SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ

http://dergipark.gov.tr/saufenbilder

A numerical method for solution of integro-differential equations of fractional order

Sertan Alkan*

07.05.2016 Geliş/Received, 10.06.2016 Kabul/Accepted

doi: https://doi.org/10.16984/saufenbilder.296796

ABSTRACT

In this study, sinc-collocation method is introduced for solving Volterra integro-differential equations of fractional order. Fractional derivative is described in the Caputo sense often used in fractional calculus. Obtained results are given to literature as two new theorems. Some numerical examples are presented to demonstrate the theoretical results.

Keywords: Integro-differential equation, sinc-collocation method, Caputo fractional derivative.

Kesirli mertebeden integro-diferansiyel denklemlerin çözümü için sayısal bir yöntem

ÖZ

Bu çalışmada, sinc sıralama yöntemi kesirli mertebeden Volterra integro-diferansiyel denklemleri yaklaşık olarak çözmek için geliştirilmiştir. Kesirli türev, kesirli analizde sıkça kullanılan Caputo anlamında tanımlanmıştır. Elde edilen sonuçlar iki yeni teorem ile verilmiştir. Bazı sayısal örnekleri teorik sonuçları göstermek için sunulmuştur.

Anahtar kelimeler: Integro-diferansiyel denklem, sinc-sıralama yöntemi, Caputo kesirli türevi.

* Sorumlu Yazar / Corresponding Author
Iskenderun Technical University, Faculty of Engineering and Natural Sciences, Department of Computer Engineering, Hatay, sertan.alkan@iste.edu.tr

1. INTRODUCTION

Many problems, in science and engineering such as earthquake engineering, biomedical engineering, fluid mechanics can be modeled by fractional integro-differential equations [34, 35, 36]. In order to better analyze these systems, it is required to obtain the solution of these equations. But, achieving the analytical solution of these equations can not be possible. Therefore, finding more accurate solutions using numerical schemes can be helpful. Some numerical algorithm for solving integro-differential equation of fractional order can be summarized as follows, but not limited to; Adomian decomposition method [1, 2, 23], Taylor expansion method [3], differential transform method [4, 5] and homotopy perturbation method [6, 7], Spectral collocation method [14], Legendre wavelets method [13], Chebyshev wavelets method [15, 29], piecewise collocation methods[20, 21], Chebyshev pseudo-spectral method [24, 28], homotopy analysis method [25, 26], variational iteration method [27].

According to best knowledge of the authors, there is no study dealing with the solution of fractional linear Volterra integro-differential equation by means of sinc-collocation method. The main advantage of the sinc-collocation method than other methods is that sinc-collocation method provides a much better rate of convergence and more e cient results in the presence of singularity [37]. For more details about the sinc-collocation method see [8, 9, 10, 12].

Particulary, in the present paper, as an original contribution to literature, sinc-collocation method is introduced for solving linear Volterra integro-differential equations of fractional order. Examined integro-differential equations in the present paper have singularities at some points. Obtained results are given in the form of two new theorems. Some numerical examples in the form of graphics and tables are given to illustrate the theoretical results.

In this study, Volterra integro-differential equations of fractional order are considered as follows:

$$\mu_2(x)y'' + \mu_1(x)y' + \mu_\alpha(x)D_x^\alpha y + \mu_0(x)y = f(x) + \lambda\int_\alpha^x K(x,t)y(t)dt \quad ,0<\alpha<1 \tag{1}$$

in which D_x^α is the Caputo sense fractional derivative. Eq.1 is subject to following nonhomogeneous boundary conditions

$$y(a)=y_0,\ \ y(b)=y_1, \quad a<x<b.$$

The structure of this paper is organized as follows; In section 2, some preliminaries and basic definitions related to fractional calculus and sinc functions are recalled. In the next section, sinc-collocation method is constructed for solving integro-differential equations of fractional order. In section 4, numerical examples are presented. Finally, conclusions and remarks are given in the section 5.

2. PRELIMINARIES AND NOTATIONS

In this section, some preliminaries and notations related to fractional calculus and sinc basis functions are given. For more details see [16, 17, 18, 19, 30, 31, 32, 33].

