b)+\beta^2\phi_n(x+2b).$$

By continuing in a similar way, we obtain the ith finite operator of $\phi_n(x)$, termed $(p,q)-$Fibonacci finite operator polynomials as follows:

$$\mathbb{Y}^{(i)}_{\lambda,\beta}\left[\phi_n;a,b\right](x)=\phi_n^{(i)}(x)=\lambda\mathbb{Y}^{(i-1)}_{\lambda,\beta}\left[\phi_n;a,b\right](x+a)+\beta\mathbb{Y}^{(i-1)}_{\lambda,\beta}\left[\phi_n;a,b\right](x+b)$$

or

$$\mathbb{Y}^{(i)}_{\lambda,\beta}\left[\phi_n;a,b\right](x)=\phi_n^{(i)}(x)=\sum_{k=0}^{i}\binom{i}{k}\lambda^{i-k}\beta^k\phi_n\left(x+bk+(i-k)a\right).$$

Now, we present our main results. Firstly we give the recurence relation satisfied by the sequence $\phi_n^{(i)}(x)$.

Theorem 2.1. *The following recurrence relation hold for $(p,q)-$Fibonacci finite operator polynomials:*

$$\phi_{n+1}^{(i)}(x)=p(x)\phi_n^{(i)}(x)+q(x)\phi_{n-1}^{(i)}(x), \quad (n\geq 1) \tag{2.1}$$

with the initial values $\phi_0^{(i)}(x)$ and $\phi_1^{(i)}(x)$.

Proof. The claim can be demonstrated by induction on i. □

The following theorem gives the Binet-like formula of the sequence $\phi_n^{(i)}(x)$.

Theorem 2.2. *The Binet-like formula for the $(p,q)-$Fibonacci finite operator polynomials is*

$$\phi_n^{(i)}(x)=\frac{\phi_1^{(i)}(x)\left(\lambda^n(x)-\mu^n(x)\right)+q(x)\phi_0^{(i)}(x)\left(\lambda^{n-1}(x)-\mu^{n-1}(x)\right)}{\lambda(x)-\mu(x)}. \tag{2.2}$$

Proof. From the (2.1), we obtain

$$\phi_n^{(i)}(x)=c_1(x)\lambda^n(x)+c_2(x)\mu^n(x),$$

where $\lambda(x)+\mu(x)=p(x)$, $\lambda(x)\mu(x)=-q(x)$, and $p^2(x)+4q(x)>0$.
Putting $n=0$, $n=1$, and solving the linear equations then we get

$$c_1(x)=\frac{\phi_1^{(i)}(x)-\mu(x)\phi_0^{(i)}(x)}{\lambda(x)-\mu(x)}$$

and

$$c_2(x)=\frac{\lambda(x)\phi_0^{(i)}(x)-\phi_1^{(i)}(x)}{\lambda(x)-\mu(x)}.$$

Then we lastly have

$$\phi_n^{(i)}(x)=\left(\frac{\phi_1^{(i)}(x)-\mu(x)\phi_0^{(i)}(x)}{\lambda(x)-\mu(x)}\right)\lambda^n(x)+\left(\frac{\lambda(x)\phi_0^{(i)}(x)-\phi_1^{(i)}(x)}{\lambda(x)-\mu(x)}\right)\mu^n(x)=\frac{\phi_1^{(i)}(x)\left(\lambda^n(x)-\mu^n(x)\right)+q(x)\phi_0^{(i)}(x)\left(\lambda^{n-1}(x)-\mu^{n-1}(x)\right)}{\lambda(x)-\mu(x)}$$

as desired. □

The following three theorems give the generating, exponential generating, and Poisson generating functions of the sequence $\phi_n^{(i)}(x)$.

