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# A NOTE ON CONNECTION FORMS ON THE QUATERNIONIC HOPF BUNDLE

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## ABSTRACT

It is known that there exists a family of connection 1-forms depending on two parameters on the standard quaternionic Hopf bundle. This family is constructed by using the canonical connection 1-form. In this work, the self duality and the anti-self duality of these connection 1-forms are investigated. Parameters for which the family of connection 1-forms are self dual and anti-self dual are determined.

**Keywords:** Principal fibre bundle, Connection 1-form, Self (anti-self) dual form, Instanton

## KUATERNİONİK HOPF DEMETİ ÜZERİNDE BAĞLANTI FORMLARI ÜZERİNE BİR NOT

## ÖZET

Standart kuaternionik Hopf demeti üzerinde iki parametreye bağlı bağlantı formlarının bir ailesinin varlığı bilinmektedir. Bu bağlantı formların ailesi kanonik konneksiyon 1-formu kullanılarak inşaa edilmiştir. Bu çalışmada, bu bağlantı 1-formlarının self dual ve anti-self dualliği araştırılmıştır. Bu bağlantı 1-formların hangi parametreler için self dual ya da anti- self dual olduğu belirlenmiştir.

**Anahtar Kelimeler:** Asli lif demeti, Bağlantı 1-formu, Self (anti-self) dual form, İnstanton

## 1. INTRODUCTION

It is known that instantons are important for the topological invariants of 4-manifolds [1,2]. Geometrically, instantons are connection 1-forms (gauge potentials) on principle fiber bundles over 4-dimensional manifolds whose curvature 2-forms (gauge fields associated with gauge potentials) are self-dual (or anti-self dual, when orientation is reversed). The natural examples of instantons are given on the standard quaternionic Hopf bundle [1,3,4]. Actually, self-dual (or anti self-dual) connection 1-forms satisfy the Yang-Mills equations. Earlier, special solutions to the Yang-Mills equations were given in [3]. These solutions (the BPST instantons) are called pseudoparticles. After this work, Trautman showed that [5] the solutions of the Yang-Mills equations (the BPST instantons) correspond to the canonical connections on the complex and quaternionic Hopf bundles $S^1 \to S^3 \to S^2$ and $S^3 \to S^7 \to S^4$. In his work, nontrivial solutions of Maxwell and Yang-Mills equations are constructed and the curvature 2-form (electromagnetic field) $\mathcal{F}$ is self dual ($*\mathcal{F} = \mathcal{F}$) on the two dimensional complex projective space with the Fubini-Study metric. On the other hand, Minami [6] produced two specific connections on $S^7$ by using Gauge transformations and showed that the gauge potentials which are the pullbacks of these two connections are identical with solutions of $SU(2)$ Yang-Mills equations. There also exist similar results for some higher dimensions. Corrigan, Devchand and Fairlie studied the gauge field equation in dimensions greater than four [7]. In [8], the Hopf bundle $S^7 \to S^{15} \to S^8$ is considered and a solution of eight dimensional Euclidean Yang-Mills field equation is obtained.

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A family of connection 1-forms depending on $n$ parameters is given in [9] on complex and quaternionic Hopf bundles $S^1 \to S^{2n-1} \to \mathbb{C}P_n$ and $S^3 \to S^{4n-1} \to \mathbb{H}P_n$, respectively. These connections are expressed in the following theorem:

**Theorem 1** The $i$Imℂ-valued 1-form

$$\omega_{(r_1,r_2,\cdots,r_n)} = i\, Im\left(\frac{\|z_1\|^{r_1}\bar{z}_1 dz_1 + \|z_2\|^{r_2}\bar{z}_2 dz_2 + \cdots + \|z_n\|^{r_n}\bar{z}_n dz_n}{\|z_1\|^{r_1+2} + \|z_2\|^{r_2+2} + \cdots + \|z_n\|^{r_n+2}}\right)$$

on $S^{2n-1}$ is a connection 1-form on the complex Hopf bundle $S^1 \to S^{2n-1} \to \mathbb{C}P_n$, where $r_1, r_2, \cdots, r_n$ are positive real numbers.