Definition 1. Let $f:[a,b]\to\mathbb{R}$ be a function, α a positive real number, n the integer satisfying $n-1\le\alpha<n$, and Γ the Euler gamma function. Then, the left Caputo fractional derivative of order of $f(x)$ is given as follows:

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_\alpha^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt. \tag{2}$$

Definition 2. The Sinc function is defined on the whole real line $-\infty<x<\infty$ by

$$sinc(x)=\begin{cases}\dfrac{\sin(\pi x)}{\pi x} & x\neq 0\\ 1 & x=0.\end{cases}$$

Definition 3. For $h>0$ and $k=0,\pm1,\pm2,\ldots$ the translated sinc function with space node are given by:

$$S(k,h)(x) = sinc\left(\frac{x-kh}{h}\right) = \begin{cases}\dfrac{\sin(\pi\frac{x-kh}{h})}{\pi\frac{x-kh}{h}} & x\neq kh\\ 1 & x=kh.\end{cases}$$

To construct approximation on the interval (a,b) the conformal map

$$\phi(z)=\ln\left(\frac{z-a}{b-z}\right).$$

is employed. The basis functions on the interval (a,b) are derived from the composite translated sinc functions

$$S_k(z) = S(k,h)(z)\ o\ \phi(z) = sinc\left(\frac{\phi(z)-kh}{h}\right).$$

The inverse map of $\omega=\phi(z)$ is

$$z=\phi^{-1}(\omega)=\frac{a+be^\omega}{1+e^\omega}.$$

The sinc grid points $z_k \in (a,b)$ will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

3. THE SINC-COLLOCATION METHOD

Let us assume an approximate solution for $y(x)$ in Eq.(1) by finite expansion of sinc basis functions for as follows;

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x), \quad n = M + N + 1 \tag{3}$$

where $S_k(x)$ is the function $S(k,h)\ o\ \phi(x)$. Here, the unknown coefficients c_k in (3) are determined by sinc-collocation method via the following theorems.

Theorem 1. The first and second derivatives of $y_n(x)$ are given by

$$\frac{d}{dx} y_n(x) = \sum_{k=-M}^{N} c_k\, \phi'(x) \frac{d}{d\phi} S_k(x) \tag{4}$$

$$\frac{d^2}{dx^2} y_n(x) = \sum_{k=-M}^{N} c_k \left(\phi''(x) \frac{d}{d\phi} S_k(x) + (\phi')^2 \frac{d^2}{d\phi^2} S_k(x) \right) \tag{5}$$

respectively.

Theorem 2. If ξ is a conformal map for the interval $[a,x]$, then α order derivative of $y_n(x)$ for $0 < \alpha < 1$ is given by

$$D_x^\alpha\big(y_n(x)\big) = \sum_{k=-M}^{N} c_k\, D_x^\alpha(S_k(x)) \tag{6}$$

where

$$D_x^\alpha(S_k(x)) \approx \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^{L} \frac{(x - x_r) S_k'(x_r)}{\xi'(x_r)}$$

Proof. If we use the definition of Caputo fractional derivative given in (2), it is written that

$$D_x^\alpha\big(y_n(x)\big) = \sum_{k=-M}^{N} c_k\, D_x^\alpha(S_k(x))$$

where

$$D_x^\alpha(S_k(x)) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} S_k'(t)\,dt.$$

Now we use quadrature rule given by (2.13) in [11] to compute the above integral which is divergent on the interval $[a,x]$. For this purpose, a conformal map and its inverse image that denotes the sinc grid points are given by

$$\xi(t) = \ln\left(\frac{t-a}{x-t}\right)$$

and

$$x_r = \xi^{-1}(rh_L) = \frac{a + xe^{rh_L}}{1 + e^{rh_L}}$$

where $h_L = \pi/\sqrt{L}$. Then, according to equality (2.13) in [11], we can write

$$D_x^\alpha(S_k(x)) \approx \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^{L} \frac{(x - x_r) S_k'(x_r)}{\xi'(x_r)}$$

This completes the proof.

Lemma 1. The following relation holds

$$\int_a^{x_j} K(x,t) y(t)\,dt \approx h \sum_{k=-M}^{N} \delta_{jk}^{(-1)} \frac{K(x_j, t_k)}{\phi'(t_k)} y_k \tag{7}$$

where

$$\sigma_{jk} = \int_0^{j-k} \frac{sin\pi t}{\pi t}\,dt$$

$$\delta_{jk}^{(-1)} = \frac{1}{2} + \sigma_{jk}$$

and y_k denotes an approximate value of $y(t_k)$.

