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Interpolative Contractive Results for m-Metric Spaces

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Abstract

In this paper, we initiate the study of fixed points for interpolative mappings in m-metric spaces. We discuss three different cases: the sum of "interpolative exponents" is less than, equal to or greater than 1. We support each of our result by examples in m-metric spaces. In the last section, we obtain our results in p-metric spaces. Finally we note that our results generalize results of [3], [4] and [5] from ordinary metric to m- and p-metrics.

Keywords: m-metric p-metric Fixed point Interpolative contractions .
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1. Introduction

After the famous fixed point theorem of Banach [2], the fixed point theory has flourished in many dimensions and has played an important role in many fields of mathematics. Following the technique of Banach, many researchers have proved fixed point results for different type of contractions as well as for different types of metric structures. Recently, a number of researchers have been working on the technique of establishing fixed point results for interpolative Kannan type contractions. In this direction, for example, Karapinar [5] proved a fixed point result for interpolative Kannan type contractions, Gabba et al. [4] proved the result for the case when the sum of "interpolative exponents" is less than 1 in the interpolative Kannan type contractions, whereas Errai et al. [3] proved such a result for the case when the sum of "interpolative exponents" is greater than or equal to 1. All these results have been proved in ordinary metric spaces. In this paper, we initiate the study of existence of fixed points for interpolative Kannan type contractions over the structure of m-metric spaces and we proved the fixed points results for different cases of "interpolative exponents".

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The idea of m-metric firstly given by Asadi et al. in [1], which constitutes a generalization of p-metric as given in the lemma below. For more results in this direction see the [7, 8, 9, 10, 11, 12, 13, 14] and references mentioned therein.

The first section contains some required definitions and basic results related to m-metric spaces and interpolative Kannan type contractions. In second section, three different fixed point results for m-metric spaces under different conditions on “interpolative exponents" are proved. We also support each of our result by examples in m-metric spaces. In the last section, we obtain our results in p-metric spaces. Although we get them as corresponding special cases of our results for m-metric spaces yet they are new in themselves. Finally we note that our results generalize results of [5], [4] and [3] respectively.

Lemma 1.1. *[1] Every p-metric is an m-metric but not conversely.*

2. Preliminaries

Definition 2.1. *[6] A partial metric on a non empty set Υ is a function $p : \Upsilon \times \Upsilon \to \mathbb{R}^+$ such that for all $x_1, x_2, x_3 \in \Upsilon$*

(p_1) $p(h_1, h_2) = p(h_1, h_1) = p(h_2, h_2) \Leftrightarrow h_1 = h_2$

(p_2) $p(h_1, h_1) \leq p(h_1, h_2)$

(p_3) $p(h_1, h_2) = p(h_2, h_1)$

(p_4) $p(h_1, h_2) \leq p(h_1, h_3) + p(h_3, h_2) - p(h_3, h_3)$.

A partial metric space is a pair (Υ, p) such that Υ is non empty set and p is a partial metric on Υ.

Definition 2.2. *[1] Let Υ be a nonempty set. Then m-metric is a function $m : \Upsilon \times \Upsilon \to \mathbb{R}^+$ satisfying the following conditions;*

(m_1) $m(h_1, h_2) = m(h_1, h_1) = m(h_2, h_2) \Leftrightarrow h_1 = h_2$

(m_2) $m_{h_1h_2} \leq m(h_1, h_2)$ *where* $m_{h_1h_2} := \min\{m(h_1, h_1), m(h_2, h_2)\}$

(m_3) $m(h_1, h_2) = m(h_2, h_1)$

(m_4) $(m(h_1, h_2) - m_{h_1h_2}) \leq (m(h_1, h_3) - m_{h_1h_3}) + (m(h_3, h_2) - m_{h_3h_2})$

for all $h_1, h_2, h_3 \in \Upsilon$. The pair (Υ, m) is called m-metric space.