Theorem 2.3. *The generating function of $(p,q)-$Fibonacci finite operator polynomials is*

$$\sum_{n=0}^{\infty}\phi_n^{(i)}(x)t^n=\frac{\phi_0^{(i)}(x)+\left(\phi_1^{(i)}(x)-p(x)\phi_0^{(i)}(x)\right)t}{1-p(x)t-q(x)t^2}.$$

Proof. Let $Q(x,t)=\sum_{n=0}^{\infty}\phi_n^{(i)}(x)t^n$ be the generating function for $(p,q)-$Fibonacci finite operator polynomials. Now, we consider

$$\sum_{n=0}^{\infty}\phi_n^{(i)}(x)t^n=\phi_0^{(i)}(x)+\phi_1^{(i)}(x)t+\phi_2^{(i)}(x)t^2+\cdots+\phi_n^{(i)}(x)t^n+\cdots.$$

Then we have

$$\begin{aligned}-p(x)tQ(x,t)&=-p(x)\phi_0^{(i)}(x)t-p(x)\phi_1^{(i)}(x)t^2-\cdots-p(x)\phi_{n-1}^{(i)}(x)t^n-\cdots\\-q(x)t^2Q(x,t)&=-q(x)\phi_0^{(i)}(x)t^2-q(x)\phi_1^{(i)}(x)t^3-\cdots-q(x)\phi_{n-2}^{(i)}(x)t^n-\cdots\end{aligned}$$

Therefore,

$$\begin{aligned}\left(1-p(x)t-q(x)t^2\right)Q(x,t)&=\phi_0^{(i)}(x)+\left(\phi_1^{(i)}(x)-p(x)\phi_0^{(i)}(x)\right)t\\&\quad+\sum_{n=2}^{\infty}\left(\phi_n^{(i)}(x)-p(x)\phi_{n-1}^{(i)}(x)-q(x)\phi_{n-2}^{(i)}(x)\right)t^n\\&=\phi_0^{(i)}(x)+\left(\phi_1^{(i)}(x)-p(x)\phi_0^{(i)}(x)\right)t.\end{aligned}$$

From the last equation, we obtain

$$Q(x,t)=\sum_{n=0}^{\infty}\phi_n^{(i)}(x)t^n=\frac{\phi_0^{(i)}(x)+\left(\phi_1^{(i)}(x)-p(x)\phi_0^{(i)}(x)\right)t}{1-p(x)t-q(x)t^2}.$$

Thus the proof is completed. □

Theorem 2.4. *The exponential generating function for $(p,q)-$Fibonacci finite operator polynomials is*

$$EQ(x,t)=\frac{\left(\phi_1^{(i)}(x)+\frac{q(x)\phi_0^{(i)}(x)}{\lambda(x)}\right)e^{\lambda(x)t}-\left(\phi_1^{(i)}(x)+\frac{q(x)\phi_0^{(i)}(x)}{\mu(x)}\right)e^{\mu(x)t}}{\lambda(x)-\mu(x)}.$$

Proof. Let $EQ(x,t)=\sum_{n=0}^{\infty}\phi_n^{(i)}(x)\frac{t^n}{n!}$ be the exponential generating function for $(p,q)-$Fibonacci finite operator polynomials. By using (2.2), we obtain

$$\begin{aligned} EQ(x,t) &= \sum_{n=0}^{\infty}\left(\frac{\phi_1^{(i)}(x)\left(\lambda^n(x)-\mu^n(x)\right)+q(x)\phi_0^{(i)}(x)\left(\lambda^{n-1}(x)-\mu^{n-1}(x)\right)}{\lambda(x)-\mu(x)}\right)\frac{t^n}{n!} \\ &= \frac{\phi_1^{(i)}(x)}{\lambda(x)-\mu(x)}\sum_{n=0}^{\infty}\left(\lambda^n(x)-\mu^n(x)\right)\frac{t^n}{n!}+\frac{q(x)\phi_0^{(i)}(x)}{\lambda(x)-\mu(x)}\sum_{n=0}^{\infty}\left(\lambda^{n-1}(x)-\mu^{n-1}(x)\right)\frac{t^n}{n!} \\ &= \frac{\phi_1^{(i)}(x)}{\lambda(x)-\mu(x)}\left(e^{\lambda(x)t}-e^{\mu(x)t}\right)+\frac{q(x)\phi_0^{(i)}(x)}{\lambda(x)-\mu(x)}\left(\frac{e^{\lambda(x)t}}{\lambda(x)}-\frac{e^{\mu(x)t}}{\mu(x)}\right) \\ &= \frac{\left(\phi_1^{(i)}(x)+\frac{q(x)\phi_0^{(i)}(x)}{\lambda(x)}\right)e^{\lambda(x)t}-\left(\phi_1^{(i)}(x)+\frac{q(x)\phi_0^{(i)}(x)}{\mu(x)}\right)e^{\mu(x)t}}{\lambda(x)-\mu(x)} \end{aligned}$$