The $Im(\mathbb{H})$-valued 1-form

$$\omega_{(s_1,s_2,\cdots,s_n)} = Im\left(\frac{\|q_1\|^{s_1}\bar{q}_1 dq_1 + \|q_2\|^{s_2}\bar{q}_2 dq_2 + \cdots + \|q_n\|^{s_n}\bar{q}_n dq_n}{\|q_1\|^{s_1+2} + \|q_2\|^{s_2+2} + \cdots + \|q_n\|^{s_n+2}}\right)$$

on $S^{4n-1}$ is a connection 1-form on the complex Hopf bundle $S^3 \to S^{4n-1} \to \mathbb{H}P_n$, where $s_1, s_2, \cdots, s_n$ are positive real numbers.

In this work, we investigate the duality properties of the two parametric family of connection 1-forms

$$\omega_{(r,s)} = Im\left(\frac{\|q_1\|^{r}\bar{q}_1 dq_1 + \|q_2\|^{s}\bar{q}_2 dq_2}{\|q_1\|^{r+2} + \|q_2\|^{s+2}}\right) \qquad (1.1)$$

to obtain new Yang-Mills solutions on the quaternionic Hopf bundle $S^3 \to S^7 \to S^4$. For $r = s = 0$, $\omega_{(0;0)}$ coincides with the canonical connection on the quaternionic Hopf bundle.

## 2. PRELIMINARIES

Now, let us explain the materials which will be used in this work. One of these, is the principal fibre bundle which is one of the most fundamental concepts of differential geometry and topology. A principal fiber bundle consists of three manifolds $G, P, M$ which are base space, total space and fiber (a Lie group), respectively; $\pi$ is a differentiable map of P onto M and $\sigma : P \times G \to P$, $\sigma(p, g) = p \cdot g$ is a right action of $G$ on $P$ such that following conditions are satisfied:

1. $\pi(p \cdot g) = \pi(p)$, for all $p \in P$ and $g \in G$.

2. (Local Triviality): For each $m \in M$, there exists an open neighborhood $m \in U$ and a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times G$ such that $\Psi(p) = (\pi(p), \psi(p))$, where $\psi : \pi^{-1}(U) \to G$ satisfies

$$\psi(p \cdot g) = \psi(p) \cdot g,$$

for all $p \in \pi^{-1}(U)$ and $g \in G$.

Another basic concept is connection 1-form, which is a Lie algebra valued 1-form, on a principal fibre bundle and has an important role in geometry, topology and mathematical physics (see [4]). Hopf bundles are specific examples of principal fibre bundles. There are canonical connection 1-forms on Hopf bundles $S^3 \to S^7 \to S^4$ and $S^1 \to S^3 \to S^2$. These connection 1-forms are explicitly expressed in [4] and are generalized to higher dimensions in [9].

Consider the quaternionic Hopf bundle $S^3 \to S^7 \to \mathbb{H}P_1 \cong S^4$, where

$$S^7 = \{(q_1, q_2)\epsilon\mathbb{H}^2 : \ \|q_1\|^2 + \|q_2\|^2 = 1\}.$$

There is a right action of $S^3$ on $S^7$ given by

$$(q_1, q_2) \cdot g = (q_1 g, q_{2g}),$$

where $(q_1, q_2)\epsilon S^7$ and $g \in S^3$. The Lie algebra of $S^3$ can be identified with $Im\mathbb{H}$. The projection map of this bundle is $\pi(q_1, q_2) = [q_1, q_2] \in \mathbb{H}P_1 \cong S^4$. Standard trivializations $(U_1, \Psi_1)$ and $(U_2, \Psi_2)$ are as follows:

$$U_1 = \{[q_1, q_2] \in \mathbb{H}P_1 : q_1 = 0\}, \quad U_2 = \{[q_1, q_2] \in \mathbb{H}P_1 : q_2 = 0\},$$