Definition 2.3. *[1] Let (Υ, m) be a m-metric space. Then*
1. a sequence (h_n) in an m-metric space converges to a point $h \in \Upsilon$ iff

$$\lim_{n\to\infty}(m(h_n, h) - m_{h_n,h}) = 0$$

2. a sequence (h_n) in an m-metric space (Υ, m) is called m-Cauchy sequence if

$$\lim_{n,j\to\infty}(m(h_n, h_j) - m_{h_n,h_j}),$$

and

$$\lim_{n,j\to\infty}(M_{h_n,h_j} - m_{h_n,h_j}),$$

exists (and are finite), where $M_{h_n,h_j} = \max(m(h_n, h_n), m(h_j, h_j))$.
3. an m-metric space (Υ, m) is said to be complete if every m-Cauchy sequence (h_n) in Υ converges to a point in Υ.

Lemma 2.4. *[1] Assume that $h_n \to h$ and $g_n \to g$ as $n \to \infty$ in an m-metric space (Υ, m). Then*

$$\lim_{n\to\infty}(m(h_n, g_n) - m_{h_n,g_n}) = m(h, g) - m_{h,g}$$

Lemma 2.5. *[1] Let (h_n) be a sequence in an m-metric space (Υ, m). If there exists $r \in [0, 1)$ such that*

$$m(h_{n+1}, h_n) \leq rm(h_n, h_{n-1}), \qquad \forall n \in \mathbb{N}, \tag{1}$$

then
(A) $\lim_{n\to\infty} m(h_n, h_{n+1}) = 0$
(B) $\lim_{n\to\infty} m(h_n, h_n) = 0$
(C) $\lim_{j,n\to\infty} m_{h_j,h_n} = 0$
(D) (h_n) is an m-Cauchy sequence.

Proof. Since from Equation (1) we have

$$m(h_{n+1}, h_n) \leq rm(h_n, h_{n-1}),$$

for all $n \in \mathbb{N}$. Thus for any fixed n we have

$$m(h_{n+1}, h_n) \leq rm(h_n, h_{n-1}) \leq r^2 m(h_{n-1}, h_{n-2}) \leq, \cdots, \leq r^{n+1} m(h_1, h_0),$$

thus

$$m(h_{n+1}, h_n) \leq r^n m(h_1, h_0)$$

by taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} m(h_{n+1}, h_n) = 0.$$

Which completes (A).
By second condition of m-metric, we have

$$m_{h_{n+1},h_n} \leq m(h_{n+1}, h_n),$$

we have

$$\lim_{n\to\infty} m_{h_{n+1},h_n} = 0,$$

or

$$\lim_{n\to\infty} \min(m(h_n, h_n), m(h_{n+1}, h_{n+1})) = 0,$$

hence

$$\lim_{n\to\infty} m(h_n, h_n) = 0.$$

It implies that (B) holds. Also, $\lim_{j\to\infty} m(h_j, h_j) = 0$, thus

$$\lim_{n,j\to\infty} m_{h_n,h_j} = \lim_{n,j\to\infty} \min(m(h_n, h_n), m(h_j, h_j)) = 0,$$

It implies that (C) holds.
Similarly for any $n, j \in \mathbb{N}$, with $n \geq j$ we have

$$\lim_{n,j\to\infty}(M_{h_n,h_j} - m_{h_n,h_j}) = 0.$$

Also by triangular inequality of m-metric

$$\lim_{n,j\to\infty}(m(h_n, h_j) - m_{h_n,h_j}) = 0.$$

Hence (h_n) is a Cauchy sequence.

□

Definition 2.6. *[5] Let (Υ, d) be a metric space. A self mapping $\Gamma : \Upsilon \to \Upsilon$ is said to be an interpolative Kannan type contraction, if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(\Gamma h, \Gamma g) \leq \lambda [d(h, \Gamma h)]^{\alpha}.[d(g, \Gamma g)]^{1-\alpha}$$

for all $h, g \in \Upsilon$ with $h \neq \Gamma h$, $g \neq \Gamma g$.

We term α as an interpolative exponent.
The following result by Karapınar is proved in [5].

