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Research Article

Generalized λ −Statistical Boundedness of Order β in Sequences of Fuzzy Numbers

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ABSTRACT

In this article, we investigate the idea of Δ_λ^m −statistical boundedness of order β for sequences of fuzzy numbers. Additionally, we provide different inclusion relations between Δ_λ^m −statistical boundedness of order β and Δ_λ^m −statistical convergence of order β.

1. Introduction

In the traditional approach to analysis of convergence, almost all of the terms of a sequence are required to belong to an arbitrarily small neighborhood of the limit. Fast [12] and Steinhaus [23] first proposed the idea of statistical convergence and later Schoenberg [21] gave a formal definition of that concept, independently. The essential tenet of statistical convergence is to loosen the restrictions of this condition and to insist that the convergence requirement be valid only for the vast majority of the elements. Later on, this concept and summability theory was associated by several mathematicians ([1],[5],[6],[9],[10],[13],[14],[18],[19],[22]). Recent years Gadjiev and Orhan [15] broaden the concept of statistical convergence into the ordered statistical convergence. Following this, Çolak [7] and Çolak and Bektaş [8] conducted research on the concept of statistical convergence. These investigations show that the principles of statistical convergence give a significant addition to the enhancement of classical analysis. In the foundational work that was authored by Zadeh [24] the idea of fuzziness was first discovered and presented to the scientific world. In 1986, Matloka [17] provided the notion of fuzzy number sequence, and then in 1995, Nuray and Savaş [20] established the statistical convergence of these sequences.

The statistical boundedness in sequences of fuzzy numbers, was first described by Aytar and Pehlivan [4]. Altinok and Mursaleen [3] then used a difference operator to generalize the statistical boundedness. In their research, Altinok and Et [2] looked into the notion of "λ −statistical boundedness of order β" in relation to sequences of fuzzy numbers. In addition to this, they investigated the monotonicity, symmetricity and solidity of the sequence class $S_\lambda^\beta B(F)$.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers that tends toward infinity such that $\lambda_1 = 1$, $\lambda_{n+1} \leq \lambda_n + 1$. We denote the set of all sequences (λ_n) defined in this way by Λ. We define λ_β −density of a subset E of $\mathbb{N}$ by

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$\delta_\lambda^\beta(\mathbb{E}) = \lim_n \frac{1}{\lambda_n^\beta} |\{k \in I_n : k \in \mathbb{E}\}|$ provided the limit exists,

where $\beta \in (0,1]$ be any real number. Clearly, the λ_β −density of any finite subset of $\mathbb{N}$ is 0 and the equality $\delta_\lambda^\beta(A^c) = 1 - \delta_\lambda^\beta(A)$ does not generally hold for values $\beta \in (0,1)$. The property $\delta_\lambda^\beta(A^c) = 1 - \delta_\lambda^\beta(A)$ holds only for $\lambda_n = n$ for all $n \in \mathbb{N}$ and for $\beta = 1$. In the case of $\lambda_n = n$ for all $n \in \mathbb{N}$, λ_β −density becomes equivalent to the β −density, in the case $\beta = 1$ reduces to the λ −density, in the special case $\beta = 1$ and $\lambda_n = n$ becomes equivalent to the natural density.

We say that x_k fulfills property $p(k)$ for λ −almost all k according to β and this is abbreviated as "$a.a.k_\lambda(\beta)$" if $x = (x_k)$ is a sequence satisfying property $p(k)$ for every k other than a set of λ_β −density zero.

A fuzzy set consists of elements with degrees of membership. The idea of membership function is the most significant aspect of characterizing and defining a fuzzy set, and it is essential to the field of fuzzy sets. If a fuzzy set u on the set of real number $\mathbb{R}$ possesses the criteria listed below, then we refer to that set as a fuzzy number:

$i)$ u is normal,

$ii)$ u is fuzzy convex,

$iii)$ u is upper semi-continuous,

$iv)$ supp$u$ = $cl\{x \in \mathbb{R}: u(x) > 0\}$ is compact.

In this sense, a fuzzy number is a specific case of a normal, convex fuzzy set of the real numbers line and is an extension of real number. For a fuzzy number u, α −level set $[u]^\alpha$ is described by

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R}: u(x) \geq \alpha\}, & if\ \alpha \in [0,1] \\ suppu, & if\ \alpha = 0 \end{cases}$$

When $[u]^\alpha$ is a closed interval for each $\alpha \in [0,1]$ and $[u]^1 \neq \emptyset$, it is obvious that u is a fuzzy number.