$$\Psi_1 : \pi^{-1}(U_1) \to U_1 \times S^3, \qquad \Psi_1(q_1, q_2) = \left([q_1, q_2], \frac{q_1}{\|q_1\|}\right),$$

$$\Psi_2 : \pi^{-1}(U_2) \to U_2 \times S^3, \qquad \Psi_2(q_1, q_2) = \left([q_1, q_2], \frac{q_2}{\|q_2\|}\right)$$

and inverses of $\Psi_1$ and $\Psi_2$ are

$$\Phi_1 = (\Psi_1)^{-1} : U_1 \times S^3 \to \pi^{-1}(U_1), \qquad \Phi_1([q_1, q_2], g) = (\|q_1\|g, q_2(q_1)^{-1}\|q_1\|g),$$

$$\Phi_2 = (\Psi_2)^{-1} : U_2 \times S^3 \to \pi^{-1}(U_2), \qquad \Phi_2([q_1, q_2], g) = (q_1(q_2)^{-1}\|q_2\|g, \|q_2\|g),$$

where $g \in S^3$. Canonical local cross-sections induced by local trivializations $(U_1, \Psi_1)$ and $(U_2, \Psi_2)$ are

$$s_1 : U_1 \to \pi^{-1}(U_1) \subset S^7, \qquad s_1([q_1, q_2]) = \left(\|q_1\|, \frac{q_2\|q_1\|}{q_1}\right),$$

$$s_2 : U_2 \to \pi^{-1}(U_2) \subset S^7, \qquad s_2([q_1, q_2]) = \left(\frac{q_1\|q_2\|}{q_2}, \|q_2\|\right).$$

The diffeomorphisms corresponding to standard coordinate neighborhoods $U_1$ and $U_2$ on $\mathbb{H}P_1$ are

$$\phi_1 : U_1 \to \mathbb{H} \cong \mathbb{R}^4, \qquad \phi_1([q_1, q_2]) = q_2 {q_1}^{-1}$$

and

$$\phi_2 : U_2 \to \mathbb{H} \cong \mathbb{R}^4, \qquad \phi_2([q_1, q_2]) = q_1 {q_2}^{-1}.$$

Thus, for all $q \in \mathbb{H} - \{0\}$, one has

$$\left(s_1 \circ {\phi_1}^{-1}\right)(q) = \frac{1}{\sqrt{1+\|q\|^2}}(1, q),$$

$$\left(s_2 \circ {\phi_2}^{-1}\right)(q) = \frac{1}{\sqrt{1+\|q\|^2}}(q, 1).$$

Details can be found in [1, 4]. It is known that the canonical connection 1-form on the quaternionic Hopf bundle $S^3 \to S^7 \to S^4$

$$\omega = Im(\bar{q}_1\, dq_1 + \bar{q}_2\, dq_2)$$

is anti-self dual (or self dual) [1, 3, 4]. In [9], duality or anti-self duality of families of connection 1-forms $\omega_{(r,s)}$ are not investigated. In this work, similar calculations are done for the two-parameters family of $Im(\mathbb{H})$-valued connection 1-forms

$$\omega_{(r,s)} = Im\left(\frac{\|q_1\|^r \bar{q}_1\, dq_1 + \|q_2\|^s \bar{q}_2\, dq_2}{\|q_1\|^{r+2} + \|q_2\|^{s+2}}\right),$$

on the quaternionic Hopf bundle $S^3 \to S^7 \to S^4$ to investigate the duality properties, where $r$ and $s$ are positive real numbers. Note that, if $r$ and $s$ are negative, then $\omega_{(r,s)}$ are not defined for $\|q_1\| = 0$, $\|q_2\| = 0$.

