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# On the unique solvability of a Cauchy problem with a fractional derivative

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**Abstract**

The unique solvability issues of the Cauchy problem with a fractional derivative is considered in the case when the coefficient at the desired function is a continuous function. The solution of the problem is found in an explicit form. The uniqueness theorem is proved. The existence theorem for a solution to the problem is proved by reducing it to a Volterra equation of the second kind with a singularity in the kernel, and the necessary and sufficient conditions for the existence of a solution to the problem are obtained.

*Keywords:* Cauchy problem fractional order differential equation regular solution Volterra integral equation of the second kind kernel singularity.
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## 1. Introduction

To date, there are a fairly large number of works devoted both to the theory of fractional calculus [31] and to its various applications [17, 22]. Fractional differential equations are valuable tools in many phenomena modeling in various fields of science and technology. In studying natural phenomena, many external factors arise, so equations with classical derivatives cannot fully describe these models. Fractional analysis has many applications in mechanics, physics and engineering sciences. Indeed, we can find many applications in the field of viscoelasticity, electrochemistry, control, porous environment, etc. [5, 6, 10, 11, 12, 23, 24, 25, 27]. Detailed description on the application of fractional calculus to various fields of science and technology at this stage are given in the monograph [32].

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The usage of a Caputo fractional derivative has some advantage over other types of fractional derivatives, because when we working with it, standard initial conditions are involved in terms of derivatives of the integer order [34, 36], which have a clear physical interpretation, for example, the initial position $y(a)$ at a point $a$ (where $y$ is an unknown function), initial velocity $y'(a)$, initial acceleration $y''(a)$ and so on. In [18] the loaded term in PDE is represented in the form of the Caputo fractional derivative, and the derivative order in the loaded term is less than the order of the differential part. It is proved that there is continuity on the right in the order of the fractional derivative. Continuity on the left is broken.

Finding analytical solutions to equations with fractional derivatives is accompanied by certain difficulties, so some researchers deal with the existence and uniqueness of the solution to the problem [2, 15, 29], while others use numerical methods to find approximate solutions [20].

Cauchy problems with a fractional derivative are considered by many authors [26], often in the case when the coefficient at the desired function is a constant [1], but the solvability issues of the problem and finding its solution in the case of a variable coefficient at the desired function have been studied rather little [19]. In [4], the problem for a nonlinear equation with a Riemann-Liouville fractional derivative is considered. Nonlinear equations with a fractional Caputo derivative were considered in the article [16], however, in the general case, the solution of the equation is not explicitly presented. In [28] fractional Cauchy problem for some nonlinear $\psi$-Caputo fractional differential equations with non local conditions are considered. In [35] there is study the regional controllability concept of a semi-linear time-fractional diffusion systems involving Caputo derivative of order $\alpha \in (0, 1)$.

In this work we focus on solving an equation

$$\partial_{0x}^{\alpha} y(x) - \psi(x) y(x) = 0,$$

where $\partial_{0x}^{\alpha}$ is a Caputo fractional derivative of an order $\alpha > 0$.

This equation has a special form of the problem from [16], however, we show that the solution of this equation can be presented explicitly, also this equation can be solved with more general assumptions about the function $y(x)$. In [13], the authors establish and prove statements about the existence and uniqueness of the equation solution in some Colombo space.

The paper is organized as follows: in Section 2 we introduce some necessary definitions and mathematical preliminaries of fractional calculus which will be needed in the forthcoming Section. In Section 2, the unique solvability of the Volterra integral equation with a singularity in the kernel is shown. The proof is carried out following [37]. For the proof, the resolvent is constructed. Then the posed Cauchy problem is equivalently reduced in to the Volterra integral equation of the same form as discussed above. After that the existence and uniqueness of the solution to the Cauchy problem is proved. The solution of the Cauchy problem is presented in an explicit form.

