Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (4) (2024), 963 – 980
DOI : 10.15672/hujms.1219762

Research Article

Quarter-symmetric connection on an almost Hermitian manifold and on a Kähler manifold

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Abstract

The paper observes an almost Hermitian manifold as an example of a generalized Riemannian manifold and examines the application of a quarter-symmetric connection on the almost Hermitian manifold. The almost Hermitian manifold with quarter-symmetric connection preserving the generalized Riemannian metric is actually the Kähler manifold. Observing the six linearly independent curvature tensors with respect to the quarter-symmetric connection, we construct tensors that do not depend on the quarter-symmetric connection generator. One of them coincides with the Weyl projective curvature tensor of symmetric metric g. Also, we obtain the relations between the Weyl projective curvature tensor and the holomorphically projective curvature tensor. Moreover, we examine the properties of curvature tensors when some tensors are hybrid.

Mathematics Subject Classification (2020). 53B05, 53B35, 53C05, 53C15

Keywords. almost Hermitian manifold, curvature tensors, hybrid tensor, Kähler manifold, quarter-symmetric connection, torsion tensor

1. Introduction

The paper deals with a non-symmetric linear connection, i.e. investigates the linear connection with the torsion tensor. In [11], the authors discussed linear connections in a generalized Riemannian manifold. Among other things, they studied connections with a totally skew-symmetric torsion tensor and connections with the Einstein metricity condition (see also [12]). Paper [17] studied the semi-symmetric metric connection and properties of the curvature tensor and determined the relations between Weyl projective curvature tensor, conformal curvature tensor, and concircular curvature tensor. S. Golab in [9] defined the quarter-symmetric connection as a generalization of the semi-symmetric connection. After the initial work, the theory of quarter-symmetric connection was expanded by many authors in various manifolds (for instance see [2,3,5,6,10,13,21]). In paper [22], M. Tripathi introduced a new linear connection with torsion tensor in a Riemannian manifold, which generalizes several semi-symmetric and quarter-symmetric connections.

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Received: 16.12.2022; Accepted: 21.08.2023

In paper [28], the authors defined a quarter-symmetric generalized metric connection on a generalized Riemannian manifold as a connection that preserves the generalized (non-symmetric) Riemannian metric. The paper determined relations for curvature tensors and studied their skew-symmetric and cyclic-symmetric properties. Now we will deal with the application of the quarter-symmetric connection on the almost Hermitian manifold. We will show that the almost Hermitian manifold with quarter-symmetric generalized metric connection is actually a Kähler manifold. Accordingly, we will find some identities for the holomorphically projective curvature tensor and the Weyl projective curvature tensor. The *holomorphically projective curvature tensor* given by equation

$$\begin{aligned}\overset{g}{P}(X,Y)Z = \overset{g}{R}(X,Y)Z &+ \frac{1}{n+2}(\overset{g}{Ric}(X,Z)Y - \overset{g}{Ric}(Y,Z)X) \\ &- \frac{1}{n+2}(\overset{g}{Ric}(X,AZ)AY - \overset{g}{Ric}(Y,AZ)AX + 2\overset{g}{Ric}(X,AY)AZ)\end{aligned} \tag{1.1}$$

is invariant under holomorphically projective mapping between two Kähler manifolds (see [19, 24]). Such mapping is a natural generalization of geodesic mapping. The *Weyl projective curvature tensor* given by equation

$$\overset{g}{W}(X,Y)Z = \overset{g}{R}(X,Y)Z + \frac{1}{n-1}(\overset{g}{Ric}(X,Z)Y - \overset{g}{Ric}(Y,Z)X) \tag{1.2}$$

is invariant under geodesic mapping between two Riemannian manifolds (for instance see [14]).

2. Preliminaries

Let $(\mathcal{M}, G = g+F)$ be a generalized Riemannian manifold, where $\mathcal{M}$ is an n-dimensional differentiable manifold, G is a non-symmetric (0,2) tensor (the so-called generalized Riemannian metric), g is the symmetric part of G and F is the skew-symmetric part of G. Tensor A is defined as a tensor associated with tensor F, i.e.

$$F(X,Y) = g(AX,Y). \tag{2.1}$$

The quarter-symmetric connection $\overset{1}{\nabla}$ preserving generalized Riemannian metric G, $\overset{1}{\nabla} G = 0$, is called *quarter-symmetric generalized metric connection* (i.e. *quarter-symmetric G-metric connection*), and it is determined by equations (see [28])

$$\overset{1}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \pi(X)AY \tag{2.2}$$

and

$$(\overset{1}{\nabla}_X g)(Y,Z) = 0, \tag{2.3}$$

$$(\overset{1}{\nabla}_X A)Y = (\overset{g}{\nabla}_X A)Y = 0, \tag{2.4}$$

where π is a 1-form associated with vector field P, i.e. $\pi(X) = g(X,P)$, and $\overset{g}{\nabla}$ is a Levi-Civita connection. A 1-form π is called the *generator* of that connection. The covariant derivative of generator π is given by equation

$$(\overset{1}{\nabla}_X \pi)(Y) = (\overset{g}{\nabla}_X \pi)(Y) + \pi(X)\pi(AY).$$

The torsion tensor of connection $\overset{1}{\nabla}$ is given by equation

$$\overset{1}{T}(X,Y) = \pi(Y)AX - \pi(X)AY,$$

from which it follows

$$\overset{1}{T}(X,Y,Z) = g(\overset{1}{T}(X,Y),Z) = \pi(Y)F(X,Z) - \pi(X)F(Y,Z).$$

The following statement gives known relations between curvature tensors $\overset{\theta}{R}$, $\theta = 0, 1, \dots, 5$, and Riemannian curvature tensor $\overset{g}{R}$.

Theorem 2.1 ([28]). *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold with the quarter-symmetric G-metric connection* (2.2). *The curvature tensors $\overset{\theta}{R}$, $\theta = 0, 1, \dots, 5$ and Riemannian curvature tensor $\overset{g}{R}$ satisfy the following relations*

$$\begin{aligned}\overset{0}{R}(X,Y)Z =&\overset{g}{R}(X,Y)Z - \frac{1}{2}(\overset{0}{D}(X,Y) - \overset{0}{D}(Y,X))AZ - \frac{1}{2}\overset{0}{D}(X,Z)AY + \frac{1}{2}\overset{0}{D}(Y,Z)AX \\ &- \frac{1}{4}\pi(Z)(\pi(Y)A^2X - \pi(X)A^2Y),\end{aligned} \tag{2.5}$$

$$\overset{1}{R}(X,Y)Z =\overset{g}{R}(X,Y)Z - \overset{1}{D}(X,Y)AZ, \tag{2.6}$$

$$\overset{2}{R}(X,Y)Z =\overset{g}{R}(X,Y)Z - \overset{2}{D}(X,Z)AY + \overset{2}{D}(Y,Z)AX, \tag{2.7}$$