Theorem 2.7. *[5] Let (Υ, d) be a complete metric space and Γ be an interpolative Kannan type contraction. Then Γ has a unique fixed point .*

In [4], Gabba et al. defined the following interpolative Kannan type contraction.

Definition 2.8. *Let (Υ, d) be a metric space, a self mapping $\Gamma : \Upsilon \to \Upsilon$ is called (λ, α, β)-interpolative Kannan type contraction if there exist $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, such that*

$$d(\Gamma h, \Gamma g) \leq \lambda [d(h, \Gamma h)]^{\alpha} [d(g, \Gamma g)]^{\beta}$$

for all $h, g \in \Upsilon$ with $h \neq \Gamma h$, $g \neq \Gamma g$.

Moreover, they proved the following fixed point theorem.

Theorem 2.9. *[4] Let (Υ, d) be a complete metric space such that $d(h, g) \geq 1$ for all $h, g \in \Upsilon$ with $h \neq g$. Let $\Gamma : \Upsilon \to \Upsilon$ be a (λ, α, β)-interpolative Kannan type contraction. Then Γ has a fixed point.*

Errai et al. [3] proved the following fixed point result for interpolative Kannan type contraction for the case $\alpha + \beta > 1$ with $\alpha, \beta \in (0, 1)$.

Theorem 2.10. *[3] Let (Υ, d) be a complete metric space and Γ a self mapping on Υ such that*

$$d(\Gamma h, \Gamma g) \leq \lambda (d(h, \Gamma h))^{\alpha} (d(g, \Gamma g))^{\beta},$$

for all $h, g \in \Upsilon$ with $h \neq \Gamma h$ and $g \neq \Gamma g$, and where $\lambda \in (0, 1)$ and $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta \geq 1$. If there exists $h \in \Upsilon$ such that $d(h, \Gamma h) \leq 1$, then Γ has a fixed point in Υ.

Note that all above results have been proved in ordinary metric space (Υ, d). No results on interpolative Kannan type contraction has been proved in m-metric spaces yet. Let us initiate this study in our next section.

3. Main results

In this section, we prove some fixed point results for interpolative Kannan type contractions for the first time. We present three results where sum of interpolative exponents is less than 1, equal to 1 and greater than 1. We also support our results by suitable examples to validate them. Let us start this section by defining m-interpolative Kannan type contraction where the sum of interpolative constants is 1 as follows.

Definition 3.1. *Let (Υ, m) be an m-metric space. We say that the self mapping $\Gamma : \Upsilon \to \Upsilon$ is an m-interpolative Kannan type contraction, if there exist constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$m(\Gamma h, \Gamma g) \leq \lambda [m(h, \Gamma h)]^{\alpha}.[m(g, \Gamma g)]^{1-\alpha}$$

for all $h, g \in \Upsilon$ with $h \neq \Gamma h, g \neq \Gamma g$ and $m(h, \Gamma h) \neq 0, m(g, \Gamma g) \neq 0$.

Theorem 3.2. *Let (Υ, m) be a complete m-metric space and $\Gamma : \Upsilon \to \Upsilon$ be an m-interpolative Kannan type contraction. Then Γ has a fixed point.*

Proof. Let $h_0 \in \Upsilon$, we set a constructive sequence (h_n) by $h_{n+1} = \Gamma(h_n) = \Gamma^n(h_0)$ for all positive integers n. Without loss of generality, we assume that $h_n \neq h_{n+1}$ for each nonnegative integer n. Indeed, if there exists a nonnegative integer n_0 such that $h_{n_0} = h_{n_0+1} = \Gamma h_{n_0}$, then h_{n_0} forms a fixed point.