## 2. Preliminaries

The integral and the Riemann – Liouville derivative of an order $\alpha$ with respect to the variable $x$ with the origin at the point $x = a$ are defined as follows [30]

**Definition 2.1.** *Let $y(t) \in L_1[a, b]$. Then,*

$$D_{ax}^{\alpha} y(x) = \begin{cases} sign(x-a) \int_a^x y(t) \frac{(|x-t|)^{-\alpha-1}}{\Gamma(-\alpha)} dt, & if \quad \alpha < 0; \\ y(x), & if \quad \alpha = 0; \\ sign^n(x-a) \frac{d^n}{dx^n} \left[ D_{ax}^{\alpha-n} y(x) \right], & if \quad n-1 < \alpha \leq n,\ n \in N. \end{cases} \quad (1)$$

For practical applications, the definition of a Caputo fractional derivative is significant. It is obtained after interchanging differentiation and integration in (1). The Caputo derivative is defined as follows [30].

**Definition 2.2.** *Let $y(t) \in AC^n[a,b]$ (i.e. $y^{(n-1)}(t)$ is an absolutely continuous function). Then,*

$$\partial_{ax}^{\alpha} y(x) = \begin{cases} sign^n(x-a) D_{ax}^{\alpha-n} D^n y(x), & if \quad n-1 < \alpha \le n, n \in N; \\ y(x), & if \quad \alpha = 0, \end{cases} \tag{2}$$

*where*

$$D^n y(x) = \frac{d^n}{dx^n} y(x). \tag{3}$$

**Definition 2.3.** *Let $x \in \mathbb{R}$ and the functions $f(x)$, $g(x)$ be defined and integrable on the interval $[0, +\infty)$. Then their convolution is the function*

$$(f * g)(x) = \int_0^x f(t) g(x-t) dt. \tag{4}$$

Let's introduce the notation

$$h_\beta(x) = \frac{x^{\beta-1}}{\Gamma(\beta)}, \ \beta > 0. \tag{5}$$

Let us show the most important properties of function (5).

**Proposition 2.4.**

$$(h_c * h_d)(x) = h_{c+d}(x), \ c > 0, \ d > 0. \tag{6}$$

*Proof.*

$$(h_c * h_d)(x) = \int_0^x \frac{t^{c-1}}{\Gamma(c)} \frac{(x-t)^{d-1}}{\Gamma(d)} dt =$$
$$= \begin{bmatrix} t = xs \\ dt = xds \end{bmatrix} = \frac{x^{c+d-1}}{\Gamma(c)\Gamma(d)} \int_0^1 s^{c-1} (1-s)^{d-1} ds =$$
$$= \frac{x^{c+d-1}}{\Gamma(c)\Gamma(d)} B(c,d) = \frac{x^{c+d-1}}{\Gamma(c)\Gamma(d)} \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)} = h_{c+d}(x).$$

Here $B(c,d)$ is a Beta-function. □

**Proposition 2.5.**

$$h_c(x) h_d(x) = \frac{\Gamma(c+d-1)}{\Gamma(c)\Gamma(d)} h_{c+d-1}(x), c > 0, \ d > 0. \tag{7}$$

*Proof.*

$$h_c(x) h_d(x) = \frac{x^{c-1}}{\Gamma(c)} \frac{x^{d-1}}{\Gamma(d)} = \frac{\Gamma(c+d-1)}{\Gamma(c)\Gamma(d)} \frac{x^{c+d-2}}{\Gamma(c+d-1)} =$$
$$= \frac{\Gamma(c+d-1)}{\Gamma(c)\Gamma(d)} h_{c+d-1}(x).$$

□

**Proposition 2.6.**

$$\frac{d^n}{dx^n} h_c(x) = \begin{cases} h_{c-n}(x), & if \quad c \in (n, +\infty), \quad n \in \mathrm{N}; \\ 0, & if \quad c = k, \ k \le n, \quad k \in \mathrm{N}. \end{cases} \tag{8}$$

*Proof.*

$$\frac{d^n}{dx^n} h_c(x) = \frac{d^n}{dx^n} \left( \frac{x^{c-1}}{\Gamma(c)} \right) = \frac{d^{n-1}}{dx^{n-1}} \left( \frac{(c-1)x^{c-2}}{(c-1)\Gamma(c-1)} \right) =$$
$$= \frac{d}{dx} \left( \frac{x^{c-n}}{\Gamma(c-n+1)} \right) = \begin{cases} h_{c-n}(x), & \text{if} \quad c \in (n, +\infty), \quad n \in \mathrm{N}; \\ 0, & \text{if} \quad c = k, \ k \le n, \quad k \in \mathrm{N}. \end{cases}$$