as desired. □

Corollary 2.5. *The Poisson generating function of the $(p,q)-$Fibonacci finite operator polynomials is*

$$PQ(x,t)=\frac{\left(\phi_1^{(i)}(x)+\frac{q(x)\phi_0^{(i)}(x)}{\lambda(x)}\right)e^{\lambda(x)t}-\left(\phi_1^{(i)}(x)+\frac{q(x)\phi_0^{(i)}(x)}{\mu(x)}\right)e^{\mu(x)t}}{e^t\left(\lambda(x)-\mu(x)\right)}.$$

Proof. The proof follows from the relation $PQ(x,t)=e^{-t}EQ(x,t)$. □

Now, we give a binomial summation formula related to $\phi_n^{(i)}(x)$.

Theorem 2.6. *The following formula holds for $n\geq 0$:*

$$\sum_{m=0}^{n}\binom{n}{m}p^{n-m}(x)q^m(x)\phi_{n-m}^{(i)}(x)=\phi_{2n}^{(i)}(x).$$

Proof. By applying the formula (2.2), we obtain

$$\begin{aligned} &\sum_{m=0}^{n}\binom{n}{m}p^{n-m}(x)q^m(x)\left(\frac{\phi_1^{(i)}(x)\left(\lambda^{n-m}(x)-\mu^{n-m}(x)\right)+q(x)\phi_0^{(i)}(x)\left(\lambda^{n-m-1}(x)-\mu^{n-m-1}(x)\right)}{\lambda(x)-\mu(x)}\right) \\ &= \frac{\phi_1^{(i)}(x)}{\lambda(x)-\mu(x)}\left((\lambda(x)p(x)+q(x))^n-(\mu(x)p(x)+q(x))^n\right) \\ &\quad +\frac{q(x)\phi_0^{(i)}(x)}{\lambda(x)-\mu(x)}\left(\frac{(\lambda(x)p(x)+q(x))^n}{\lambda(x)}-\frac{(\mu(x)p(x)+q(x))^n}{\mu(x)}\right) \\ &= \phi_{2n}^{(i)}(x) \end{aligned}$$

as desired. □

3. Determinantal Expression of $(p,q)-$Fibonacci Finite Operator Polynomials

In this section, we deal with the determinantal expression of $(p,q)-$Fibonacci finite operator polynomials. First of all, we require the following powerful lemma to express the $(p,q)-$Fibonacci finite operator polynomials in terms of a tridiagonal determinant.

Lemma 3.1. ([4, p. 40]) Let $a(t)$ and $b(t)\neq 0$ be differentiable functions, let $A_{(n+1)\times 1}(t)$ be an $(n+1)\times 1$ matrix whose elements $a_{k,1}(t)=a^{(k-1)}(t)$ for $1\leq k\leq n+1$, let $B_{(n+1)\times n}(t)$ be an $(n+1)\times n$ matrix whose elements

$$b_{i,j}(t)=\begin{cases}\binom{i-1}{j-1}b^{(i-j)}(t), & i-j\geq 0;\\ 0, & i-j<0\end{cases}$$

for $1\leq i\leq n+1$ and $1\leq j\leq n$, and let $\left|C_{(n+1)\times(n+1)}(t)\right|$ indicate the lower Hessenberg determinant of the $(n+1)\times(n+1)$ lower Hessenberg matrix

$$C_{(n+1)\times(n+1)}(t)=\begin{bmatrix} A_{(n+1)\times 1}(t) & B_{(n+1)\times n}(t)\end{bmatrix}.$$