Now for $n = 1$, we have

$$m(h_2, h_1) = m(\Gamma h_1, \Gamma h_0) \leq \lambda[m(h_1, \Gamma h_1)]^{\alpha}.[m(h_0, \Gamma h_0)]^{1-\alpha}.$$

This yields

$$[m(h_2, h_1)]^{1-\alpha} \leq \lambda[m(h_0, \Gamma h_0)]^{1-\alpha}$$

and hence

$$m(h_2, h_1) \leq \lambda^{1/1-\alpha} m(h_0, h_1) \leq \lambda m(h_0, h_1).$$

In a similar fashion, we can write

$$m(h_{n+1}, h_n) = m(\Gamma h_n, \Gamma h_{n-1}) \leq \lambda[m(h_n, \Gamma h_n)]^{\alpha}.[m(h_{n-1}, \Gamma h_{n-1})]^{1-\alpha}$$

for any natural number n, and so

$$m(h_{n+1}, h_n) \leq \lambda^{1/1-\alpha} m(h_{n-1}, h_n) \leq \lambda m(h_{n-1}, h_n).$$

By Lemma 2.5, we get

$$\lim_{n\to\infty} m(h_n, h_{n+1}) = 0 \tag{2}$$

and so (h_n) is an m-Cauchy sequence. Since (Υ, m) is complete so (h_n) converges to, say, $h \in \Upsilon$ with respect to convergence in m-metric. Also we have

$$m(\Gamma h_n, \Gamma h) \leq \lambda[m(h_n, \Gamma h_n)]^{\alpha}.[m(h, \Gamma h)]^{1-\alpha} = \lambda[m(h_n, h_{n+1})]^{\alpha}.[m(h, \Gamma h)]^{1-\alpha}.$$

Letting n tend to infinity and using the facts that $\lim_{n\to\infty} m(h_n, h_{n+1}) = 0$ and $m(h, \Gamma h) < \infty$, we have

$$\lim_{n\to\infty} m(\Gamma h_n, \Gamma h) = 0.$$

Then by using the condition (m_2) of m-metric, we get

$$\lim_{n\to\infty} m_{\Gamma h_n, \Gamma h} = 0.$$

Thus we obtain

$$\lim_{n\to\infty} (m(\Gamma h_n, \Gamma h) - m_{\Gamma h_n, \Gamma h}) = 0.$$

This by definition of convergence in m-metric implies that Γh_n converges to Γh w.r.t the m-metric. Again by (2),

$$m(h_n, \Gamma h_n) = m(h_n, h_{n+1})$$

yields

$$\lim_{n\to\infty} m(h_n, \Gamma h_n) = 0.$$

An application of condition (m_2) of m-metric gives

$$\lim_{n\to\infty} (m(\Gamma h_n, h_n) - m_{\Gamma h_n, h_n}) = 0.$$

Since h_n and Γh_n converge to h and Γh respectively, a use of Lemma 2.4 provides us with

$$\lim_{n\to\infty} (m(\Gamma h_n, h_n) - m_{\Gamma h_n, h_n}) = m(\Gamma h, h) - m_{\Gamma h, h} = 0,$$

or

$$m(\Gamma h, h) = m_{\Gamma h, h}.$$

Again $\Gamma h_n = h_{n+1}$ converges to h w.r.t m-metric gives by Lemma 2.4 that

$$0 = \lim_{n\to\infty}(m(h_n, \Gamma h_n) - m_{h_n, \Gamma h_n}) = m(h, h) - m_{h, \Gamma h}.$$

That is to say

$$m(h, h) = m_{h, \Gamma h}.$$

Similarly, from

$$0 = \lim_{n\to\infty}(m(h_n, \Gamma h_n) - m_{h_n, \Gamma h_n}) = \lim_{n\to\infty}(m(\Gamma h_{n-1}, \Gamma h_n) - m_{h_n, \Gamma h_n}) = m(\Gamma h, \Gamma h) - m_{h, \Gamma h},$$

we get

$$m(\Gamma h, \Gamma h) = m_{h, \Gamma h}.$$

Consequently,

$$m(\Gamma h, \Gamma h) = m(h, \Gamma h) = m(h, h) = m_{h, \Gamma h}.$$

thus by condition (m_1) of m-metric, we have $h = \Gamma h$, that is h is the fixed point of Γ. □

In order to validate our above result, we now present the following example.