Kizmaz [16] defined the difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, which consist of any real-valued sequences $x = (x_k)$ such that $\Delta x = \Delta^1 x = (x_k - x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0. Et and Çolak [11] expanded the concept of difference sequences by making the difference m times such that $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ for $(m = 1,2,3, \ldots)$.

Within the scope of this investigation, we broaden the application of the concept of " $\lambda-$ statistical boundedness of order β " to sequences of fuzzy numbers and present various inclusion relations by making use of the generalized difference operator Δ^m. Moreover, in order to contribute to the field of the fuzzy numbers theory, we present certain relation theorems as a means of filling in the gaps that currently exist.

2. Main Results

In this part, we define and investigate the idea of Δ_λ^m − statistical boundedness of order β for fuzzy sequences, where β denotes any real integer such that $\beta \in (0,1]$.

Definition 1. Let $\lambda = (\lambda_n) \in \Lambda$, $\beta \in (0,1]$ and $X = (X_k)$ be a fuzzy sequence. A sequence $X = (X_k)$ is said to be a Δ_λ^m −statistically Cauchy sequence of order β if there exists a natural number $N(= N(\varepsilon))$ for every $\varepsilon > 0$ such that $d(X_k, X_N) < \varepsilon$ for $a.a.k_\lambda(\beta)$. i.e.

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k \in I_n : d(\Delta^m X_k, X_N) \geq \varepsilon\}| = 0.$$

Definition 2. Let $\lambda = (\lambda_n) \in \Lambda$, $\beta \in (0,1]$ and $X = (X_k)$ be a fuzzy sequence. It is said that a fuzzy sequence (X_k) is Δ_λ^m −statistically bounded above of order β if there is some value u satisfying

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k \in I_n : \Delta^m X_k > u\} \cup \{k \in I_n : \Delta^m X_k \nsim u\}| = 0.$$

Similarly, if we can find a fuzzy number u satisfying

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k \in I_n : \Delta^m X_k < u\} \cup \{k \in I_n : \Delta^m X_k \nsim u\}| = 0.$$

then a sequence (X_k) is said to be Δ_λ^m −statistically bounded below of order β. Here, we use the symbol $\nsim$ to show incomparable elements in $L(\mathbb{R})$.

A sequence $X = (X_k)$ of fuzzy numbers is Δ_λ^m −statistical bounded of order β if and only if (X_k) is both Δ_λ^m −statistical bounded above of order β and Δ_λ^m −statistical bounded below of order β. We will refer to the set of all Δ_λ^m −statistically bounded sequences of order of β fuzzy number sequences as $S^\beta B(\Delta_\lambda^m, F)$. On the other hand, $SB(\Delta_\lambda^m, F)$ will be used to designate the set of all Δ_λ^m − statistically bounded fuzzy sequences, $S^\beta B(\Delta^m, F)$ will be used to denote the set of all Δ^m − statistically bounded fuzzy sequences of order β, and $SB(\Delta^m, F)$ will be used to denote the set of all Δ^m − statistically bounded fuzzy sequences.

Theorem 3. Let β be a constant such that $\beta \in (0,1]$ and sequence $\lambda = (\lambda_n)$ belongs to space Λ. Then, if any sequence (X_k) is Δ^m −bounded, then it is Δ_λ^m −statistically bounded of order β, but the opposite is not always correct.

Proof. It is known that an empty set has zero $\beta-$ density, so the first part of the proof is straightforward. To see the inverse, we take a sequence $X = (X_k)$ such that

$$X_k(x) = \begin{cases} \left.\begin{matrix} \frac{1}{2}(x-k+2), & \text{for } k-2 \le x \le k \\ \frac{1}{2}(k+2-x), & \text{for } k \le x \le k+2 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k = n^2 \ (n = 1,2,3,\dots) \\ \left.\begin{matrix} \frac{1}{2}(x+1), & \text{for } -1 \le x \le 1 \\ \frac{1}{2}(3-x), & \text{for } 1 \le x \le 3 \\ 0, & \text{otherwise} \end{matrix}\right\} := X_0 & \text{if } k \ne n^2. \end{cases}$$

and we obtain

$$[X_k]^\alpha = \begin{cases} [2\alpha + k - 2, k + 2 - 2\alpha] & \text{if } k = n^2 \\ [2\alpha - 1, 3 - 2\alpha] & \text{if } k \ne n^2. \end{cases}$$