Proof. See [12]

Replacing each term of (1) with the approximation given in (3)-(7), multiplying the resulting equation by $\{(1/\phi)^2\}$, we obtain the following system

$$\sum_{k=-M}^{N}\left[c_k\left\{\begin{array}{c}\sum_{i=0}^{2} g_i(x)\frac{d^i}{d\phi^i}S_k + g_3(x)D_x^{\alpha}\big(S_k(x)\big)\\ +g_4(x)\delta_{jk}^{(-1)}\frac{K(x,t_k)}{\phi'(t_k)}\end{array}\right\}\right]$$

$$=\left(f(x)\left(\frac{1}{\phi'(x)}\right)^2\right)$$

where

$$g_0(x)=\mu_0(x)\left(\frac{1}{\phi'(x)}\right)^2$$

$$g_1(x)=\left[\mu_1(x)\left(\frac{1}{\phi'(x)}\right)-\mu_2(x)\left(\frac{1}{\phi'(x)}\right)'\right]$$

$$g_2(x)=\mu_2(x)$$

$$g_3(x)=\mu_\alpha(x)\left(\frac{1}{\phi'(x)}\right)^2$$

$$g_4(x)=-\lambda h\left(\frac{1}{\phi'(x)}\right)^2.$$

We know from [12] that

$$\delta_{jk}^{(0)}=\delta_{kj}^{(0)},\qquad \delta_{jk}^{(1)}=-\delta_{kj}^{(1)},\qquad \delta_{jk}^{(2)}=\delta_{kj}^{(2)}$$

then setting $x=x_j$, we obtain the following theorem.

Theorem 3. If the assumed approximate solution of boundary value problem (1) is (3), then the discrete sinc-collocation system for the determination of the unknown coefficients $\{c_k\}_{k=-M}^{N}$ is given by

$$\sum_{k=-M}^{N}\left[c_k\left\{\begin{array}{c}\sum_{i=0}^{2}\frac{g_i(x_j)}{h^i}\delta_{jk}^{(i)} + g_3(x_j)D_x^{\alpha}\left(S_k(x_j)\right)\\ +g_4(x_j)\delta_{jk}^{(-1)}\frac{K(x_j,t_k)}{\phi'(t_k)}\end{array}\right\}\right] \quad (8)$$

$$=\left(f(x_j)\left(\frac{1}{\phi'(x_j)}\right)^2\right)\quad ,j=-M,\dots N$$

We now introduce some notations to rewrite in the matrix form for system (8). Let $\boldsymbol{D}(y)$ denotes a diagonal matrix whose diagonal elements are $y(x_{-M})$, $y(x_{-M+1})$,…, $y(x_N)$ and non-diagonal elements are zero, let

$$\mathbf{G}=D_x^{\alpha}\big(S_k(x_j)\big)$$

and

$$\mathbf{E}=\frac{K(x_j,t_k)}{\big(\phi'(x_j)\big)^2\,\phi'(t_k)}$$

denote a matrix and also let $\mathbf{I}^{(i)}$ denote the matrices

$$\mathbf{I}^{(i)}=\left[\delta_{jk}^{(i)}\right],\quad i=-1,0,1,2$$

where $\mathbf{D},\mathbf{G},\mathbf{E}$, $\mathbf{I}^{(-1)},\mathbf{I}^{(0)},\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ are square matrices of order $n\times n$. In order to calculate unknown coefficients c_k in linear system (8), we rewrite this system by using the above notations in matrix form as

$$\mathbf{Ac}=\mathbf{B} \quad (9)$$

where

$$\mathbf{A}=\sum_{i=0}^{2}\frac{1}{h^i}\mathbf{D}(g_i)\mathbf{I}^{(i)}+\mathbf{D}(g_3)\mathbf{G}+\mathbf{D}(g_4)\big(\mathbf{E}\circ\mathbf{I}^{(-1)}\big)$$

$$\mathbf{B}=\left(\begin{array}{c}\left(f(x_{-M})\left(\frac{1}{\phi'(x_{-M})}\right)^{\mathbf{2}}\right),\left(f(x_{-M+1})\left(\frac{1}{\phi'(x_{-M+1})}\right)^{\mathbf{2}}\right)\\ ,\dots,\left(f(x_N)\left(\frac{1}{\phi'(x_N)}\right)^{\mathbf{2}}\right)\end{array}\right)^T$$

$$\mathbf{c}=(c_{-M},c_{-M+1},\dots,c_N)^T$$

The notation "$\circ$" denotes the Hadamard matrix multiplication. Now we have linear system of n equations in the n unknown coefficients given by (9). We can nd the unknown coefficients c_k by solving this system.