□

## 3. Main Results

*3.1. Problem statement*

For $\alpha > 0$, $n-1 < \alpha \le n$, $n \in N$, $\psi(x) \in C[0,a]$, $a = const$, $0 < x \le a$, find a regular solution to the equation

$$\partial_{0x}^{\alpha} y(x) - \psi(x) y(x) = 0, \tag{9}$$

satisfying the conditions

$$\lim_{x \to 0+} D^k y(x) = b_k, \quad b_k \in R,\ k = 0,\ 1,\ ...,\ n-1. \tag{10}$$

**Definition 3.1.** *The function $y(x)$ is called a regular solution of equation (9) for $0 < x \le a$, if $D^{n-1}y(x) \in AC[0,a]$ and $y(x)$ satisfies equation (9).*

Here $AC[0,a]$ is the class of absolutely continuous functions $\varphi(x)$ on the segment $[0,a]$, which coincides with the class of functions transformed from Lebesgue summable functions [3]:

$$\varphi(x) \in AC[0,a] \Leftrightarrow \varphi(x) = const + D_{0x}^{-1} r(x), \ \ r(x) \in L(0,a).$$

*3.2. Volterra integral equation of the 2nd kind with a singularity in the kernel*

**Lemma 3.2.** *An equation*

$$\varphi(t) = f(t) + \lambda \int_a^t \frac{H(t,s)}{(t-s)^{\beta}} \varphi(s) ds, \quad 0 \le \beta < 1, \tag{11}$$

*is solvable and has a unique solution for $t \in (a;b]$, which can be represented as*

$$\varphi(t) = f(t) + \lambda \int_a^t R(t,s;\lambda) f(s)\, ds, \tag{12}$$

*where $f(t)$ is a continuous function when $a \le t \le b$,*
*$H(t,s)$ is a continuous function when $a \le t \le b$, $a \le s \le t$,*

$$R(t,s,\lambda) = \sum_{\nu=0}^{\infty} \lambda^{\nu} K_{\nu+1}(t,s),$$

$$K_{\nu+1}(t,s) = \int_s^t K(t,s_1) K_{\nu}(s_1,s) ds_1,$$

$$K_1(t,s) = K(t,s) = \frac{H(t,s)}{(t-s)^{\beta}}, \quad \nu \in \mathrm{N}.$$

*Proof.* Let us use the path indicated in [37]. We find $K_n(t,s)$.

$$K_2(t,s) = \int_s^t \frac{H(t,\tau)}{(t-\tau)^{\beta}} \frac{H(\tau,s)}{(\tau-s)^{\beta}} d\tau = \left| \begin{array}{c} \tau = s + (t-s)z \\ d\tau = (t-s)dz \end{array} \right| = (t-s)^{1-2\beta} F_2(t,s),$$

where

$$F_2(t,s) = \int_0^1 \frac{H(t, s+(t-s)z)}{(1-z)^{\beta}} \frac{H(s+(t-s)z, s)}{z^{\beta}} dz,$$
$$F_1(t,s) = H(t,s).$$

The function $F_2(t,s)$ is bounded because the integral on the right-hand side converges absolutely for $\beta < 1$ according to the theorem from [9, 33].

$$K_3(t,s) = \int_s^t \frac{H(t,\tau)}{(t-\tau)^{\beta}} \frac{F_2(\tau,s)}{(\tau-s)^{2\beta-1}} d\tau = (t-s)^{2-3\beta} F_3(t,s),$$

$$F_3(t,s)=\int_0^1 \frac{H(t,s+(t-s)z)}{(1-z)^\beta}\frac{F_2(s+(t-s)z,s)}{z^{2\beta-1}}dz.$$

Similarly, we get

$$K_n(t,s)=(t-s)^{n(1-\beta)-1}F_n(t,s), \tag{13}$$

where

$$F_n(t,s)=\int_0^1 \frac{H(t,s+(t-s)z)}{(1-z)^\beta}\frac{F_{n-1}(s+(t-s)z,s)}{z^{2+(n-1)\beta-n}}dz$$

is a bounded continuous function.