After routine operations, α −level sets and membership functions of (ΔX_k) and $(\Delta^2 X_k)$ can be found as follows:

$$[\Delta X_k]^\alpha = \begin{cases} [4\alpha + k - 5, k - 4\alpha + 3] & \text{if } k = n^2 \\ [4\alpha - k - 4, 4 - 4\alpha - k] & \text{if } k + 1 = n^2 \\ [4\alpha - 4, 4 - 4\alpha] & \text{otherwise} \end{cases}$$

$$\triangle X_k(x) = \begin{cases} \left.\begin{matrix} \frac{1}{4}(x-k+5), & k-5 \le x \le k-1 \\ \frac{1}{4}(-x+k+3), & k-1 \le x \le k+3 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k = n^2 \ n = (1,2,3,\dots.) \\ \left.\begin{matrix} \frac{1}{4}(x+k+4), & -k-4 \le x \le -k \\ \frac{1}{4}(-x-k+4), & -k \le x \le k+4 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k + 1 = n^2 \\ \begin{matrix} \frac{1}{4}(x+4), & -4 \le x \le 0 \\ \frac{1}{4}(-x+4), & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{matrix} & \text{if } k \ne n^2 \end{cases}$$

and

$$[\Delta^2 X_k]^\alpha = \begin{cases} [8\alpha + k - 9, k - 8\alpha + 7] & \text{if } k = n^2 \\ [8\alpha + k - 7, k - 8\alpha + 9] & \text{if } k + 1 = n^2 \\ [8\alpha - 2k - 8, -2k - 8\alpha + 8] & \text{if } k + 2 = n^2 \\ [8\alpha - 8, 8 - 8\alpha] & \text{otherwise} \end{cases}$$

$$\triangle^2 X_k(x) = \begin{cases} \left.\begin{matrix} \frac{1}{8}(x-k+9), & k-9 \le x \le k-1 \\ \frac{1}{8}(-x+k+7), & k-1 \le x \le k+7 \\ 0 & \text{otherwise} \end{matrix}\right\} & \text{if } k = n^2 \ n = (1,2,3,\dots.) \\ \left.\begin{matrix} \frac{1}{8}(x-k+7), & k-7 \le x \le k+1 \\ \frac{1}{8}(-x+k+9), & k+1 \le x \le k+9 \\ 0 & \text{otherwise} \end{matrix}\right\} & \text{if } k + 1 = n^2 \\ \left.\begin{matrix} \frac{1}{8}(x+2k+8), & -2k-8 \le x \le -2k \\ \frac{1}{8}(-x-2k+8), & -2k \le x \le -2k+8 \\ 0 & \text{otherwise} \end{matrix}\right\} & \text{if } k + 2 = n^2 \\ \left.\begin{matrix} \frac{1}{8}(x+8), & -8 \le x \le 0 \\ \frac{1}{8}(-x+8), & 0 \le x \le 8 \\ 0 & \text{otherwise} \end{matrix}\right\} & \text{otherwise } k \ne n^2 \end{cases}$$

In the same manner, if we keep taking the difference m times for $m \in \mathbb{N}$, we can readily demonstrate that $X = (X_k) \in S^\beta B(\Delta_\lambda^m, F)$ for $\beta > \frac{1}{2}$, but it is (X_k) is not $SB(\Delta^m, F)$ since

$$\delta^\beta(\{k \in \mathbb{N}: \Delta_\lambda^m X_k > X_0\} \cup \{k \in \mathbb{N}: \Delta_\lambda^m X_k \nsim X_0\}) = 0$$

and

$$\delta^\beta(\{k \in \mathbb{N}: \Delta_\lambda^m X_k < X_0\} \cup \{k \in \mathbb{N}: \Delta_\lambda^m X_k \nsim X_0\}) = 0,$$

where $[X_0]^\alpha = [2^{m+1}(\alpha - 1), 2^{m+1}(1 - \alpha)]$, specifically, when $\beta = 1$ and $\lambda_n = n$ (See Fig. 1 for $m = 1$).