Let $M$ be a 4-dimensional Riemannian manifold. It is known that in 4-dimension the Hodge star operator is an involution on 2-forms, that is

$$*: \wedge^2(M) \to \wedge^2(M) \text{ and } *^2 = \text{id}\,.$$

The 2-forms associated to the eigenvalue +1 are called self dual and the 2-forms associated to the eigenvalue -1 are called anti-self dual forms. Then, the space of 2-forms decomposes into self dual and anti-self dual parts. Let $\{e_0, e_1, e_2, e_3\}$ be a local orthonormal frame on an open set $U \subset M$ and $e^i$ the 1-form dual to $e_i$, for $0 \leq i \leq 3$. In local coordinates

$$e^0 \wedge e^1 - e^2 \wedge e^3, \qquad e^0 \wedge e^2 + e^1 \wedge e^3, \qquad e^0 \wedge e^3 - e^1 \wedge e^2$$

is a basis for self dual forms and

$$e^0 \wedge e^1 + e^2 \wedge e^3, \qquad e^0 \wedge e^2 - e^1 \wedge e^3, \qquad e^0 \wedge e^3 + e^1 \wedge e^2$$

is a basis for anti-self dual forms.

## 3. THE DUALITY OF CONNECTIONS

It is proved in [1, 3, 4] that pullbacks of the canonical connection 1-form $\omega$ to $\mathbb{R}^4 \cong \mathbb{H}$ by $s_1 \circ {\phi_1}^{-1}$ and $s_2 \circ {\phi_2}^{-1}$ are anti-self-dual. In this section, we compute pullbacks of connection 1-forms $\omega_{(r,s)}$ to $\mathbb{H}$ by the canonical local cross section $s_1$ and the standard chart $(U_1, \Psi_1)$ on $\mathbb{H}P_1$ by similar arguments to [4]. Let $k := s_1 \circ {\phi_1}^{-1}: \mathbb{H} \to \pi^{-1}(U_1)$. Then, one can evaluate

$$\left(k^* \omega_{(r,s)}\right)_q (v_q) = \frac{\|q\|^s}{(1 + \|q\|^2)^{\frac{s-r}{2}} + \|q\|^{s+2}} Im(\bar{q} v),$$

where $q \in \mathbb{H}$ and $v_q \in T_q\mathbb{H}$. Hence, we have

$$\mathcal{A}_{(r,s)} \coloneqq k^* \omega_{(r,s)} = i\, Im\,(g(q) dq),$$

where

$$g(q) = f(q)\bar{q}, \qquad f(q) = \frac{\|q\|^s}{(1 + \|q\|^2)^{\frac{s-r}{2}} + \|q\|^{s+2}}.$$

Thus, if $q \neq 0$, then $\mathcal{A}_{(r,s)}$ is defined for all real numbers $r, s$. If $\eta = Im(g(q)dq)$ is an $Im\mathbb{H}$-valued 1-form on $\mathbb{R}^4$, one can easily check that [4]

$$d\eta + \eta \wedge \eta = Im\{dg \wedge dq + g(q)dq \wedge g(q)dq\}.$$

After tedious calculations, curvature 2-form $\mathcal{F}_{(r;s)}$ of $\mathcal{A}_{(r;s)}$ is

$$\begin{aligned}\mathcal{F}_{(r,s)} = {} & A(dq_0 \wedge dq_1 - dq_2 \wedge dq_3)\boldsymbol{i} + A(dq_0 \wedge dq_2 + dq_1 \wedge dq_3)\boldsymbol{j} \\ & A(dq_0 \wedge dq_3 - dq_1 \wedge dq_2)\boldsymbol{k} + B(dq_0 \wedge dq_1 + dq_2 \wedge dq_3)\bar{q}\boldsymbol{i}q \\ & B(dq_0 \wedge dq_2 - dq_1 \wedge dq_3)\bar{q}\boldsymbol{j}q + B(dq_0 \wedge dq_3 + dq_1 \wedge dq_2)\bar{q}\boldsymbol{k}q\end{aligned}$$

where

$$\begin{aligned}A &= 2\left(f(q) + \frac{K}{2}\|q\|^2\right) - \|q\|^2\left(f(q)^2 + \frac{K}{2}\right) \\ &= \frac{\left((4+s) + (4+r)\|q\|^2\right)\|q\|^s(1+\|q\|^2)^{\frac{s-r}{2}-1}}{2\left((1+\|q\|^2)^{\frac{s-r}{2}} + \|q\|^{s+2}\right)^2}\end{aligned}$$

and

$$\begin{aligned}B &= f(q)^2 + \frac{K}{2} \\ &= \frac{(s + r\|q\|^2)\|q\|^{s-2}(1+\|q\|^2)^{\frac{s-r}{2}-1}}{2\left((1+\|q\|^2)^{\frac{s-r}{2}} + \|q\|^{s+2}\right)^2}.\end{aligned}$$