Let's check (13) by the mathematical induction. For $n=2$ the assertion is true. Assume that this expression is true for $n=k$. Let's check the truth of the statement for $n=k+1$.

$$K_{k+1}(t,s)=\int_s^t K(t,\tau)K_k(\tau,s)d\tau=\int_s^t \frac{H(t,\tau)}{(t-\tau)^\beta}(\tau-s)^{k(1-\beta)-1}F_k(\tau,s)d\tau=$$

$$=(t-s)^{(k+1)(1-\beta)-1}\int_0^1 \frac{H(t,s+(t-s)z)}{(1-z)^\beta}\frac{F_k(s+(t-s)z,s)}{z^{2+k\beta-(k+1)}}dz=$$

$$=(t-s)^{(k+1)(1-\beta)-1}F_{k+1}(t,s).$$

This implies the validity of the formula (13).

It can be seen that for $n(1-\beta)-1>0$ the kernels $K_n(t,s)$, $K_{n+1}(t,s)$, etc. are bounded and continuous for $a\le t\le b,\ a\le s\le t$.

At the same time, (11) can be reduced to a similar equation with the kernel $K_n(t,s)$ by convoluting both parts of (11) with $\lambda K(t,s)$.

$$\lambda\int_a^t \varphi(s)K(t,s)ds=\lambda^2\int_a^t K(t,s)ds\int_a^s K(s,t_1)\varphi(t_1)dt_1+$$

$$+\lambda\int_a^t f(s)K(t,s)ds=\lambda^2\int_a^t \varphi(s)K_2(t,s)ds+\lambda\int_a^t f(s)K(t,s)ds;$$

$$\varphi(t)=\lambda^2\int_a^t \varphi(s)K_2(t,s)ds+f_2(t),$$

$$f_2(t)=\lambda\int_a^t f(s)K(t,s)ds+f(t);$$

$$\lambda\int_a^t \varphi(s)K(t,s)ds=$$

$$=\lambda^3\int_a^t \varphi(s)K_3(t,s)ds+\lambda^2\int_a^t f(s)K_2(t,s)ds+\lambda\int_a^t f(s)K(t,s)ds;$$

$$\varphi(t)=\lambda^3\int_a^t \varphi(s)K_3(t,s)ds+f_3(t),$$

$$f_3(t)=\lambda^2\int_a^t \varphi(s)K_2(t,s)ds+\lambda\int_a^t f(s)K(t,s)ds+f(t)=$$

$$=\lambda\int_a^t f_2(s)K(t,s)ds+f(t).$$

Continuing the process we get

$$\varphi(t)=\lambda^n\int_a^t \varphi(s)K_n(t,s)ds+f_n(t),$$
$$f_n(t)=\lambda\int_a^t f_{n-1}(s)K(t,s)ds+f(t),\quad f_1(t)=f(t). \tag{14}$$

Describing this sequence, we get

$$f_n(t) = f(t) + \sum_{k=1}^{n-1} \lambda^k \int_a^t K_k(t,s) f(s) ds.$$

We see that equation (14), which is equivalent to (11), has a unique continuous solution due to the boundedness of the $K_n(t,s)$, $f_n(t)$ by the theorem from [37]. Indeed, let

$$|f(t)| \le B, \quad |H(t,s)| \le M_1, \quad A = |\lambda| M_1 \Gamma(1-\beta)(b-a)^{(1-\beta)}$$

then

$$|f_2(t)| = |\lambda| \left| \int_a^t f(s) \frac{H(t,s)}{(t-s)^\beta} ds + f(t) \right| \le |\lambda| B M_1 \frac{(b-a)^{1-\beta}}{1-\beta} + B.$$

Similarly, the boundedness of $f_n(t)$ is shown. Now let's show that the solution can be represented as, (12).