4. COMPUTATIONAL EXAMPLES

In this section, some numerical examples whose exact solutions are known are presented to show the accuracy of the introduced method by **MATHEMATICA** 10. In all examples, $d=\pi/2,\ \alpha=\beta=1/2,\ N=M$ are taken into account $\boldsymbol{E_{M,L}}$ shows the error between the exact solution and numerical solution by sinc-collocation method. Also, $\boldsymbol{R_{M,L}}$ in example 2 indicates the experimental rate of convergence that calculates the following formula like [22]

$$\boldsymbol{R_{M,L}}=\frac{\log\left[\boldsymbol{E_{M/2,L/2}}/\boldsymbol{E_{M,L}}\right]}{\log 2}$$

Example 1. Consider linear fractional Volterra integro-differential equation in the following form

$$y''(x) + D_x^{0.5}y(x) + y(x) = f(x) - 2\int_0^x K(x,t)y(t)dt \qquad (10)$$

subject to the nonhomogeneous boundary conditions

$$y(0) = 2, \quad y(1) = 3$$

where $f(x) = \frac{1}{3}(-x^7 + x^6 - 4x^4 + 7x^3 + 18x + 6) + \frac{6}{\Gamma(3.5)}x^{2.5}$ and $K(x,t) = t^2(x-1)$. The exact solution of Eq.10 is $y(x) = x^2 + 2$. In this problem, firstly, let us convert nonhomogeneous boundary conditions to homogeneous ones by following transformation

$$u(x) = y(x) - x - 2$$

Obtained numerical results are presented in the table 1 after applying the sinc-collocation method. Also,

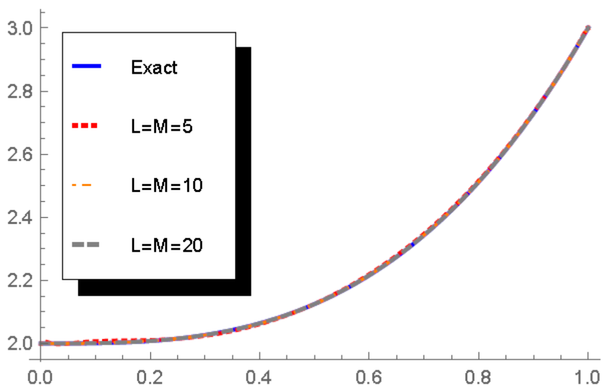

Figure 1 The graphics of the exact and approximate solutions for Example 1

the graphics of the exact and approximate solutions for different values of L and M are given in Figure 1.

Table 1 Numerical results for Example 1

x	Exact s	$E_{20,20}$	$E_{10,10}$	$E_{5,5}$
0	2	0	0	0
0.1	2.001	1.02×10^{-5}	2.89×10^{-4}	6.16×10^{-3}
0.2	2.008	8.13×10^{-6}	4.35×10^{-4}	2.88×10^{-3}
0.3	2.027	1.57×10^{-5}	1.43×10^{-4}	2.36×10^{-3}
0.4	2.064	8.38×10^{-6}	4.54×10^{-4}	2.59×10^{-3}
0.5	2.125	4.39×10^{-6}	1.40×10^{-4}	3.38×10^{-4}
0.6	2.216	8.39×10^{-7}	1.80×10^{-4}	3.28×10^{-3}
0.7	2.343	2.18×10^{-5}	3.96×10^{-5}	4.24×10^{-3}
0.8	2.512	2.21×10^{-5}	2.65×10^{-4}	3.40×10^{-3}
0.9	2.729	1.62×10^{-5}	1.60×10^{-4}	2.14×10^{-3}
1	3	0	0	0