If similar calculations are done for $s_2 \circ {\phi_2}^{-1}$, we get curvature 2-form on $\mathbb{H} \cong \mathbb{R}^4$ as

$$\begin{aligned}\tilde{\mathcal{F}}_{(r;s)} = {} & \tilde{A}(dq_0 \wedge dq_1 - dq_2 \wedge dq_3)\boldsymbol{i} + \tilde{A}(dq_0 \wedge dq_2 + dq_1 \wedge dq_3)\boldsymbol{j} \\ & \tilde{A}(dq_0 \wedge dq_3 - dq_1 \wedge dq_2)\boldsymbol{k} + \tilde{B}(dq_0 \wedge dq_1 + dq_2 \wedge dq_3)\bar{q}\boldsymbol{i}q \\ & \tilde{B}(dq_0 \wedge dq_2 - dq_1 \wedge dq_3)\bar{q}\boldsymbol{j}q + \tilde{B}(dq_0 \wedge dq_3 + dq_1 \wedge dq_2)\bar{q}\boldsymbol{k}q\end{aligned}$$

where

$$\tilde{A} = \frac{\left((4+r) + (4+s)\|q\|^2\right)\|q\|^r(1+\|q\|^2)^{\frac{r-s}{2}-1}}{2\left((1+\|q\|^2)^{\frac{r-s}{2}} + \|q\|^{r+2}\right)^2}$$

and

$$\tilde{B} = \frac{(r + s\|q\|^2)\|q\|^{r-2}(1+\|q\|^2)^{\frac{r-s}{2}-1}}{2\left((1+\|q\|^2)^{\frac{r-s}{2}} + \|q\|^{r+2}\right)^2}.$$

Note that, if $r$ (or $s$) is a negative real number,

$$\omega_{(r,s)} = Im\left(\frac{\|q_1\|^r \bar{q}_1\, dq_1 + \|q_2\|^s \bar{q}_2\, dq_2}{\|q_1\|^{r+2} + \|q_2\|^{s+2}}\right)$$

can not be a global connection 1-form on $S^7$. But locally, the pullback of the connection 1-form $\omega_{(r,s)}$ with respect to $s_1 \circ {\phi_1}^{-1}$ (or $s_2 \circ {\phi_2}^{-1}$), $\mathcal{A}_{(r;s)}$ $\left(\tilde{\mathcal{A}}_{(r;s)}\right)$ is a connection 1-form on $\mathbb{H} - \{0\}$.

As a result, the 2-form $\mathcal{F}_{(r;s)}$ is self-dual if and only if

$$A = \frac{\left((4+s)+(4+r)\|q\|^2\right)\|q\|^s(1+\|q\|^2)^{\frac{s-r}{2}-1}}{2\left((1+\|q\|^2)^{\frac{s-r}{2}}+\|q\|^{s+2}\right)^2} = 0$$

and anti-self dual if and only if

$$B = \frac{(s+r\|q\|^2)\|q\|^s(1+\|q\|^2)^{\frac{s-r}{2}-1}}{2\left((1+\|q\|^2)^{\frac{s-r}{2}}+\|q\|^{s+2}\right)^2} = 0.$$

Therefore, we state the theorem below:

**Theorem 2** Followings hold for the curvature 2-form $\mathcal{F}_{(r,s)}$ :
1. $\mathcal{F}_{(r,s)}$ is anti-self dual if and only if $r = s = 0$.
2. $\mathcal{F}_{(r,s)}$ is self dual if and only if $r = s = -4$.