$$\overset{3}{R}(X,Y)Z =\overset{g}{R}(X,Y)Z - \overset{2}{D}(X,Y)AZ + \overset{3}{D}(Y,Z)AX, \tag{2.8}$$

$$\overset{4}{R}(X,Y)Z =\overset{g}{R}(X,Y)Z - \overset{3}{D}(X,Y)AZ + \overset{3}{D}(Y,Z)AX - \pi(Z)(\pi(Y)A^2X - \pi(X)A^2Y), \tag{2.9}$$

$$\begin{aligned}\overset{5}{R}(X,Y)Z =&\overset{g}{R}(X,Y)Z - \frac{1}{2}(\overset{2}{D}(X,Y) - \overset{3}{D}(Y,X))AZ - \frac{1}{2}\overset{3}{D}(X,Z)AY + \frac{1}{2}\overset{2}{D}(Y,Z)AX \\ &+ \frac{1}{2}\pi(Y)(\pi(X)A^2Z - \pi(Z)A^2X),\end{aligned} \tag{2.10}$$

where

$$\overset{0}{D}(X,Y) = (\overset{g}{\nabla}_X\pi)(Y) + \frac{1}{2}\pi(X)\pi(AY) + \frac{1}{2}\pi(Y)\pi(AX), \tag{2.11}$$

$$\overset{1}{D}(X,Y) = (\overset{g}{\nabla}_X\pi)(Y) - (\overset{g}{\nabla}_Y\pi)(X), \tag{2.12}$$

$$\overset{2}{D}(X,Y) = (\overset{g}{\nabla}_X\pi)(Y) + \pi(Y)\pi(AX), \tag{2.13}$$

$$\overset{3}{D}(X,Y) = (\overset{g}{\nabla}_X\pi)(Y) + \pi(X)\pi(AY) = (\overset{1}{\nabla}_X\pi)(Y). \tag{2.14}$$

The corresponding (0,4) curvature tensors are defined by relations

$$\overset{\theta}{R}(X,Y,Z,W) = g(\overset{\theta}{R}(X,Y)Z,W), \theta = 0,1,\dots,5 \quad \text{and}$$
$$\overset{g}{R}(X,Y,Z,W) = g(\overset{g}{R}(X,Y)Z,W).$$

The corresponding Ricci tensors are defined by relations

$$\overset{\theta}{Ric}(Y,Z) = Trace\{X \to \overset{\theta}{R}(X,Y)Z\}, \theta = 0,1,\dots,5 \quad \text{and}$$
$$\overset{g}{Ric}(Y,Z) = Trace\{X \to \overset{g}{R}(X,Y)Z\}.$$

3. Almost Hermitian manifolds

Depending on the properties of tensor A, we can observe various examples of the generalized Riemannian manifold (see [11]). An *almost Hermitian manifold* $(\mathcal{M}, g, A)$ is an n-dimensional Riemannian manifold $(\mathcal{M}, g)$ (where $n = 2k \geq 4$) equipped with almost complex structure A which satisfies

$$A^2 = -I, \ g(AX, AY) = g(X,Y). \tag{3.1}$$

The fundamental 2-form F (the so-called the *Kähler form*) is defined by $F(X,Y) = g(AX,Y)$. The following equations also apply to the almost Hermitian manifold

$$F(AX,Y) = -F(X,AY) = -g(X,Y) \quad \text{and} \quad F(AX,AY) = F(X,Y). \tag{3.2}$$

From equations (2.1), (3.1), (3.2), we conclude that

$$G(AX,Y) = -G(X,AY) = -G(Y,X) \quad \text{and} \quad G(AX,AY) = G(X,Y). \tag{3.3}$$

An almost Hermitian manifold $(\mathcal{M}, g, A)$ can be considered as a generalized Riemannian manifold $(\mathcal{M}, G = g + F)$ that satisfies relations (3.1) (see [11, 18]), where the skew-symmetric part F of basic tensor G is defined with $F(X,Y) = g(AX,Y)$. We will observe an almost Hermitian manifold $(\mathcal{M}, g, A)$ with a quarter-symmetric G-metric connection (2.2). Actually, such a connection preserves the almost Hermitian structure (g, A), i.e. $\overset{1}{\nabla} g = \overset{1}{\nabla} A = 0$. A linear connection that preserves the almost Hermitian structure is called the *almost Hermitian connection* (also known as a *natural connection*) (for instance, see [8, 27]), which means that quarter-symmetric G-metric connection is an almost Hermitian connection.

Theorem 3.1. *The torsion tensor* $\overset{1}{T}$ *of quarter-symmetric connection* (2.2) *on an almost Hermitian manifold satisfies the following relations*

$$A\overset{1}{T}(AX,AY) = A\overset{1}{T}(X,Y) - \overset{1}{T}(AX,Y) - \overset{1}{T}(X,AY),$$
$$\overset{1}{T}(X,Y,Z) = \overset{1}{T}(AX,AY,Z) + \overset{1}{T}(AX,Y,AZ) + \overset{1}{T}(X,AY,AZ),$$
$$\underset{XYZ}{\sigma} \overset{1}{T}(X,Y,Z) = \underset{XYZ}{\sigma} (\overset{1}{T}(AX,Y,AZ) + \overset{1}{T}(X,AY,AZ)),$$

where $\underset{XYZ}{\sigma}$ *denote the cyclic sum with respect to the vector fields* X, Y, Z.

Proof. These relations can be proven by using the property of skew-symmetric 2-form F in almost Herimitian manifolds, i.e. by using equations (3.1) and (3.2). □

An almost Hermitian manifold is a *Kähler manifold* if $\overset{g}{\nabla} A = 0$. From equation (2.4), we see that structure tensor A is parallel with respect to the Levi-Civita connection, and it implies the following statement.

Theorem 3.2. *The almost Hermitian manifold* $(\mathcal{M}, g, A)$ *with quarter-symmetric connection* (2.2) *preserving the generalized Riemannian metric* G *is the Kähler manifold.*

Following the previous theorem, further consideration will be related with the Kähler manifold. For this manifold, the term "generalized metric (i.e. G-metric) connection" is equivalent to the term "metric A-connection" (note that many papers use the term "metric F-connection", as F is used to denote (1,1) structure tensor).

The Riemannian curvature tensor $\overset{g}{R}$ on the Kähler manifold $(\mathcal{M}, g, A)$ satisfies the following relations (for instance see [4, 14])

$$\overset{g}{R}(X,Y)AZ = A\overset{g}{R}(X,Y)Z, \tag{3.4}$$
$$\overset{g}{R}(X,Y,AZ,AW) = \overset{g}{R}(AX,AY,Z,W), \tag{3.5}$$
$$\overset{g}{R}(X,AY,AZ,W) = \overset{g}{R}(AX,Y,Z,AW), \tag{3.6}$$
$$\overset{g}{R}(AX,AY,AZ,AW) = \overset{g}{R}(X,Y,Z,W), \tag{3.7}$$
$$\overset{g}{R}(X,Y,Z,AW) = -\overset{g}{R}(X,Y,AZ,W). \tag{3.8}$$

Moreover, if the (0,2) type tensor B is *hybrid*, then it holds (for instance see [16], [23, pp. 31])

$$B(AX,Y) = -B(X,AY).$$

On the Kähler manifold, we have

$$B(AX, AY) = B(X, Y).$$

For example, on the Kähler manifold, tensors g and F are hybrid (see [24, pp. 192]), from which it follows that generalized Riemannian metric G is hybrid (this is shown by equations (3.1), (3.2) and (3.3)). Also, on the Kähler manifold, the Ricci tensor $\overset{g}{Ric}$ is a hybrid tensor (see [24, pp. 68]), i.e. satisfies relation

$$\overset{g}{Ric}(X, AY) = -\overset{g}{Ric}(AX, Y). \tag{3.9}$$

In the following theorem, we state the results we will use to study curvature tensor properties.