Example 2. Now, let us consider following singular Volterra integro-differential equation of fractional order

$$y''(x) + \frac{1}{x}D_x^{0.3}y(x) + \frac{1}{x-1}y(x) = f(x) + \int_0^x K(x,t)y(t)dt$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 0$$

where $f(x) = x^{11} - \frac{1}{30}x^6 + \frac{1}{20}x^5 + x^3 + \frac{24}{\Gamma(4.7)}x^{2.7} + 12x^2 - \frac{6}{\Gamma(3.7)}x^{1.7} - 6x$ and $K(x,t) = x - t$. The exact solution of this problem is $y(x) = x^3(x-1)$. For this problem, numerical solutions are presented in Table 2 and Table 3, and plotting of the numerical solutions are given in Figure 2.

Table 2 Numerical results for Example 2

x	Exact so	$E_{20,20}$	$E_{10,10}$	$E_{5,5}$
0	2	0	0	0
0.1	-0.0009	5.82×10^{-7}	1.41×10^{-4}	5.46×10^{-4}
0.2	-0.0064	5.28×10^{-6}	1.01×10^{-4}	1.56×10^{-3}
0.3	-0.0189	2.71×10^{-6}	8.35×10^{-5}	2.23×10^{-3}
0.4	-0.0384	5.40×10^{-6}	1.06×10^{-4}	8.78×10^{-4}
0.5	-0.0625	6.95×10^{-6}	2.70×10^{-4}	2.39×10^{-3}
0.6	-0.0864	1.61×10^{-6}	6.05×10^{-4}	5.46×10^{-3}
0.7	-0.1029	1.02×10^{-5}	2.39×10^{-4}	5.21×10^{-3}
0.8	-0.1024	2.84×10^{-6}	4.35×10^{-4}	1.83×10^{-6}
0.9	-0.0729	6.72×10^{-6}	3.88×10^{-4}	3.73×10^{-3}
1	0	0	0	0

Table 3 Maximum absolute errors and rate of convergence for Example 2

M, L	Maximum absolute errors $E_{M,L}$	Rate of convergence $R_{M,L}$
5	5.46×10^{-2}	
10	6.05×10^{-4}	3.17
20	1.61×10^{-5}	5.23
40	1.24×10^{-7}	7.02

Example 3. Finally, consider the problem

$$y''(x) + x^2D_x^{0.7}y(x) + xy(x) = f(x) - \int_0^x K(x,t)y(t)dt$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 0$$

where $f(x) = \frac{1}{7}x^7 - \frac{2}{5}x^6 + \frac{24}{\Gamma(4.3)}x^{5.3} - \frac{6}{5}x^5 + \frac{2}{3}x^4 - \frac{2}{\Gamma(2.3)}x^{3.3} + x^3 - 12x^2 + 2$ and $K(x,t) = 2x - t^2$.The exact solution of this problem $y(x) = x^2(1-x^2)$. The numerical solutions and graphs of the solutions are presented in Table 4 and Figure 3.