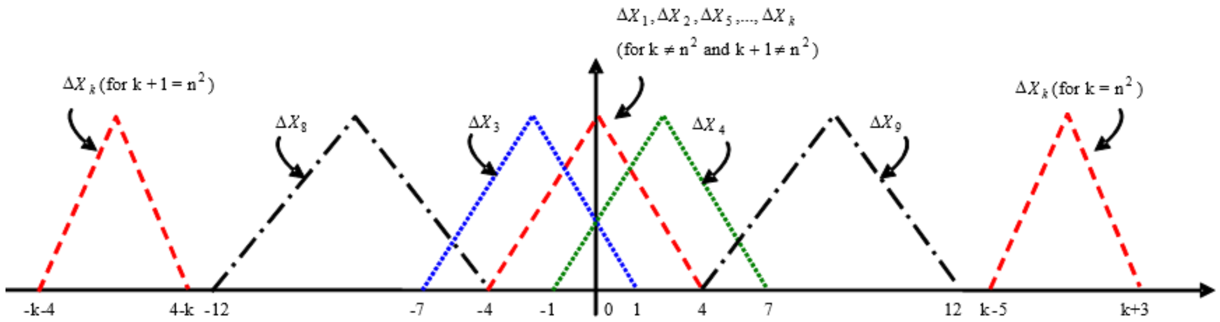

Figure 1. X_k *is* Δ^m *− statistically bounded of order* β, *but not* Δ^m *− bounded for* $m = 1$

Theorem 4. Let β be a constant such that $\beta \in (0,1]$ and sequence $\lambda = (\lambda_n)$ belongs to space Λ. For any fuzzy sequence (X_k), if $(X_k) \in S^\beta(\Delta_\lambda^m, F)$, then $(X_k) \in S^\beta B(\Delta_\lambda^m, F)$, but the opposite is not correct.

Proof. Let fuzzy sequence $X = (X_k)$ belongs to sequence class $S^\beta(\Delta_\lambda^m, F)$. Then, we can talk about the existence of some number X_0 in fuzzy number space satisfying equality

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k \in I_n : d(\Delta^m X_k, X_0) \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. Now we can write

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\beta} \left|\{k \in I_n : d\left(\Delta^m X_k, \overline{0}\right) \geq X_0 + \varepsilon\}\right|$$

$$\leq \lim_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k \in I_n : d(\Delta^m X_k, X_0) \geq \varepsilon\}|.$$

For the aforementioned inequality, since fuzzy sequence $(X_k) \in S^\beta(\Delta_\lambda^m, F)$, then it can be seen that the right side approaches 0.

To see the inverse, we take a sequence $X = (X_k)$ such that

$X_k(x) =$

$$\begin{cases} \left.\begin{matrix} \frac{1}{2}(x-1), & \text{for } 1 \leq x \leq 3 \\ \frac{1}{2}(5-x), & \text{for } 3 \leq x \leq 5 \\ 0, & \text{otherwise} \end{matrix}\right\} := L_1 & \text{if } k \text{ is odd} \\ \left.\begin{matrix} \frac{1}{2}(x-7), & \text{for } 7 \leq x \leq 9 \\ \frac{1}{2}(11-x), & \text{for } 9 \leq x \leq 11 \\ 0, & \text{otherwise} \end{matrix}\right\} := L_2 & \text{if } k \text{ is even} \end{cases}$$

and we obtain

$$[X_k]^\alpha = \begin{cases} [2\alpha + 1, 5 - 2\alpha] & \text{if } k \text{ is odd} \\ [2\alpha + 7, 11 - 2\alpha] & \text{if } k \text{ is even} \end{cases}$$

After routine operations, α −level sets and membership functions of (ΔX_k), $(\Delta^2 X_k)$ and $(\Delta^m X_k)$ can be found as follows:

$$[\triangle X_k]^\alpha = \begin{cases} [4\alpha - 10, -4\alpha - 2], & \text{if } k \text{ is odd} \\ [4\alpha + 2, -4\alpha + 10], & \text{if } k \text{ is even} \end{cases}$$

$\triangle X_k(x) =$

$$\begin{cases} \left.\begin{matrix} \frac{1}{4}(x+10), & -10 \leq x \leq -6 \\ \frac{1}{4}(-x-2), & -6 \leq x \leq -2 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k \text{ is odd} \\ \left.\begin{matrix} \frac{1}{4}(x-2), & 2 \leq x \leq 6 \\ \frac{1}{4}(-x+10), & 6 \leq x \leq 10 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k \text{ is even} \end{cases}$$

$$[\triangle^2 X_k]^\alpha = \begin{cases} [8\alpha - 20, -8\alpha - 4], & \text{if } k \text{ is odd} \\ [8\alpha + 4, -8\alpha + 20], & \text{if } k \text{ is even} \end{cases}$$