Then the nth derivative of $\frac{a(t)}{b(t)}$ can be calculated by

$$\frac{d^n}{dx^n}\left[\frac{a(t)}{b(t)}\right]=(-1)^n\frac{\left|C_{(n+1)\times(n+1)}(t)\right|}{b^{n+1}(t)}. \tag{3.1}$$

We note for interested readers that this lemma has been widely used in [6, 11, 14, 15, 16, 19, 22, 23].

The following theorem expresses $\phi_n^{(i)}(x)$ in terms of the determinant of tridiagonal matrices.

Theorem 3.2. *For $n \geq 0$, $(p,q)-$Fibonacci finite operator polynomials can be expressed determinantally as*

$$\phi_n^{(i)}(x) = \frac{1}{n!}\begin{vmatrix} \phi_0^{(i)}(x) & -1 & 0 & 0 & \cdots & 0 & 0 \\ \phi_1^{(i)}(x) - p(x)\phi_0^{(i)}(x) & p(x)\binom{1}{0} & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2q(x)\binom{2}{0} & p(x)\binom{2}{1} & -1 & \cdots & 0 & 0 \\ 0 & 0 & 2q(x)\binom{3}{1} & p(x)\binom{3}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & p(x)\binom{n-1}{n-2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 2q(x)\binom{n}{n-2} & p(x)\binom{n}{n-1} \end{vmatrix}. \tag{3.2}$$

Proof. Taking $a(t) = \phi_0^{(i)}(x) + \left(\phi_1^{(i)}(x) - p(x)\phi_0^{(i)}(x)\right)t$ and $b(t) = 1 - p(x)t - q(x)t^2$ in (3.1) leads to

$$\begin{aligned} & \frac{d^n}{dt^n}\left[\frac{\phi_0^{(i)}(x) + \left(\phi_1^{(i)}(x) - p(x)\phi_0^{(i)}(x)\right)t}{1 - p(x)t - q(x)t^2}\right] \\ = \; & \frac{(-1)^n}{\left(1 - p(x)t - q(x)t^2\right)^{n+1}} \times \\ & \begin{vmatrix} \phi_0^{(i)}(x) + \left(\phi_1^{(i)}(x) - p(x)\phi_0^{(i)}(x)\right)t & 1 - p(x)t - q(x)t^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \phi_1^{(i)}(x) - p(x)\phi_0^{(i)}(x) & -(p(x)+2q(x)t)\binom{1}{0} & 1 - p(x)t - q(x)t^2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -2q(x)\binom{2}{0} & -(p(x)+2q(x)t)\binom{2}{1} & 1 - p(x)t - q(x)t^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -2q(x)\binom{3}{1} & -(p(x)+2q(x)t)\binom{3}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(p(x)+2q(x)t)\binom{n-2}{n-3} & 1 - p(x)t - q(x)t^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -2q(x)\binom{n-1}{n-3} & -(p(x)+2q(x)t)\binom{n-1}{n-2} & 1 - p(x)t - q(x)t^2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -2q(x)\binom{n}{n-2} & -(p(x)+2q(x)t)\binom{n}{n-1} \end{vmatrix}. \end{aligned}$$