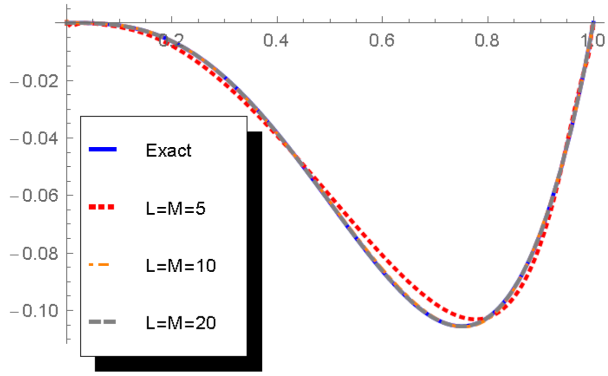

Figure 2 The graphics of the exact and approximate solutions for Example 2

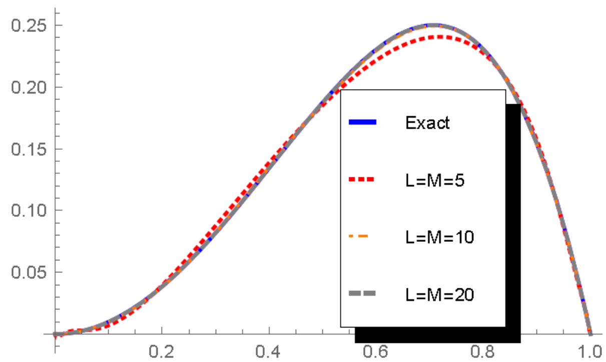

Figure 3 The graphics of the exact and approximate solutions for Example 3

Table 4 Numerical results for Example 3

x	Exact sol.	$E_{20,20}$	$E_{10,10}$	$E_{5,5}$
0	0	0	0	0
0.1	0.0099	1.26×10^{-7}	9.85×10^{-5}	2.83×10^{-3}
0.2	0.0384	1.38×10^{-6}	3.69×10^{-4}	1.02×10^{-3}
0.3	0.0819	5.58×10^{-6}	3.12×10^{-4}	5.81×10^{-3}
0.4	0.1344	1.63×10^{-5}	5.58×10^{-4}	4.36×10^{-3}
0.5	0.1875	2.17×10^{-5}	2.41×10^{-4}	2.10×10^{-3}
0.6	0.2304	7.28×10^{-5}	1.07×10^{-3}	8.70×10^{-3}
0.7	0.2499	7.51×10^{-5}	7.36×10^{-4}	9.82×10^{-3}
0.8	0.2304	1.36×10^{-4}	1.65×10^{-4}	3.05×10^{-3}
0.9	0.1539	1.28×10^{-4}	9.20×10^{-4}	2.65×10^{-3}
1	0	0	0	0

5. CONCLUSION (SONUÇLAR)

In recent years several numerical methods have been applied to integro-differential equations of fractional order. In this study, we have applied sinc-collocation method to a class of Volterra integro-differential equation of fractional order to obtain the approximate solutions. In order to illustrate the accuracy of the present method, we have compared the obtained results with the exact ones. With respect to comparisons it has seen that sinc-collocation method provides a good approximate solution. Additionally, according to comparison results one may say that proposed method promises for solving many other types of integro differential equations.

REFERENCES

[1] Momani, S., Noor, M. A. (2006). Numerical methods for fourth-order fractional integro-differential equations. Applied Mathematics and Computation, 182(1), 754-760.

[2] Momani, S., Qaralleh, R. (2006). An e cient method for solving systems of fractional integro-differential equations. Computers & Mathematics with Applications, 52(3), 459-470.

[3] Huang, L., Li, X. F., Zhao, Y., Duan, X. Y. (2011). Approximate solution of fractional integro-differential equations by Taylor expansion method. Computers & Mathematics with Applications, 62(3), 1127-1134.

[4] Nazari, D., Shahmorad, S. (2010). Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions. Journal of Compu-tational and Applied Mathematics, 234(3), 883-891.

[5] Arikoglu, A., Ozkol, I. (2009). Solution of fractional integro-differential equations by using fractional differential transform method. Chaos, Solitons & Fractals, 40(2), 521-529.

[6] Saeed, R. K., Sdeq, H. M., (2010). Solving a system of linear fredholm fractional integro-differential equations using homotopy perturbation method. Australian Journal of Basic and Applied Sciences, 4(4), 633-638.

[7] Nawaz, Y. (2011). Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations. Computers & Mathematics with Applications, 61(8), 2330-2341.

[8] Alkan, S., (2015).A new solution method for nonlinear fractional integro-differential equations, Dis-crete and Continuous Dynamical Systems - Series S, 8(6), 1065-1077.

[9] Alkan, S., Yildirim, K., Secer, A. (2016). An e cient algorithm for solving fractional differential equations with boundary conditions, Open Physics, 14(1), 6-14.

[10] Alkan, S., Secer, A. (2015). Solution of nonlinear fractional boundary value problems with nonhomo-geneous boundary conditions, Applied and Computational Mathematics, 14(3),284-295.

[11] Secer, A., Alkan, S., Akinlar, M. A., Bayram, M. (2013). Sinc-Galerkin method for approximate solutions of fractional order boundary value problems. Boundary Value Problems, 2013(1), 281.

[12] Zarebnia, M., Nikpour, Z. (2009). Solution of linear Volterra integro-differential equations via Sinc functions. International Journal of Applied Mathematics and Computation, 2(1), 001-010.

[13] Rawashdeh, E. A. (2011). Legendre wavelets method for fractional integro-differential equations. Ap-plied Mathematical Sciences, 5(2), 2467-2474.