$\triangle^2 X_k(x) =$

$$\begin{cases} \left.\begin{matrix} \frac{1}{8}(x+20), & -20 \leq x \leq -12 \\ \frac{1}{8}(-x-4), & -12 \leq x \leq -4 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k \text{ is odd} \\ \left.\begin{matrix} \frac{1}{8}(x-4), & 4 \leq x \leq 12 \\ \frac{1}{8}(-x+20), & 12 \leq x \leq 20 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k \text{ is even} \end{cases}$$

and

$[\Delta^m X_k]^\alpha =$

$$\begin{cases} [2^{m-1}(4\alpha - 10), 2^{m-1}(-4\alpha - 2)] & \text{if } k \text{ is odd} \\ [2^{m-1}(4\alpha + 2), 2^{m-1}(10 - 4\alpha)] & \text{if } k \text{ is even} \end{cases}$$

$$\triangle^m X_k(x) =$$

$$\begin{cases} \left.\begin{matrix} \frac{1}{4}(2^{1-m}x+10), & -10.2^{m-1} \le x \le -6.2^{m-1} \\ \frac{1}{4}(-2^{1-m}x-2), & -6.2^{m-1} \le x \le -2.2^{m-1} \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k \text{ is odd} \\ \left.\begin{matrix} \frac{1}{2}(2^{-m}x-1), & 2^m \le x \le 3.2^m \\ \frac{1}{2}(-2^{-m}x+5), & 3.2^m \le x \le 5.2^m \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k \text{ is even} \end{cases}$$

Then, we conclude that $X = (X_k) \notin S^\beta(\Delta_\lambda^m, F)$, but it $(X_k) \in S^\beta B(\Delta_\lambda^m, F)$ in the special case $\lambda_n = n$, for all $n \in \mathbb{N}$ (See Fig. 2).

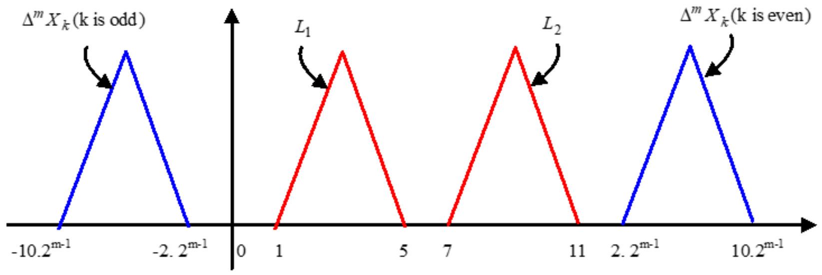

Figure 2. (X_k) *is* Δ_λ^m -statistically bounded of order β, but not Δ_λ^m – statistically convergent of order β for $\lambda_n = n$

Corollary 5. Let β be a constant such that $\beta \in (0,1]$, $(X_k) \in L(\mathbb{R})$and sequence $\lambda = (\lambda_n)$ belongs to space Λ.

$i)$ If $(X_k) \in S^\beta(\Delta_\lambda^m, F)$, then $(X_k) \in S^\beta B(\Delta_\lambda^m, F)$,

$ii)$ If $(X_k) \in S^\beta(\Delta_\lambda^m, F)$, then $(X_k) \in SB(\Delta_\lambda^m, F)$,

$iii)$ If $(X_k) \in S(\Delta_\lambda^m, F)$, then $(X_k) \in SB(\Delta_\lambda^m, F)$.

The opposite of above statements is not correct.

Theorem 6. Let the parameters β and γ are fixed real numbers such that $0 < \beta \le \gamma \le 1$, $(X_k) \in L(\mathbb{R})$ and $\lambda = (\lambda_n) \in \Lambda$. If $(X_k) \in S^\beta B(\Delta_\lambda^m, F)$, then $(X_k) \in S^\gamma B(\Delta_\lambda^m, F)$, but the opposite is not correct.