Theorem 3.3. *Let $(\mathcal{M}, g, A)$ be a Kähler manifold.*

(1) *If $\overset{g}{\nabla}\pi$ is hybrid tensor, then $\overset{1}{D}$ is also hybrid, i.e. it holds that*

$$\overset{1}{D}(AX, Y) = -\overset{1}{D}(X, AY),\ \overset{1}{D}(AX, AY) = \overset{1}{D}(X, Y), \tag{3.10}$$

where $\overset{1}{D}$ is given by (2.12).

(2) *If $\overset{g}{\nabla}\pi$ and $\pi \otimes \pi$ are hybrid tensors, then $\overset{\theta}{D}$ are hybrid, $\theta = 0, 2, 3$, i.e. it holds that*

$$\overset{\theta}{D}(AX, Y) = -\overset{\theta}{D}(X, AY),\ \overset{\theta}{D}(AX, AY) = \overset{\theta}{D}(X, Y),\ \theta = 0, 2, 3, \tag{3.11}$$

where $\overset{\theta}{D}$ are given by (2.11), (2.13), (2.14).

Proof. We will prove equations (3.11) for $\theta = 2$. From equation (2.13), on the Kähler manifold, we have

$$\overset{2}{D}(AX, Y) + \overset{2}{D}(X, AY) = (\overset{g}{\nabla}_{AX}\pi)(Y) + (\overset{g}{\nabla}_X\pi)(AY) + \pi(AX)\pi(AY) - \pi(X)\pi(Y). \tag{3.12}$$

If $\overset{g}{\nabla}\pi$ and $\pi \otimes \pi$ are hybrid tensors, then it holds

$$(\overset{g}{\nabla}_{AX}\pi)(Y) = -(\overset{g}{\nabla}_X\pi)(AY),\ \pi(AX)\pi(Y) = -\pi(X)\pi(AY) \tag{3.13}$$

and

$$(\overset{g}{\nabla}_{AX}\pi)(AY) = (\overset{g}{\nabla}_X\pi)(Y),\ \pi(AX)\pi(AY) = \pi(X)\pi(Y). \tag{3.14}$$

Applying equations (3.13) and (3.14) to (3.12), we get

$$\overset{2}{D}(AX, Y) + \overset{2}{D}(X, AY) = 0.$$

If we replace X with AX in the previous equation and using $A^2 = -I$, we obtain the second equation of (3.11). □

Remark 3.4. Theorem 3.2 is the equivalent form of Theorem 4.2 in [25]: In order that the covariant derivative of the complex structure tensor of a Hermitian manifold with respect to the quarter-symmetric metric connection vanish, it is necessary and sufficient that the Hermitian manifold be a Kähler manifold.

4. Curvature properties of quarter-symmetric connection on Kähler manifold

In this section, we will consider the properties of the curvature tensor on the Kähler manifold with a quarter-symmetric connection (2.2). The papers [1, 7, 15, 20, 26] studied the quarter-symmetric connection on the Kähler manifold. For example, in paper [26], K. Yano and T. Imai proved that the Kähler manifold with a quarter-symmetric metric A-connection (2.2) is flat if curvature tensor $\overset{1}{R}$ vanishes.

Using the linearly independent curvature tensors with respect to quarter-symmetric connection (2.2), below we will construct the tensors that are independent of the choice of quarter-symmetric connection generator.

4.1. Curvature tensor of the first kind

The curvature tensor of the first kind on the Kähler manifold with a quarter-symmetric metric A-connection (2.2) is given by equation

$$\overset{1}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z - \overset{1}{D}(X,Y)AZ, \tag{4.1}$$

where $\overset{1}{D}$ is tensor given by (2.12). By contracting with respect to vector field X in equation (4.1), we obtain

$$\overset{1}{Ric}(Y,Z) = \overset{g}{Ric}(Y,Z) - \overset{1}{D}(AZ,Y).$$

If we replace Z with AZ in the previous equation, we have

$$\overset{1}{Ric}(Y,AZ) = \overset{g}{Ric}(Y,AZ) - \overset{1}{D}(A^2Z,Y),$$

from which we obtain

$$\overset{1}{D}(Z,Y) = \overset{1}{Ric}(Y,AZ) - \overset{g}{Ric}(Y,AZ). \tag{4.2}$$

By substituting (4.2) into (4.1), we obtain

$$\overset{1}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z - (\overset{1}{Ric}(Y,AX) - \overset{g}{Ric}(Y,AX))AZ.$$

By separating the elements of connections $\overset{1}{\nabla}$ and $\overset{g}{\nabla}$, we get the relation

$$\overset{1}{R}(X,Y)Z + \overset{1}{Ric}(Y,AX)AZ = \overset{g}{R}(X,Y)Z + \overset{g}{Ric}(Y,AX)AZ \tag{4.3}$$

and based on that, we will formulate the following theorem.

Theorem 4.1. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. Tensor*

$$\overset{1}{H}(X,Y)Z = \overset{1}{R}(X,Y)Z + \overset{1}{Ric}(Y,AX)AZ \tag{4.4}$$

is independent of generator π*.*

In this part, we will also deal with some other properties of the curvature tensors on the Kähler manifold, depending on the quarter-symmetric connection generator properties. We now state the properties of curvature tensors of the first kind.