[14] Ma, X., Huang, C. (2014). Spectral collocation method for linear fractional integro-differential equa-tions. Applied Mathematical Modelling, 38(4), 1434-1448.

[15] Zhu, L., Fan, Q. (2012). Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet. Communications in Nonlinear Science and Numerical Simulation, 17(6), 2333-2341.

[16] Podlubny, I., Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Vol. 198. Academic press, 1998.

[17] Lund, J., Bowers, K. L., Sinc methods for quadrature and differential equations, SIAM, 1992.

[18] Samko, S. G., Kilbas, A. A., Marichev, O. I., Fractional integrals and derivatives. Theory and Appli-cations, Gordon and Breach, Yverdon, 1993.

[19] Stenger, F., Handbook of Sinc numerical methods, CRC Press, 2010.

[20] Zhao, J., Xiao, J., Ford, N. J. (2014). Collocation methods for fractional integro-differential equations with weakly singular kernels. Numerical Algorithms, 65(4), 723-743.

[21] Rawashdeh, E. A. (2006). Numerical solution of fractional integro-differential equations by collocation method. Applied Mathematics and Computation, 176(1), 1-6.

[22] Aziz, I., Fayyaz, M. (2013). A new approach for numerical solution of integro-differential equations via Haar wavelets. International Journal of Computer Mathematics, 90(9), 1971-1989.

[23] Mittal, R. C., Nigam, R. (2008). Solution of fractional integro-differential equations by Adomian decomposition method. The International Journal of Applied Mathematics and Mechanics, 4(2), 87-94.

[24] Sweilam, N. H., Khader, M. M. (2010). A Chebyshev pseudo-spectral method for solving fractional-order integro-differential equations. The ANZIAM Journal, 51(04), 464-475.

[25] Abbasbandy, S., Hashemi, M. S., Hashim, I. (2013). On convergence of homotopy analysis method and its application to fractional integro-differential equations. Quaestiones Mathematicae, 36(1), 93-105.

[26] Zhang, X., Tang, B., He, Y. (2011). Homotopy analysis method for higher-order fractional integro-differential equations. Computers & Mathematics with Applications, 62(8), 3194-3203.

[27] Nawaz, Y. (2011). Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations. Computers & Mathematics with Applications, 61(8), 2330-2341.

[28] Khader, M. M., Sweilam, N. H. (2013). On the approximate solutions for system of fractional integro-differential equations using Chebyshev pseudo-spectral method. Applied Mathematical Modelling, 37(24), 9819-9828.

[29] Yuanlu, L. (2010). Solving a nonlinear fractional di erential equation using Chebyshev wavelets. Com-munications in Nonlinear Science and Numerical Simulation, 15(9), 2284-2292.

[30] Atangana, A., Koca, I. (2016). On the new fractional derivative and application to nonlinear Baggs and Freedman model. J. Nonlinear Sci. Appl., 9(2016), 2467-2480.

[31] Koca, I., Atangana, A. (2016). Analysis of a nonlinear model of interpersonal relationships with time fractional derivative. Journal of Mathematical Analysis, 7(2), 1-11.

[32] Atangana, A. (2016). On the new fractional derivative and application to nonlinear Fishers reaction-di usion equation. Applied Mathematics and Computation, 273, 948-956.

[33] Atangana, A., Koca, I. (2016). Chaos in a simple nonlinear system with Atangana Baleanu derivatives with fractional order. Chaos, Solitons and Fractals, 1-8.

[34] Carpinteri, A., Cornetti, P., Sapora, A. (2014). Nonlocal elasticity: an approach based on fractional calculus. Meccanica, 49(11), 2551-2569.

[35] Tejado, I., Valrio, D., Valrio, N. (2015). Fractional Calculus in Economic Growth Modelling: The Span-ish Case. In CONTROLO2014Proceedings of the 11th Portuguese Conference on Automatic Control, Springer International Publishing, 449-458.

[36] Meilanov, R. P., Magomedov, R. A. (2014). Thermodynamics in Fractional Calculus. Journal of En-gineering Physics and Thermophysics, 87(6), 1521-1531.

[37] Mohsen, A., El-Gamel, M. (2008). On the Galerkin and collocation methods for two-point boundary value problems using sinc bases. Computers & Mathematics with Applications, 56(4), 930-941.