Proof. Let $0 < \beta \le \gamma \le 1$, then

$$\delta^\gamma(\{k \in \mathbb{N}: \Delta_\lambda^m X_k > u\} \cup \{k \in \mathbb{N}: \Delta_\lambda^m X_k \nsim u\})$$
$$\subseteq \delta^\beta(\{k \in \mathbb{N}: \Delta_\lambda^m X_k > u\} \cup \{k \in \mathbb{N}: \Delta_\lambda^m X_k \nsim u\})$$

and similarly

$$\delta^\gamma(\{k \in \mathbb{N}: \Delta_\lambda^m X_k < u\} \cup \{k \in \mathbb{N}: \Delta_\lambda^m X_k \nsim u\})$$
$$\subseteq \delta^\gamma(\{k \in \mathbb{N}: \Delta_\lambda^m X_k < u\} \cup \{k \in \mathbb{N}: \Delta_\lambda^m X_k \nsim u\})$$

This is the first part of the proof. Define the sequence (X_k) in the following way for the second half of the proof:

$$X_k(x) =$$

$$\begin{cases} \left.\begin{matrix} \frac{1}{2}(x-k+1), & k-1 \le x \le k+1 \\ \frac{1}{2}(-x+k+3), & k+1 \le x \le k+3 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k = n^3 \\ \left.\begin{matrix} \frac{x}{2}, & 0 \le x \le 2 \\ 2-\frac{x}{2}, & 2 \le x \le 4 \\ 0, & \text{otherwise} \end{matrix}\right\} := X_0 & \text{if } k \ne n^3 \end{cases}$$

and we obtain

$$[X_k]^\alpha = \begin{cases} [2\alpha+k-1, k+3-2\alpha], & \text{if } k = n^3 \\ [2\alpha, 4-2\alpha], & \text{if } k \ne n^3 \end{cases}$$

After routine operations, α –level sets and membership functions of (ΔX_k) can be found as follows:

$$[\Delta X_k]^\alpha =$$

$$\begin{cases} [4\alpha+k-5, k-4\alpha+3] & \text{if } k = n^3 \\ [4\alpha-k-4, 4-4\alpha-k] & \text{if } k+1 = n^3 \\ [4\alpha-4, 4-4\alpha] & \text{otherwise} \end{cases}$$

$\triangle X_k(x) =$

$$\begin{cases} \left.\begin{matrix} \frac{1}{4}(x-k+5), & k-5 \le x \le k-1 \\ \frac{1}{4}(-x+k+3), & k-1 \le x \le k+3 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k = n^3 \\ \left.\begin{matrix} \frac{1}{4}(x+k+4), & -k-4 \le x \le -k \\ \frac{1}{4}(-x-k+4), & -k \le x \le -k+4 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k+1 = n^3 \\ \begin{matrix} \frac{1}{4}(x+4), & -4 \le x \le 0 \\ \frac{1}{4}(-x+4), & 0 \le x \le 4 \\ 0, & \text{otherwise} \end{matrix} & \text{if } k \ne n^3 \end{cases}$$

Then, we can say that $(X_k) \notin S^\beta B(\Delta_\lambda^m, F)$ for $0 < \beta \le \frac{1}{3}$, but $(X_k) \in S^\gamma B(\Delta_\lambda^m, F)$ for $\frac{1}{3} < \gamma \le 1$ in the special case $\lambda_n = n.$, then $(X_k) \in SB(\Delta_\lambda^m, F)$

Corollary 7.

i) The sequence classes $S^\beta B(\Delta_\lambda^m, F)$ and $S^\gamma B(\Delta_\lambda^m, F)$ are equivalent ⇔ $\beta = \gamma$.

ii) If $(X_k) \in S^\beta B(\Delta_\lambda^m, F)$, then $(X_k) \in SB(\Delta_\lambda^m, F)$ for $0 < \beta \le 1$.

Since it's easy to show that each of the following results is true, we're just going to say them without giving proof.

Theorem 8. Let β be a constant such that $\beta \in (0,1]$, $(X_k) \in L(\mathbb{R})$ and $\lambda = (\lambda_n) \in \Lambda$. If $(X_k) \in S^\beta(\Delta_\lambda^m, F)$, then it is $\Delta_\lambda^m -$ statistically Cauchy sequence of order β.

Theorem 9. Let β be a constant such that $\beta \in (0,1]$, $(X_k) \in L(\mathbb{R})$ and $\lambda = (\lambda_n) \in \Lambda$. Every Δ_λ^m −statistically Cauchy fuzzy sequence of order β is Δ_λ^m −statistical bounded of order β.