Theorem 4.2. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. Then, we have:*

(1) *If* $\overset{g}{\nabla}\pi$ *is hybrid tensor, then the curvature tensor of the first kind and structure tensor A satisfies the following relations*

$$\overset{1}{R}(X,Y,AZ,AW) = \overset{1}{R}(AX,AY,Z,W),$$
$$\overset{1}{R}(X,AY,AZ,W) = \overset{1}{R}(AX,Y,Z,AW),$$
$$\overset{1}{R}(AX,AY,AZ,AW) = \overset{1}{R}(X,Y,Z,W).$$

(2) *The curvature tensor of the first kind and structure tensor A satisfies the following relations*

$$\overset{1}{R}(X,Y)AZ = A\overset{1}{R}(X,Y)Z,$$
$$\overset{1}{R}(X,Y,Z,AW) = -\overset{1}{R}(X,Y,AZ,W).$$

Proof. From equation (2.6), we obtain the (0,4) type curvature tensor of the first kind

$$\overset{1}{R}(X,Y,Z,W) = \overset{g}{R}(X,Y,Z,W) - \overset{1}{D}(X,Y)F(Z,W).$$

From here, we have

$$\begin{aligned}\overset{1}{R}(X,Y,AZ,AW) &= \overset{g}{R}(X,Y,AZ,AW) - \overset{1}{D}(X,Y)F(AZ,AW)\\ &= \overset{g}{R}(X,Y,AZ,AW) - \overset{1}{D}(X,Y)F(Z,W),\end{aligned} \tag{4.5}$$

where we used equation (3.2). On the other hand, we have

$$\overset{1}{R}(AX,AY,Z,W) = \overset{g}{R}(AX,AY,Z,W) - \overset{1}{D}(AX,AY)F(Z,W). \tag{4.6}$$

After subtracting equation (4.6) from (4.5) and using (3.5), we get

$$\overset{1}{R}(X,Y,AZ,AW) - \overset{1}{R}(AX,AY,Z,W) = (\overset{1}{D}(AX,AY) - \overset{1}{D}(X,Y))F(Z,W).$$

From equation (3.10), we see that

$$\overset{1}{R}(X,Y,AZ,AW) = \overset{1}{R}(AX,AY,Z,W)$$

if $\overset{g}{\nabla}\pi$ is hybrid. Other relations are proved analogously. □

4.2. Curvature tensor of the second kind

Using the curvature tensor of the second kind, we can get a new tensor on the Kähler manifold that is independent of quarter-symmetric connection generator π.

Theorem 4.3. *Let* $(\mathcal{M},g,A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2). *Tensor*

$$\overset{2}{H}(X,Y)Z = \overset{2}{R}(X,Y)Z + \overset{2}{Ric}(AX,Z)AY - \overset{2}{Ric}(AY,Z)AX \tag{4.7}$$

is independent of generator π.

Proof. The curvature tensor of the second kind with respect to quarter-symmetric connection (2.2) reads

$$\overset{2}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z - \overset{2}{D}(X,Z)AY + \overset{2}{D}(Y,Z)AX, \tag{4.8}$$

where $\overset{2}{D}$ is $(0,2)$ type tensor given by (2.13). By contracting vector field X in equation (4.8), we have

$$\overset{2}{Ric}(Y,Z) = \overset{g}{Ric}(Y,Z) - \overset{2}{D}(AY,Z), \tag{4.9}$$

where we used that the structure tensor A is trace-free, i.e. $Trace\{X \to AX\} = 0$. From equation (4.9), we have

$$\overset{2}{D}(A^2Y, Z) = \overset{g}{Ric}(AY, Z) - \overset{2}{Ric}(AY, Z)$$

and further

$$\overset{2}{D}(Y, Z) = \overset{2}{Ric}(AY, Z) - \overset{g}{Ric}(AY, Z). \tag{4.10}$$

By combining equations (4.8) and (4.10), we find

$$\overset{2}{R}(X, Y)Z = \overset{g}{R}(X, Y)Z - (\overset{2}{Ric}(AX, Z) - \overset{g}{Ric}(AX, Z))AY + (\overset{2}{Ric}(AY, Z) - \overset{g}{Ric}(AY, Z))AX,$$

from which

$$\begin{aligned}\overset{2}{R}(X, Y)Z + \overset{2}{Ric}(AX, Z)AY - \overset{2}{Ric}(AY, Z)AX = \overset{g}{R}(X, Y)Z + \overset{g}{Ric}(AX, Z)AY \\ - \overset{g}{Ric}(AY, Z)AX.\end{aligned} \tag{4.11}$$

□

Depending on generator π property, the curvature tensor of the second kind and structure tensor A have the following properties.

Theorem 4.4. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. Then, we have:*

(1) *If* $\overset{g}{\nabla}\pi$ *and* $\pi \otimes \pi$ *are hybrid, then the curvature tensor of the second kind and structure tensor A satisfies the following relations*

$$\begin{aligned}\overset{2}{R}(X, Y, AZ, AW) &= \overset{2}{R}(AX, AY, Z, W), \\ \overset{2}{R}(X, AY, AZ, W) &= \overset{2}{R}(AX, Y, Z, AW), \\ \overset{2}{R}(AX, AY, AZ, AW) &= \overset{2}{R}(X, Y, Z, W).\end{aligned}$$

(2) *The curvature tensor of the second kind and the structure tensor A satisfies the following relations*

$$\begin{aligned}\overset{2}{R}(X, Y)AZ &= A\overset{2}{R}(X, Y)Z, \\ \overset{2}{R}(X, Y, Z, AW) &= -\overset{2}{R}(X, Y, AZ, W),\end{aligned}$$

if and only if

$$\overset{2}{D}(X, Z)Y + \overset{2}{D}(X, AZ)AY = \overset{2}{D}(Y, Z)X + \overset{2}{D}(Y, AZ)AX,$$

where $\overset{2}{D}$ *given with* (2.13)*.*

Proof. The (0,4) type curvature tensor of the second kind is given by equation

$$\overset{2}{R}(X, Y, Z, W) = \overset{g}{R}(X, Y, Z, W) - \overset{2}{D}(X, Z)F(Y, W) + \overset{2}{D}(Y, Z)F(X, W),$$

from which it follows

$$\begin{aligned}\overset{2}{R}(X, AY, AZ, W) &= \overset{g}{R}(X, AY, AZ, W) + \overset{2}{D}(X, AZ)g(Y, W) + \overset{2}{D}(AY, AZ)F(X, W), \\ \overset{2}{R}(AX, Y, Z, AW) &= \overset{g}{R}(AX, Y, Z, AW) - \overset{2}{D}(AX, Z)g(Y, W) + \overset{2}{D}(Y, Z)F(X, W),\end{aligned}$$

where we used relations (3.2). By subtracting the previous two equations and using (3.6), we get

$$\overset{2}{R}(X,AY,AZ,W)-\overset{2}{R}(AX,Y,Z,AW)=(\overset{2}{D}(X,AZ)+\overset{2}{D}(AX,Z))g(Y,W)\\ +(\overset{2}{D}(AY,AZ)-\overset{2}{D}(Y,Z))F(X,W).$$

If $\overset{g}{\nabla}\pi$ and $\pi\otimes\pi$ are hybrid, then the relation (3.11) holds, and we verified that

$$\overset{2}{R}(X,AY,AZ,W)=\overset{2}{R}(AX,Y,Z,AW).$$

□

4.3. Curvature tensor of the third kind

The curvature tensor of the third kind $\overset{3}{R}$ with respect to quarter-symmetric connection (2.2) is given by equation

$$\overset{3}{R}(X,Y)Z=\overset{g}{R}(X,Y)Z-\overset{2}{D}(X,Y)AZ+\overset{3}{D}(Y,Z)AX, \tag{4.12}$$

where $\overset{2}{D}$ and $\overset{3}{D}$ are $(0,2)$ type tensors given by (2.13) and (2.14), respectively. If we contract equation (4.12) with respect to X, then we obtain the relation between Ricci tensors $\overset{3}{Ric}$ and $\overset{g}{Ric}$

$$\overset{3}{Ric}(Y,Z)=\overset{g}{Ric}(Y,Z)-\overset{2}{D}(AZ,Y),$$

from which we get the following relation

$$\overset{2}{D}(Z,Y)=\overset{3}{Ric}(Y,AZ)-\overset{g}{Ric}(Y,AZ). \tag{4.13}$$