We will look for a solution by the method of successive approximations in the form of a series

$$\varphi(x) = \varphi_0(x) + \lambda \varphi_1(x) + ... + \lambda^\nu \varphi_\nu(x) + ... \tag{15}$$

With the initial approximation $\varphi_0(t) = f(t)$, we get

$$\varphi(t) = f(t) + \sum_{\nu=1}^{\infty} \lambda^\nu \int_a^t K_\nu(t,s) f(s) ds. \tag{16}$$

Let's show that the series (16) converges uniformly.

$$|\lambda K_1(x,t)| \le \frac{\lambda M_1}{(t-s)^\beta};$$

$$|\lambda^2 K_2(t,s)| = |\lambda^2| (t-s)^{1-2\beta} \int_0^1 \frac{H(t, s+(t-s)z)}{(1-z)^\beta} \frac{H(s+(t-s)z, s)}{z^\beta} dz \le$$

$$\le |\lambda^2| (t-s)^{1-2\beta} M_1^2 \frac{(\Gamma(1-\beta))^2}{\Gamma(2(1-\beta))};$$

$$|\lambda^3 K_3(t,s)| \le |\lambda^3| M_1^3 \frac{(\Gamma(1-\beta))^3}{\Gamma(3(1-\beta))} (t-s)^{2-3\beta}.$$

Continuing the calculations, we get

$$|\lambda^n K_n(t,s)| \le \frac{(|\lambda| M_1 \Gamma(1-\beta))^n}{\Gamma(n(1-\beta))} (t-s)^{n(1-\beta)-1}.$$

Hence

$$|\varphi(t)| = \left| f(t) + \sum_{n=1}^{\infty} \lambda^n \int_a^t K_n(t,s) f(s) ds \right| \le$$

$$\le B + \sum_{n=1}^{\infty} \int_a^t |f(s)| \frac{(|\lambda| M_1 \Gamma(1-\beta))^n}{\Gamma(n(1-\beta))} (t-s)^{n(1-\beta)-1} ds \le$$

$$\le B + B \sum_{n=1}^{\infty} \frac{(|\lambda| M_1 \Gamma(1-\beta)(b-a)^{(1-\beta)})^n}{\Gamma(n(1-\beta)+1)} \le B + B \sum_{n=1}^{\infty} \frac{A^n}{\Gamma(n(1-\beta)+1)}.$$

By d'Alembert's test [21], we have

$$\lim_{n\to\infty}\left|\frac{A^{n+1}\Gamma(n(1-\beta)+1)}{\Gamma((n+1)(1-\beta)+1)A^n}\right| = \lim_{n\to\infty}\left|\frac{A\Gamma(n(1-\beta)+1)}{\Gamma((n+1)(1-\beta)+1)}\right| =$$

$$= [\alpha = 1-\beta,\ \alpha n = t,\ t\to+\infty] = \lim_{t\to\infty}\left|\frac{A\Gamma(t+1)}{\Gamma(t+1+a)}\right| = \lim_{t\to\infty}\left|\frac{A}{(t+1)^a}\right| = 0.$$

Series (16) converges uniformly for $a < x \le b,\ a < t \le x$ according to the Weierstrass test [9], and is also a solution to (11). When calculating the limit, the asymptotics of the Gamma function was used. If we take into account that for $n(1-\beta)-1>0$ the series $\sum_{\nu=n-1}^{\infty}\lambda^\nu K_{\nu+1}(t,s)$ converges uniformly, we get:

$$\varphi(t) = f\left(t\right) + \sum_{\nu=1}^{\infty}\lambda^\nu\int_a^t K_\nu(t,s)f(s)ds =$$

$$= f\left(t\right) + \sum_{\nu=1}^{n-1}\lambda^\nu\int_a^t K_\nu(t,s)f(s)ds + \sum_{\nu=n}^{\infty}\lambda^\nu\int_a^t K_\nu(t,s)f(s)ds =$$

$$= f\left(t\right) + \int_a^t\sum_{\nu=1}^{n-1}\lambda^\nu K_\nu(t,s)f(s)ds + \int_a^t\sum_{\nu=n}^{\infty}\lambda^\nu K_\nu(t,s)f(s)ds =$$

$$= f\left(t\right) + \lambda\int_a^t R(t,s,\lambda)f\left(s\right)ds.$$

Lemma 3.2 is proven. □

**Remark 3.3.** *In [14], the authors considers a similar singular integral equation of the second kind of the Volterra type, but the method of successive approximations is not applicable to it.*