Theorem 10. Let β be a constant such that $\beta \in (0,1]$, $(X_k) \in L(\mathbb{R})$ and $\lambda = (\lambda_n) \in \Lambda$. Every $\Delta_\lambda^m -$ bounded sequence of fuzzy numbers is Δ_λ^m −statistically bounded of order β.

Proof. The first part of the proof is straightforward. Define the sequence (X_k) in the following way for the second half of the proof:

$X_k(x) =$

$$\begin{cases} \left.\begin{matrix} x-2^k+1, & \text{for } 2^k-1 \le x \le 2^k \\ 2^k+1-x, & \text{for } 2^k \le x \le 2^k+1 \\ 0, & \text{otherwise} \end{matrix}\right\} & \begin{matrix} \text{if } k = 3^n \\ (n = 1,2,\dots) \end{matrix} \\ \left.\begin{matrix} x-1, & \text{for } 1 \le x \le 2 \\ -x+3, & \text{for } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k \ne 3^n \end{cases}$$

and we obtain

$[X_k]^\alpha =$

$$\begin{cases} [\alpha+2^k-1, -\alpha+2^k+1], & \begin{matrix} \text{if } k = 3^n \\ (n = 1,2,3,\dots) \end{matrix} \\ [\alpha+1, -\alpha+3], & \text{if } k \ne 3^n \end{cases}$$

After routine operations, α −level sets and membership functions of (ΔX_k) can be found as follows:

$[\Delta X_k]^\alpha =$

$$\begin{cases} [2\alpha+2^k-4, 2^k-2\alpha] & \text{if } k = 3^n \\ [2\alpha-2^{k+1}, 4-2\alpha-2^{k+1}] & \text{if } k+1 = 3^n \\ [2\alpha-2, 2-2\alpha] & \text{otherwise} \end{cases}$$

$\triangle X_k(x) =$

$$\begin{cases} \left.\begin{matrix} \frac{1}{2}(x-2^k+4), & 2^k-4 \le x \le 2^k-2 \\ \frac{1}{2}(-x+2^k), & 2^k-2 \le x \le 2^k \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k = 3^n \\ \left.\begin{matrix} \frac{1}{2}(x+2^{k+1}), & -2^{k+1} \le x \le -2^{k+1}+2 \\ \frac{1}{2}(-x-2^{k+1}+4), & -2^{k+1}+2 \le x \le -2^{k+1}+4 \\ 0, & \text{otherwise} \end{matrix}\right\} & \text{if } k+1 = 3^n \\ \begin{matrix} \frac{1}{2}(x+2), & -2 \le x \le 0 \\ \frac{1}{4}(-x+2), & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{matrix} & \text{if } k \ne 3^n \end{cases}$$

In the same manner, if we keep taking the difference m times for $m \in \mathbb{N}$, we can readily demonstrate that $(X_k) \in S^\beta B(\Delta_\lambda^m, F)$, but it is not Δ_λ^m −bounded for $\lambda_n = n$, $\beta = 1$ and $m = 1$ (See Fig. 3).

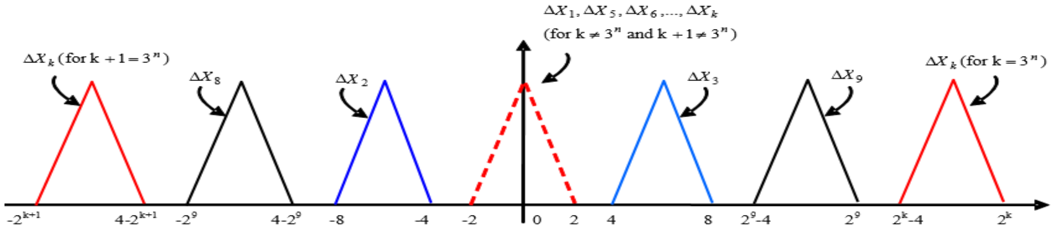

Figure 3. (X_k) *is* $\Delta_\lambda^m -$ *statistically bounded of order* β*, but not* $\Delta_\lambda^m -$ *bounded for* $m = 1$ *and* $\lambda_n = n$

3. Conclusion

After Matloka [17] gave the definition of fuzzy sequence, many studies were carried out on this subject and a relationship was established with the summability theory. Now in this paper, we defined the sequence class $S^{\beta}B(\Delta_{\lambda}^{m},F)$ by making use of the generalized difference operator Δ^{m} and any sequence (λ_n). Furthermore, we presented various inclusion relations between this sequence class and other ones as a means of filling in the gaps that currently exist.

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