On the other hand, if we contract equation (4.12) with respect to vector field Z, then we get

$$'\overset{3}{R}(X,Y)=\overset{3}{D}(Y,AX), \tag{4.14}$$

where we used $Trace\{Z\to\overset{g}{R}(X,Y)Z\}=0$ and denoted $'\overset{3}{R}(X,Y)=Trace\{Z\to\overset{3}{R}(X,Y)Z\}$. Further, it follows that

$$\overset{3}{D}(Y,X)=-'\overset{3}{R}(AX,Y), \tag{4.15}$$

where we take into account that $A^2=-I$. By replacing equations (4.13) and (4.15) into (4.12), we have

$$\overset{3}{R}(X,Y)Z=\overset{g}{R}(X,Y)Z-(\overset{3}{Ric}(Y,AX)-\overset{g}{Ric}(Y,AX))AZ-'\overset{3}{R}(AZ,Y)AX,$$

and further

$$\overset{3}{R}(X,Y)Z+\overset{3}{Ric}(Y,AX)AZ+'\overset{3}{R}(AZ,Y)AX=\overset{g}{R}(X,Y)Z+\overset{g}{Ric}(Y,AX)AZ. \tag{4.16}$$

Finally, we have proved the following theorem.

Theorem 4.5. *Let* $(\mathcal{M},g,A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. Tensor*

$$\overset{3}{H}(X,Y)Z=\overset{3}{R}(X,Y)Z+\overset{3}{Ric}(Y,AX)AZ+'\overset{3}{R}(AZ,Y)AX$$

is independent of generator π*.*

By comparing equations (4.3) and (4.16), we conclude that

$$\overset{1}{H}(X,Y)Z = \overset{3}{H}(X,Y)Z.$$

Based on expressions for tensor $\overset{2}{D}$, i.e. from equations (4.10) and (4.13), it follows that

$$\overset{2}{Ric}(X,Y) = \overset{3}{Ric}(Y,X).$$

In the following statement, we state the properties of the curvature tensor of the third kind, which can be proved similarly to the properties of the previous tensors.

Theorem 4.6. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. Then, we have:*

(1) *If* $\overset{g}{\nabla}\pi$ *and* $\pi\otimes\pi$ *are hybrid, then the curvature tensor of the third kind and structure tensor* A *satisfies the following relations*

$$\overset{3}{R}(X,Y,AZ,AW) = \overset{3}{R}(AX,AY,Z,W),$$
$$\overset{3}{R}(X,AY,AZ,W) = \overset{3}{R}(AX,Y,Z,AW),$$
$$\overset{3}{R}(AX,AY,AZ,AW) = \overset{3}{R}(X,Y,Z,W).$$

(2) *The curvature tensor of the third kind and structure tensor* A *satisfies the following relations*

$$\overset{3}{R}(X,Y)AZ = A\overset{3}{R}(X,Y)Z,$$
$$\overset{3}{R}(X,Y,Z,AW) = -\overset{3}{R}(X,Y,AZ,W),$$

if and only if

$$\overset{3}{D}(Y,Z)X = -\overset{3}{D}(Y,AZ)AX,$$

where $\overset{3}{D}$ *given by* (2.14)*.*

4.4. Curvature tensor of the fourth kind

The equation of the curvature tensor of the fourth kind $\overset{4}{R}$ on the Kähler manifold with a quarter-symmetric connection (2.2) take the form

$$\overset{4}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z - \overset{3}{D}(X,Y)AZ + \overset{3}{D}(Y,Z)AX + \pi(Z)(\pi(Y)X - \pi(X)Y), \quad (4.17)$$

where $\overset{3}{D}$ is given by equation (2.14). If we contract with respect to vector X in equation (4.17), then we obtain the relation between the Ricci tensor of the fourth kind and the Ricci tensor of metric g

$$\overset{4}{Ric}(Y,Z) = \overset{g}{Ric}(Y,Z) - \overset{3}{D}(AZ,Y) + (n-1)\pi(Y)\pi(Z). \quad (4.18)$$

On the other hand, by contracting equation (4.17) with respect to Z, we have the following equation

$$'\overset{4}{R}(X,Y) = \overset{3}{D}(Y,AX), \quad (4.19)$$

from which we obtain

$$\overset{3}{D}(Y,X) = -'\overset{4}{R}(AX,Y), \quad (4.20)$$

where $'\overset{4}{R}(X,Y) = Trace\{Z \to \overset{4}{R}(X,Y)Z\}$. From (4.18) and (4.20), we have

$$\pi(Y)\pi(Z) = \frac{1}{n-1}(\overset{4}{Ric}(Y,Z) - \overset{g}{Ric}(Y,Z) - '\overset{4}{R}(AY,AZ)). \tag{4.21}$$

By substituting equations (4.20) and (4.21) into (4.17), after simple rearranging, we obtain

$$\begin{aligned}
&\overset{4}{R}(X,Y)Z - '\overset{4}{R}(AY,X)AZ + '\overset{4}{R}(AZ,Y)AX \\
&\quad - \frac{1}{n-1}(\overset{4}{Ric}(Y,Z)X - \overset{4}{Ric}(X,Z)Y - '\overset{4}{R}(AY,AZ)X + '\overset{4}{R}(AX,AZ)Y) \\
&\quad = \overset{g}{R}(X,Y)Z + \frac{1}{n-1}(\overset{g}{Ric}(X,Z)Y - \overset{g}{Ric}(Y,Z)X) = \overset{g}{W}(X,Y)Z,
\end{aligned}$$

where $\overset{g}{W}$ is the Weyl projective curvature tensor (1.2). The tensor of the left-hand side of the previous equation is independent of the choice of a 1-form π.

Theorem 4.7. *Let $(\mathcal{M}, g, A)$ be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. Tensor*

$$\begin{aligned}
\overset{4}{H}(X,Y)Z =& \overset{4}{R}(X,Y)Z - '\overset{4}{R}(AY,X)AZ + '\overset{4}{R}(AZ,Y)AX \\
& - \frac{1}{n-1}(\overset{4}{Ric}(Y,Z)X - \overset{4}{Ric}(X,Z)Y - '\overset{4}{R}(AY,AZ)X + '\overset{4}{R}(AX,AZ)Y)
\end{aligned}$$

is independent of generator π and it is equal to the Weyl projective curvature tensor $\overset{g}{W}$.

Immediately, we have the following corollary.