Example 3.3. *Let $\Upsilon = [1/4, \infty)$ and the mapping $m : \Upsilon \times \Upsilon \to R^+$ be defined as follows.*

$$m(h, g) = \begin{cases} h & ; \ h = g \\ h + g & ; \ h \neq g. \end{cases}$$

Also define a self mapping $\Gamma : \Upsilon \to \Upsilon$ as follows.

$$\Gamma h = \begin{cases} 2 & ; \ h \in [1/4, 4), \\ 1/4 & ; \ h \in [4, \infty). \end{cases}$$

Note that 2 is the fixed point of Γ.

We first prove that (Υ, m) is an m-metric space.

Since the conditions $(m_1), (m_2)$ and (m_3) of m-metric follow trivially from definition of m-metric, it suffices to establish (m_4). For this, we have to consider the following possibilities. Let $h, g, z \in \Upsilon$.

If $h < g < z$, then

$$z = m(h, z) - m_{h,z} \leq (m(h, g) - m_{h,g}) + (m(g, z) - m_{g,z}) = g + z.$$

If $h < g$, $z < g$ and $h < z$, then

$$z = m(h, z) - m_{h,z} < (m(h, g) - m_{h,g}) + (m(g, z) - m_{g,z}) = 2g.$$

If $h < g$, $z < g$ and $h > z$, then

$$h = m(h, z) - m_{h,z} < (m(h, g) - m_{h,g}) + (m(g, z) - m_{g,z}) = 2g.$$

Similarly, we can explore all other possibilities for $h, g, z \in \Upsilon$ and establish (m_4) thereby proving (Υ, m) an m-metric space.

Now we discuss following three cases to prove that Γ is m-interpolative Kannan type contraction of Theorem (3.2) for $\alpha = 1/2$ and $\lambda = 8/9$.

Case 1. If $h, g \in [1/4, 4)$, then we have $m(\Gamma h, \Gamma g) = m(2, 2) = 2$ and for $h \neq 2$ and $g \neq 2$, we have

$$\lambda m(h, \Gamma h)^{1/2} m(g, \Gamma g)^{1/2} = \lambda(h + 2)^{1/2}(g + 2)^{1/2} \geq (9/4)\lambda = 2$$

Case 2. *If* $h \in [1/4, 4)$ *and* $g \in [4, \infty)$, *then* $m(\Gamma h, \Gamma g) = m(2, 1/4) = 9/4 = 2.25$, *and for* $h \neq 2$, *we have*

$$\lambda m(h, \Gamma h)^{1/2} m(g, \Gamma g)^{1/2} = \lambda(h+2)^{1/2}(g+1/4)^{1/2} \geq (8/9)(9/4)^{1/2}(17/4)^{1/2} = 2.7$$

Case 3. *If* $h, g \in [4, \infty)$, *then*

$$m(\Gamma h, \Gamma g) = m(1/4, 1/4) = 1/4$$

Thus

$$\lambda m(h, \Gamma h)^{1/2} m(g, \Gamma g)^{1/2} = \lambda(h+1/4)^{1/2}(g+1/4)^{1/2} \geq (8/9)(17/4) = 3.7$$

Hence in all the cases, Γ *is an* m*-interpolative Kannan type contraction, so by Theorem 3.2,* Γ *has a fixed point and it actually is* 2.

The above example shows that Γ has one fixed point, whereas the next example will show that Γ may have more than one(actually infinite many) fixed points.