*3.3. Reducing the Cauchy problem with the Caputo fractional derivative in the equation to a Volterra integral equation of the 2nd kind*

**Theorem 3.4.** *If $y(x)$ is a regular solution to equation (9), then the Cauchy problem (9) – (10) is equivalently reduced in to a Volterra integral equation of the second kind*

$$y = \sum_{k=0}^{n-1} b_k h_{k+1}(x) + (D_{0x}^{-\alpha}(\psi y))(x). \tag{17}$$

*Proof.* Let us rewrite the definition of the left-hand fractional Caputo derivative with origin at the zero point as a convolution

$$\partial_{0x}^{\alpha} y(x) = \begin{cases} \left(D^n y * h_{n-\alpha}\right)(x), & \text{if}\quad n-1<\alpha<n, \quad n\in N; \\ D^n y(x), & \text{if}\quad \alpha = n. \end{cases}$$

Consider equation (9) for $n-1<\alpha<n$ and write it as a Laplace convolution

$$\left(D_{0x}^n y * h_{n-\alpha}\right)(x) - \psi(x)y(x) = 0 \tag{18}$$

It follows from the condition $D^{n-1}y(x) \in AC[0,a]$ that $D^n y(x) \in L(0,a)$, and due to $h_{n-\alpha}(x) \in L(0,a)$, it follows that $\left(D_{0x}^n y * h_{n-\alpha}\right)(x) \in L(0,a)$. The function $\psi(x)y(x) \in L(0,a)$, hence equation (18) can be integrated.

In (18) equation, we replace $x$ by $s$, multiply both sides of the resulting equality by $h_\alpha(x-s)$ and integrate over $s$ from 0 to $x$:

$$\left(\left(D^n y * h_{n-\alpha}\right) * h_\alpha\right)(x) - \left((\psi y) * h_\alpha\right)(x) = \left(0 * h_\alpha\right)(x).$$

Using the associativity property of the function convolution, we obtain:

$$(D^n y * (h_{n-\alpha} * h_\alpha))(x) - ((\psi y) * h_\alpha)(x) = 0,$$
$$(D^n y * h_n)(x) - ((\psi y) * h_\alpha)(x) = 0,$$
$$\left(D_{0x}^{-n} D^n y\right)(x) - \left(D_{0x}^{-\alpha} \psi y\right)(x) = 0.$$

Using the generalized Newton-Leibniz formula [3], the last equality can be rewritten as:

$$y(x) - \sum_{k=1}^{n} h_{1-k+n}(x)(D^{n-k}y)(0) - \left(D_{0x}^{-\alpha} \psi y\right)(x) = 0.$$

Changing the order of summation $j = n - k$, we get:

$$y(x) - \sum_{j=0}^{n-1} h_{1+j}(x) D^j y(0) - \left(D_{0x}^{-\alpha} \psi y\right)(x) = 0.$$

Using conditions (10), we get:

$$y(x) = \sum_{k=0}^{n-1} b_k h_{1+k}(x) + \left(D_{0x}^{-\alpha} \psi y\right)(x)$$

or

$$y(x) = \sum_{k=0}^{n-1} b_k h_{1+k}(x) + \int_0^x \psi(t) y(t) h_\alpha(x-t) dt. \tag{19}$$

Consider equation (9) with $\alpha = n$, then this equation will take the form

$$D^n y(x) - \psi(x) y(x) = 0. \tag{20}$$

Convolving both sides of equation (20) with the function $h_n(x)$, we get

$$y(x) - (D_{0x}^{-n} \psi y)(x) = \sum_{k=0}^{n-1} b_k h_{1+k}(x)$$

or

$$y(x) = \sum_{k=0}^{n-1} b_k h_{1+k}(x) + (D_{0x}^{-n} \psi y)(x). \tag{21}$$

As a result, we can combine (21) and (19) by writing them as (17).