Corollary 4.8. *Let $(\mathcal{M}, g, A)$ be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. If Ricci tensor $\overset{4}{Ric}$ and tensor $'\overset{4}{R}$ vanish on this manifold, then the curvature tensor of the fourth kind and the Weyl projective curvature tensor are equal, i.e. $\overset{4}{R} = \overset{g}{W}$.*

From equations (4.14) and (4.19), we obtain the relation

$$'\overset{3}{R} = '\overset{4}{R}.$$

Using relations (3.4)-(3.8), we can easily prove some relations for the curvature tensor of the fourth kind.

Theorem 4.9. *Let $(\mathcal{M}, g, A)$ be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. Then, we have:*

(1) *If $\overset{g}{\nabla}\pi$ and $\pi \otimes \pi$ are hybrid, then the curvature tensor of the fourth kind and structure tensor A satisfies the following relations*

$$\overset{4}{R}(X,Y,AZ,AW) = \overset{4}{R}(AX,AY,Z,W),$$
$$\overset{4}{R}(X,AY,AZ,W) = \overset{4}{R}(AX,Y,Z,AW),$$
$$\overset{4}{R}(AX,AY,AZ,AW) = \overset{4}{R}(X,Y,Z,W).$$

(2) *The curvature tensor of the fourth kind and structure tensor A satisfies the following relations*

$$\overset{4}{R}(X,Y)AZ = A\overset{4}{R}(X,Y)Z,$$
$$\overset{4}{R}(X,Y,Z,AW) = -\overset{4}{R}(X,Y,AZ,W),$$

if and only if

$$\overset{4}{D}(Y,Z)AX + \pi(X)\pi(Z)AY = -\overset{4}{D}(Y,AZ)X + \pi(X)\pi(AZ)Y,$$

where $\overset{4}{D}(Y,Z) = (\overset{g}{\nabla}_Y\pi)(AZ) - 2\pi(Y)\pi(Z)$.

4.5. Curvature tensor of the fifth kind

We will prove the following theorem using the curvature tensor of the fifth kind.

Theorem 4.10. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. Tensor*

$$\begin{aligned}\overset{5}{H}(X,Y)Z = \overset{5}{R}(X,Y)Z &+ \frac{1}{n-1}(\overset{5}{Ric}(X,Y)Z - \overset{5}{Ric}(Y,Z)X)\\ &- \frac{1}{2(n-1)}(\overset{1}{Ric}(X,Y)Z - \overset{1}{Ric}(Y,Z)X)\\ &- \frac{1}{2(n-1)}({}'\overset{3}{R}(AY,AX)Z - {}'\overset{3}{R}(AZ,AY)X)\\ &+ \frac{1}{2}(\overset{1}{Ric}(Y,AX)AZ - \overset{3}{Ric}(Z,AY)AX - {}'\overset{3}{R}(AZ,X)AY)\end{aligned} \tag{4.22}$$

is independent of generator π.

Proof. If we take into account that

$$\overset{1}{D}(X,Y) = \overset{2}{D}(X,Y) - \overset{3}{D}(Y,X),$$

where $\overset{1}{D}$, $\overset{2}{D}$, $\overset{3}{D}$ are given by (2.12), (2.13), (2.14), respectively, then the curvature tensor of the fifth kind on the Kähler manifold with a quarter-symmetric metric A-connection (2.2) takes the following form

$$\begin{aligned}\overset{5}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z &- \frac{1}{2}\overset{1}{D}(X,Y)AZ - \frac{1}{2}\overset{3}{D}(X,Z)AY + \frac{1}{2}\overset{2}{D}(Y,Z)AX\\ &- \frac{1}{2}\pi(Y)(\pi(X)Z - \pi(Z)X).\end{aligned} \tag{4.23}$$

By contracting with respect to vector field X in the previous equation gives

$$\overset{5}{Ric}(Y,Z) = \overset{g}{Ric}(Y,Z) - \frac{1}{2}\overset{1}{D}(AZ,Y) - \frac{1}{2}\overset{3}{D}(AY,Z) + \frac{n-1}{2}\pi(Y)\pi(Z).$$

From here, by using equations (4.2) and (4.15), it follows that

$$\pi(Y)\pi(Z) = \frac{1}{n-1}(2\overset{5}{Ric}(Y,Z) - \overset{1}{Ric}(Y,Z) - {}'\overset{3}{R}(AZ,AY) - \overset{g}{Ric}(Y,Z)). \tag{4.24}$$

By substituting equations (4.2), (4.13), (4.15) and (4.24) into (4.23), after rearranging, we obtain

$$\begin{aligned}
&\overset{5}{R}(X,Y)Z + \frac{1}{n-1}(\overset{5}{Ric}(X,Y)Z - \overset{5}{Ric}(Y,Z)X)\\
&\quad - \frac{1}{2(n-1)}(\overset{1}{Ric}(X,Y)Z - \overset{1}{Ric}(Y,Z)X)\\
&\quad - \frac{1}{2(n-1)}({}'\overset{3}{R}(AY,AX)Z - {}'\overset{3}{R}(AZ,AY)X)\\
&\quad + \frac{1}{2}(\overset{1}{Ric}(Y,AX)AZ - \overset{3}{Ric}(Z,AY)AX - {}'\overset{3}{R}(AZ,X)AY)\\
&\quad = \overset{g}{R}(X,Y)Z + \frac{1}{2(n-1)}(\overset{g}{Ric}(X,Y)Z - \overset{g}{Ric}(Y,Z)X)\\
&\qquad + \frac{1}{2}(\overset{g}{Ric}(AX,Y)AZ - \overset{g}{Ric}(AY,Z)AX)
\end{aligned} \tag{4.25}$$

and thereby, we proved the theorem. □

Analogously, we can prove the following theorem.

Theorem 4.11. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. Then, we have:*

(1) *If* $\overset{g}{\nabla}\pi$ *and* $\pi\otimes\pi$ *are hybrid, then the curvature tensor of the fifth kind and structure tensor A satisfies the following relations*

$$\begin{aligned}
\overset{5}{R}(X,Y,AZ,AW) &= \overset{5}{R}(AX,AY,Z,W),\\
\overset{5}{R}(X,AY,AZ,W) &= \overset{5}{R}(AX,Y,Z,AW),\\
\overset{5}{R}(AX,AY,AZ,AW) &= \overset{5}{R}(X,Y,Z,W).
\end{aligned}$$

(2) *The curvature tensor of the fifth kind and structure tensor A satisfies the following relations*

$$\begin{aligned}
\overset{5}{R}(X,Y)AZ &= A\overset{5}{R}(X,Y)Z,\\
\overset{5}{R}(X,Y,Z,AW) &= -\overset{5}{R}(X,Y,AZ,W).
\end{aligned}$$

if and only if

$$\overset{3}{D}(X,Z)Y + \overset{3}{D}(X,AZ)AY = (\overset{2}{D}(Y,Z) + \pi(Y)\pi(AZ))X + (\overset{2}{D}(Y,AZ) - \pi(Y)\pi(Z))AX,$$

where $\overset{2}{D}$, $\overset{3}{D}$ *given by* (2.13), (2.14)*, respectively.*

4.6. Curvature tensor of the zero kind

By the similar procedure as in the previous cases, using the curvature tensor of the zero kind, we can prove the following theorem.