## 4. DISCUSSION AND CONCLUSION

Potentials of most interest in physics are those whose field strengths satisfy the Bianchi identity, that is

$$D_{\mathcal{A}}\,\mathcal{F} = d\mathcal{A} + [\mathcal{A},\mathcal{F}] = 0,$$

for a potential $\mathcal{A}$ with associated gauge field $\mathcal{F}$. After tedious calculations, one can show that field strengths $\mathcal{F}_{(r,s)}$ of potentials $\mathcal{A}_{(r,s)}$ satisfy the Bianchi identity. Now, for $q \neq 0$ ($q \in \mathbb{H}$), we calculate the squared norm $\left\|\mathcal{F}_{(r,s)}(q)\right\|^2$ of $\mathcal{F}_{(r,s)}(q)$ as

$$\left\|\mathcal{F}_{(r,s)}(q)\right\|^2 = 3\frac{\|q\|^{2s}(1+\|q\|^2)^{s-r-2}}{\left((1+\|q\|^2)^{\frac{s-r}{2}}+\|q\|^{s+2}\right)^4}\left[\left((4+s)+(4+r)\|q\|^2\right)^2+(s+r\|q\|^2)^2\right].$$

If we integrate $\left\|\mathcal{F}_{(r,s)}(q)\right\|^2$ over $\mathbb{H}-\{0\}$, we obtain a global measure of the total field strength. To integrate, one can choose the standard spherical coordinates on $\mathbb{R}^4$ as

$$\begin{aligned} q_0 &= \rho\sin\varphi\sin\upsilon\cos\theta, \\ q_1 &= \rho\sin\varphi\sin\upsilon\sin\theta, \\ q_2 &= \rho\sin\varphi\cos\upsilon, \\ q_3 &= \rho\cos\varphi, \end{aligned}$$

where $\rho = \|q\| > 0$, $0 \le \varphi, \upsilon, \theta \le \pi$, then

$$\begin{aligned} S_{YM}\left[\mathcal{A}_{(r,s)}\right] &= \int_{\mathbb{R}^4-\{0\}}\left\|\mathcal{F}_{(r,s)}(q)\right\|^2 \\ &= 6\pi^2\int_0^\infty \frac{\rho^{2s+3}(1+\rho^2)^{s-r-2}}{\left((1+\rho^2)^{\frac{s-r}{2}}+\rho^{s+2}\right)^4}\left[\left((4+s)+(4+r)\rho^2\right)^2+(s+r\rho^2)^2\right]d\rho. \end{aligned} \tag{4.1}$$

Using the result outlined in Theorem 1, one can easily perform integration (4.1) when $r = s$. In that case, the squared norm reduces to the simple expression

$$\left\|\mathcal{F}_{(s,s)}(q)\right\|^2 = 3((4+s)^2+s^2)\frac{\|q\|^{2s+3}}{(1+\|q\|^{s+2})^4}$$

whose Yang-Mills functional can be exactly integrated as

$$S_{YM}[\mathcal{A}_{(s,s)}] = \int_{\mathbb{R}^4-\{0\}} \left\|\mathcal{F}_{(s,s)}(q)\right\|^2 = \begin{cases} 2\pi^2\left(\dfrac{s^2+4s+8}{s+2}\right), & if\ s>-2 \\ -2\pi^2\left(\dfrac{s^2+4s+8}{s+2}\right), & if\ \ s<-2 \end{cases}$$

By taking derivative of the Yang-Mills action $S_{YM}[\mathcal{A}_{(r,s)}]$, we observe that $s = 0$ and $s = -4$ are minimum values of the Yang-Mills action. As a result, potentials $\mathcal{A}_{(s,s)}$ minimize the Yang-Mills functional. At these extremum values of $s$, gauge fields $\mathcal{F}_{(s,s)}$ also satisfies

$$D_{\mathcal{A}} * \mathcal{F}_{(s,s)} = 0.$$

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