For $n \in \mathbb{N}$, if we let $t \to 0$ in the above equation then we obtain

$$\phi_n^{(i)}(x) = \frac{1}{n!}\lim_{t\to 0}\frac{d^n}{dt^n}\left[\frac{\phi_0^{(i)}(x)+\left(\phi_1^{(i)}(x)-p(x)\phi_0^{(i)}(x)\right)t}{1-p(x)t-q(x)t^2}\right]$$

$$= \frac{1}{n!}\begin{vmatrix} \phi_0^{(i)}(x) & -1 & 0 & 0 & \cdots & 0 & 0 \\ \phi_1^{(i)}(x)-p(x)\phi_0^{(i)}(x) & p(x)\binom{1}{0} & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2q(x)\binom{2}{0} & p(x)\binom{2}{1} & -1 & \cdots & 0 & 0 \\ 0 & 0 & 2q(x)\binom{3}{1} & p(x)\binom{3}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & p(x)\binom{n-1}{n-2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 2q(x)\binom{n}{n-2} & p(x)\binom{n}{n-1} \end{vmatrix}$$

as desired. □

Remark 3.3. After deriving the determinantal formula in Theorem 3.2, one can give different proofs for Theorem 3.2. For varied and elegant proofs see [1, 2, 3, 7, 8, 9, 10]. Additionally, several corollaries of Theorem 3.2 can be obtained by using Table 2.

4. Regaining the Recurrence of $(p,q)-$Fibonacci Finite Operator Polynomials

In this section, we regain, in an alternative way, recurrence relation for the $(p,q)-$Fibonacci finite operator polynomials given in Theorem 2.1.

Theorem 4.1. *For $n \geq 3$, the following recurrence relation holds:*

$$\phi_{n+1}^{(i)}(x) = p(x)\phi_n^{(i)}(x)+q(x)\phi_{n-1}^{(i)}(x),$$

Proof. Let $A_0 = 1$ and

$$A_n = \begin{vmatrix} s_{1,1} & s_{1,2} & 0 & \cdots & 0 & 0 \\ s_{2,1} & s_{2,2} & s_{2,3} & \cdots & 0 & 0 \\ s_{3,1} & s_{3,2} & s_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{n-2,1} & s_{n-2,2} & s_{n-2,3} & \cdots & s_{n-2,n-1} & 0 \\ s_{n-1,1} & s_{n-1,2} & s_{n-1,3} & \cdots & s_{n-1,n-1} & s_{n-1,n} \\ s_{n,1} & s_{n,2} & s_{n,3} & \cdots & s_{n,n-1} & s_{n,n} \end{vmatrix},$$

for $n \in \mathbb{N}$. In [5], Cahill et al. were demonstrated that the sequence A_n for $n \geq 0$ provides $A_1 = e_{1,1}$ and

$$A_n = \sum_{r=1}^{n}(-1)^{n-r}s_{n,r}\left(\prod_{j=r}^{n-1}s_{j,j+1}\right)A_{r-1}, \tag{4.1}$$

for $n \geq 2$ where the empty product is comprehended to be 1. If we apply the recurrence relation (4.1) to Theorem (3.2), we have

$$(n-1)!\phi_{n-1}^{(i)}(x) = 2q(x)\binom{n-1}{n-3}(n-3)!\phi_{n-3}^{(i)}(x)+p(x)\binom{n-1}{n-2}(n-2)!\phi_{n-2}^{(i)}(x) = (n-1)!\left(p(x)\phi_{n-2}^{(i)}(x)+q(x)\phi_{n-3}^{(i)}(x)\right).$$

For $n \geq 3$, it is apparent from the last equation that

$$\phi_{n+1}^{(i)}(x) = p(x)\phi_n^{(i)}(x)+q(x)\phi_{n-1}^{(i)}(x),$$

as desired. □

5. Conclusion

In this present paper, we derived several interesting formulas related to $(p,q)-$Fibonacci finite operator polynomials and gave determinantal representations of these finite operator polynomials in Section 2 and Section 3 respectively. In the last section, we regained, by taking a different tack, recurrence relation for the $(p,q)-$Fibonacci finite operator polynomials. The outcomes of this paper may potentially be used in different areas of applied sciences such as engineering and physics. We believe that researchers may find interesting connections between special polynomials by applying finite operators to special polynomials in future works.

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