Theorem 4.12. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2). *Tensor*

$$\begin{aligned}\overset{0}{H}(X,Y)Z = \overset{0}{R}(X,Y)Z &+ \frac{1}{n-1}(\overset{0}{Ric}(X,Y)Z - \overset{0}{Ric}(Y,Z)X)\\ &- \frac{1}{2(n-1)}(\overset{1}{Ric}(X,Z)Y - \overset{1}{Ric}(Y,Z)X)\\ &- \frac{1}{4(n-1)}(\overset{3}{Ric}(Z,X)Y - \overset{3}{Ric}(Z,Y)X)\\ &- \frac{1}{4(n-1)}({}'\overset{3}{R}(AZ,AX)Y - {}'\overset{3}{R}(AZ,AY)X)\\ &+ \frac{1}{4}(2\overset{1}{Ric}(Y,AX)AZ + \overset{3}{Ric}(Z,AX)AY - \overset{3}{Ric}(Z,AY)AX)\\ &- \frac{1}{4}({}'\overset{3}{R}(AZ,X)AY - {}'\overset{3}{R}(AZ,Y)AX)\end{aligned} \tag{4.26}$$

is independent of generator π.

Proof. Based on equations (2.11), (2.12), (2.13) and (2.14), we have

$$\overset{1}{D}(X,Y) = \overset{0}{D}(X,Y) - \overset{0}{D}(Y,X) \quad \text{and} \quad 2\overset{0}{D}(X,Y) = \overset{2}{D}(X,Y) + \overset{3}{D}(X,Y). \tag{4.27}$$

In view of equations (2.5), (3.1) and (4.27), the curvature tensor of the zero kind on the Kähler manifold with a quarter-symmetric metric A-connection (2.2) takes the form

$$\begin{aligned}\overset{0}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z - \frac{1}{2}\overset{1}{D}(X,Y)AZ - \frac{1}{4}(\overset{2}{D}(X,Z) + \overset{3}{D}(X,Z))AY\\ + \frac{1}{4}(\overset{2}{D}(Y,Z) + \overset{3}{D}(Y,Z))AX + \frac{1}{4}\pi(Z)(\pi(Y)X - \pi(X)Y),\end{aligned} \tag{4.28}$$

where (0,2) type tensors $\overset{1}{D}$, $\overset{2}{D}$, $\overset{3}{D}$ are given by (2.12), (2.13), (2.14), respectively. By contracting with respect to X in the previous equation, we obtain

$$\begin{aligned}\overset{0}{Ric}(Y,Z) = \overset{g}{Ric}(Y,Z) - \frac{1}{2}D(AZ,Y) - \frac{1}{4}(\overset{2}{D}(AY,Z) + \overset{3}{D}(AY,Z))\\ + \frac{n-1}{4}\pi(Y)\pi(Z).\end{aligned} \tag{4.29}$$

If we replace equations (4.2), (4.13), (4.15) into (4.29), then we get

$$\pi(Y)\pi(Z) = \frac{1}{n-1}(4\overset{0}{Ric}(Y,Z) - 2\overset{1}{Ric}(Y,Z) - \overset{3}{Ric}(Z,Y) - {}'\overset{3}{R}(AZ,AY) - \overset{g}{Ric}(Y,Z)). \tag{4.30}$$

Finally, by substituting (4.2), (4.13), (4.15) and (4.30) into equation (4.28), we obtain

$$\begin{aligned}\overset{0}{H}(X,Y)Z = \overset{g}{R}(X,Y)Z &+ \frac{1}{4(n-1)}(\overset{g}{Ric}(X,Z)Y - \overset{g}{Ric}(Y,Z)X)\\ &+ \frac{1}{4}(2\overset{g}{Ric}(AX,Y)AZ + \overset{g}{Ric}(AX,Z)AY - \overset{g}{Ric}(AY,Z)AX)\end{aligned} \tag{4.31}$$

where $\overset{0}{H}$ is given by (4.26). □

Now, we can give some other properties of the curvature tensor of the zero kind depending on generator π.

Theorem 4.13. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2). *Then, we have:*

(1) *If* $\overset{g}{\nabla}\pi$ *and* $\pi\otimes\pi$ *are hybrid, then the curvature tensor of the zero kind and structure tensor* A *satisfies the following relations*

$$\begin{aligned}\overset{0}{R}(X,Y,AZ,AW)&=\overset{0}{R}(AX,AY,Z,W),\\ \overset{0}{R}(X,AY,AZ,W)&=\overset{0}{R}(AX,Y,Z,AW),\\ \overset{0}{R}(AX,AY,AZ,AW)&=\overset{0}{R}(X,Y,Z,W).\end{aligned}$$

(2) *If*

$$(\overset{g}{\nabla}_X\pi)(Y)+\pi(X)\pi(AY)+\frac{1}{2}\pi(AX)\pi(Y)=0,$$

then the curvature tensor of the zero kind and structure tensor A *satisfies the following relations*

$$\begin{aligned}\overset{0}{R}(X,Y)AZ&=A\overset{0}{R}(X,Y)Z,\\ \overset{0}{R}(X,Y,Z,AW)&=-\overset{0}{R}(X,Y,AZ,W).\end{aligned}$$

Proof. Equation (2.5) implies the following

$$\begin{aligned}\overset{0}{R}(X,Y,Z,AW)=&\overset{g}{R}(X,Y,Z,AW)-\frac{1}{2}(\overset{0}{D}(X,Y)-\overset{0}{D}(Y,X))g(Z,W)-\frac{1}{2}\overset{0}{D}(X,Z)g(Y,W)\\ &+\frac{1}{2}\overset{0}{D}(Y,Z)g(X,W)-\frac{1}{4}\pi(Z)(\pi(Y)F(X,W)-\pi(X)F(Y,W)),\\ \overset{0}{R}(X,Y,AZ,W)=&\overset{g}{R}(X,Y,AZ,W)+\frac{1}{2}(\overset{0}{D}(X,Y)-\overset{0}{D}(Y,X))g(Z,W)\\ &-\frac{1}{2}\overset{0}{D}(X,AZ)F(Y,W)+\frac{1}{2}\overset{0}{D}(Y,AZ)F(X,W)\\ &+\frac{1}{4}\pi(AZ)(\pi(Y)g(X,W)-\pi(X)g(Y,W)).\end{aligned}$$