Let's show that equation (17) is also reduced to (9) using [3].

$$D_{0x}^{\alpha-n} D^n y(x) = D_{0x}^{\alpha-n} D^n \left[\sum_{k=0}^{n-1} b_k h_{1+k}(x)\right] + D_{0x}^{\alpha-n} D^n \left[D_{0x}^{-\alpha}(\psi y)\right](x),$$
$$\partial_{0x}^{\alpha} y(x) = D_{0x}^{\alpha-n} D_{0x}^{n-\alpha}(\psi y)(x).$$

For $\alpha = n$, we immediately get

$$D^n y(x) = \psi(x) y(x).$$

If $n - 1 < \alpha < n$, then using the generalized Newton-Leibniz formula we get

$$\partial_{0x}^{\alpha} y(x) = \psi(x) y(x) - h_{n-\alpha}(x) \lim_{x \to 0+} \left[D_{0x}^{n-\alpha-1}(\psi y)(x)\right].$$

Taking into account that $D^{n-1} y(x) \in AC[0, a]$, it follows that $y(x) \in C[0, a]$ due to theorems from [8] and [9], it is also known $\psi(x) \in C[0, a]$ from the problem statement. We write

$$\lim_{x \to 0+} \left[D_{0x}^{n-\alpha-1}(\psi y)(x)\right] = 0.$$

As a result, we get the original equation (17).

Theorem 3.4 is proved. □

*3.4. Existence and uniqueness of a solution to the Cauchy problem (9) – (10)*

**Theorem 3.5.** *The Cauchy problem (9) – (10) has at most one regular solution, which can be represented as*

$$y(x) = f(x) + \int_0^x R(x,t,1) f(t)\, dt, \tag{22}$$

*where*

$$f(x) = \sum_{k=0}^{n-1} b_k h_{1+k}(x)$$

*and* $R(x,t,1)$ *is the resolvent of equation (17).*

*Proof.* Expression (22) is the only solution to equation (17) by virtue of Lemma 3.2, therefore it also satisfies the equation (9) by Theorem 3.4. Let's denote it by $y_1$.

Suppose there is a second regular solution $y_2$ of the Cauchy problem (9) – (10).

Then $v = y_1 - y_2$ is a solution to the equation

$$\partial_{0x}^{\alpha} v(x) - \psi(x) v(x) = 0, \tag{23}$$

with initial conditions

$$D^k v(0) = (D^k (y_1 - y_2))(0) = D^k y_1(0) - D^k y_2(0) = 0, k = 0,\ 1,\ ...,\ n-1 \tag{24}$$

The solution of equation (23) can be represented as

$$v(x) = f(x) + \int_0^x R(x,t,1) f(t)\, dt,$$

where

$$f(x) = \sum_{k=0}^{n-1} D^k v(0) h_{1+k}(x) = 0,$$

whence it follows that $v = y_1 - y_2 = 0$ and $y_1 = y_2$, which proves the uniqueness of the regular solution to the Cauchy problem (9) – (10). □

**Theorem 3.6.** *Function (22) is a regular solution to Cauchy problem (9) – (10) if and only if* $D^{n-1}y(x) \in AC[0,a]$ *and the function* $y(x)$ *is a solution to integral equation (17).*

*Proof.* Indeed, if $D^{n-1}y(x) \in AC[0,a]$ and $y(x)$ is a solution to integral equation (17), then equation (17) is equivalently reduced to equation (9). This follows from the proof of Theorem 3.4. Then function (22) is a regular solution of equation (9), which is unique by Theorem 3.5. Conversely, if equation (9) has a solution, then it must be regular, since otherwise $D^n y(x)$ will not be an integrable function, and as a result, the equation loses its meaning. Thus, function (22) is a solution to equation (9) due to Theorem 3.4 and Theorem 3.5.

Theorem 3.6 is proved. □

**Corollary 3.7.** *The solution to Cauchy problem (9) – (10) exists if and only if the solution to integral equation (17) satisfies the condition:* $D^{n-1}y(x) \in AC[0,a]$.

*Proof.* The assertion of the lemma immediately follows from the proof of Theorem 3.6. □

## 4. Conclusion

In the future, we can consider the Cauchy problem with the origin at an arbitrary point, construct a general solution for the right-handed and left-handed Caputo fractional derivatives, and express the solution as a power series. Also, the studied problem can be considered in a broader sense, taking the Caputo derivative as a special case of the Dzhrbashyan-Nersesyan operator [7].

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