Adding the previous equations and using equations (2.11) and (3.8), we obtain

$$\begin{aligned}\overset{0}{R}(X,Y,Z,AW)=&-\overset{0}{R}(X,Y,AZ,W)\\ &-\frac{1}{2}((\overset{g}{\nabla}_X\pi)(Z)+\pi(X)\pi(AZ)+\frac{1}{2}\pi(AX)\pi(Z))g(Y,W)\\ &+\frac{1}{2}((\overset{g}{\nabla}_Y\pi)(Z)+\pi(Y)\pi(AZ)+\frac{1}{2}\pi(AY)\pi(Z))g(X,W)\\ &-\frac{1}{2}((\overset{g}{\nabla}_X\pi)(AZ)-\pi(X)\pi(Z)+\frac{1}{2}\pi(AX)\pi(AZ))F(Y,W)\\ &+\frac{1}{2}((\overset{g}{\nabla}_Y\pi)(AZ)-\pi(Y)\pi(Z)+\frac{1}{2}\pi(AY)\pi(AZ))F(X,W).\end{aligned}$$

If we assume that

$$(\overset{g}{\nabla}_X\pi)(Y)+\pi(X)\pi(AY)+\frac{1}{2}\pi(AX)\pi(Y)=0$$

then it holds

$$\overset{0}{R}(X,Y,Z,AW)=-\overset{0}{R}(X,Y,AZ,W).$$

□

5. Some identities obtained from $\overset{\theta}{H}$ tensors

Based on the results above, we can see that only tensor $\overset{4}{H}$ is equivalent to the well-known Weyl projective curvature tensor. By combining the remaining tensors $\overset{\theta}{H}$, $\theta = 0, 1, 2, 3, 5$, we will obtain some identities for the Weyl projective curvature tensor and the holomorphically projective curvature tensor. First, we will present Weyl projective curvature tensor as a linear combination of tensors $\overset{\theta}{H}$.

Theorem 5.1. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. The following relations hold*

$$4\overset{0}{H}(X,Y)Z - 2\overset{1}{H}(X,Y)Z - \overset{2}{H}(X,Y)Z = \overset{g}{W}(X,Y)Z,$$
$$2\overset{5}{H}(X,Y)Z - \overset{1}{H}(X,Y)Z + \overset{1}{H}(Y,Z)X = \overset{g}{W}(X,Z)Y,$$

where $\overset{0}{H}$, $\overset{1}{H}$, $\overset{2}{H}$, $\overset{5}{H}$ *are given by* (4.26), (4.4), (4.7), (4.22), *respectively.*

Proof. With help of equations (4.3) and (4.25), we have

$$2\overset{5}{H}(X,Y)Z - \overset{1}{H}(X,Y)Z + \overset{1}{H}(Y,Z)X = \overset{g}{R}(X,Y)Z + \overset{g}{R}(Y,Z)X + \frac{1}{n-1}(\overset{g}{Ric}(X,Y)Z - \overset{g}{Ric}(Y,Z)X).$$

By using the first Bianchi identity and skew-symmetric property of Riemannian curvature tensor $\overset{g}{R}$, we get

$$2\overset{5}{H}(X,Y)Z - \overset{1}{H}(X,Y)Z + \overset{1}{H}(Y,Z)X = \overset{g}{R}(X,Z)Y + \frac{1}{n-1}(\overset{g}{Ric}(X,Y)Z - \overset{g}{Ric}(Z,Y)X) = \overset{g}{W}(X,Z)Y.$$

□

If we use equation (3.9), then from (4.31), we get

$$\overset{0}{H}(X,Y)Z = \frac{n+2}{4}\overset{g}{P}(X,Y)Z - \frac{n-2}{4}\overset{g}{W}(X,Y)Z, \tag{5.1}$$

where $\overset{g}{W}$ is the Weyl projective curvature tensor (1.2) and $\overset{g}{P}$ is the holomorphically projective curvature tensor given by equation (1.1). From the previous equation, we can conclude the following.

Theorem 5.2. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. If tensor* $\overset{0}{H}$*, given by* (4.26)*, vanishes, then it holds that*

$$\overset{g}{P}(X,Y)Z = \frac{n-2}{n+2}\overset{g}{W}(X,Y)Z.$$

From equations (4.3) and (4.11), we obtain identity

$$2\overset{1}{H}(X,Y)Z + \overset{2}{H}(X,Y)Z = (n+2)\overset{g}{P}(X,Y)Z - (n-1)\overset{g}{W}(X,Y)Z$$

from which we conclude that the following statement holds.

Theorem 5.3. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric A-connection* (2.2)*. If tensors* $\overset{1}{H}$ *and* $\overset{2}{H}$*, given by* (4.4) *and* (4.7)*, respectively, vanish, then it holds that*

$$\overset{g}{P}(X,Y)Z = \frac{n-1}{n+2}\overset{g}{W}(X,Y)Z.$$

Using theorems 4.7 and 5.1, based on equation (5.1), we can represent the holomorphically projective curvature tensor as a linear combination of tensors $\overset{\theta}{H}$, $\theta = 0, 1, \dots, 5$.

Corollary 5.4. *Let* $(\mathcal{M}, g, A)$ *be a Kähler manifold with a quarter-symmetric metric* A*-connection* (2.2)*. The following relations hold*

$$\begin{aligned}
\overset{g}{P}(X,Y)Z &= \frac{4}{n+2}\overset{0}{H}(X,Y)Z + \frac{n-2}{n+2}\overset{4}{H}(X,Y)Z,\\
\overset{g}{P}(X,Y)Z &= \frac{4(n-1)}{n+2}\overset{0}{H}(X,Y)Z - \frac{2(n-2)}{n+2}\overset{1}{H}(X,Y)Z - \frac{n-2}{n+2}\overset{2}{H}(X,Y)Z,\\
\overset{g}{P}(X,Y)Z &= \frac{4}{n+2}\overset{0}{H}(X,Y)Z + \frac{n-2}{n+2}(2\overset{5}{H}(X,Z)Y - \overset{1}{H}(X,Z)Y + \overset{1}{H}(Z,Y)X).
\end{aligned}$$

6. Conclusion and further work

Observing a Kähler manifold with a quarter-symmetric metric A-connection, we determined a tensor that are independent of generator π. By using newly obtained tensors $\overset{\theta}{H}$, $\theta = 0, 1, \dots, 5$, we established some relationships between the Weyl projective curvature tensor and the holomorphically projective curvature tensor. Also, we presented them as a linear combination of tensors $\overset{\theta}{H}$. Analogously, the identities for the second holomorphically projective curvature tensor obtained by M. Prvanović in [19] can be determined.

On the other hand, we observed the case when $\overset{g}{\nabla}\pi$ and $\pi \otimes \pi$ are hybrid tensor and we determined which properties are satisfied by all linearly independent curvature tensors.

In future work, we will try to find some more properties of the tensors $\overset{\theta}{H}$, as well as their application. This research on the quarter-symmetric connection will be continued on an almost para-Hermitian and on a para-Kähler manifold.

Acknowledgment. The financial support of this research by the projects of the Ministry of Education, Science and Technological Development of the Republic of Serbia (project no. 451-03-47/2023-01/200124 for Milan Lj. Zlatanović and project no. 451-03-47/2023-01/200123 for Miroslav D. Maksimović) and by project of Faculty of Sciences and Mathematics, University of Priština in Kosovska Mitrovica (internal-junior project IJ-2303).

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