Example 3.4. *Let* $\Upsilon = [0, \infty)$ *and the mapping* $m : \Upsilon \times \Upsilon \to \mathbb{R}^+$ *be defined as follows*

$$m(h, g) = |h - g| + a$$

where "a" is any non-negative real number. Also define a self mapping $\Gamma : \Upsilon \to \Upsilon$ *as follows.*

$$\Gamma h = \begin{cases} 1 & ; h \in [0, 1/2), \\ h & ; h \in [1/2, 200), \\ 1/h & ; h \in [200, \infty). \end{cases}$$

Note that Γ *has infinite fixed points when* $h \in [1/2, 200)$. *We first prove that* (Υ, m) *is an* m*-metric space. Since the conditions* $(m_1), (m_2)$ *and* (m_3) *of* m*-metric follow trivially from definition of* m*-metric, it suffices to establish* (m_4). *For any* $h, g, z \in \Upsilon$ *we have*

$$m(h, z) - m_{h,z} = |h - z| + a - \min(a, a)$$

$$m(h, z) - m_{h,z} = |h - z| \leq [|h - g| + a] - a + [|g - z| + a] - a.$$

Hence

$$m(h, z) - m_{h,z} \leq (m(h, g) - m_{h,g}) + (m(g, z) - m_{g,z})$$

thereby proving (Υ, m) *an* m*-metric space.*

Now we discuss following three cases to prove that Γ *is* m*-interpolative Kannan type contraction of Theorem* (3.2) *for* $\alpha = 1/2$ *and* $\lambda = 3/4$.

Case 1. *If* $h, g \in [0, 1/2)$, *then we have* $m(\Gamma h, \Gamma g) = m(1, 1) = a$ *and*

$$\lambda m(h, \Gamma h)^{1/2} m(g, \Gamma g)^{1/2} = \lambda(|h - 1| + a)^{1/2}(|g - 1| + a)^{1/2} \geq (3/4)(a + 1/2)$$

Also $3/4(a + 1/2) \geq a$, *for all* $a \in [0, 3/2]$, *thus the required interpolative condition holds for all* $a \in [0, 3/2]$.

Case 2. *If* $h \in [0, 1/2)$ *and* $g \in [200, \infty)$, *then* $m(\Gamma h, \Gamma g) = m(1, 1/g) = |1 - 1/g| + a \leq 1 + a$, *and*

$$\lambda m(h, \Gamma h)^{1/2} m(g, \Gamma g)^{1/2} = \lambda(|h - 1| + a)^{1/2}(|g - 1/g| + a)^{1/2}$$

$$\geq (3/4)(1/2 + a)^{1/2}(200 - 1/200 + a)^{1/2}.$$

Also the relation holds when

$$(3/4)(1/2 + a)^{1/2}(200 - 1/200 + a)^{1/2} \geq 1 + a$$

it gives $0 \le a \le 253.706$, *hence the required interpolative condition holds for all* $a \in [0, 253.706]$.
Case 3. *If* $h, g \in [200, \infty)$, *then*

$$m(\Gamma h, \Gamma g) = m(1/h, 1/g) = |1/h - 1/g| + a \le 1/200 + a.$$

Thus

$$\lambda m(h, \Gamma h)^{1/2} m(g, \Gamma g)^{1/2} = 3/4(|h - 1/h| + a)^{1/2}(|g - 1/g| + a)^{1/2} \ge (3/4)(200 - 1/200 + a).$$

Thus $(3/4)(200 - 1/200 + a) \ge 1/200 + a$ *when* $0 \le a \le 599.965$. *So the required interpolative condition holds for all* $a \in [0, 599.965]$.
Hence from all the above three cases we conclude that the interpolative condition of Definition (3.5) *hold when* $a \in [0, 3/2]$. *Thus for such values of* a *in all the cases,* Γ *is an* m*-interpolative Kannan type contraction, so by Theorem 3.2,* Γ *have fixed points and its actually are all the points in interval* $[1/2, 200)$.

For our second theorem, we need to define the following m-interpolative Kannan type contraction where the sum of interpolative constants is assumed to be less than 1.

Definition 3.5. *Let* (Υ, m) *be an* m*-metric space. A self mapping* $\Gamma : \Upsilon \to \Upsilon$ *is called* (λ, α, β)*-*m*-interpolative Kannan type contraction if there exist* $\lambda \in [0, 1)$ *and* $\alpha, \beta \in (0, 1)$ *with* $\alpha + \beta < 1$ *such that*

$$m(\Gamma h, \Gamma g) \le \lambda [m(h, \Gamma h)]^{\alpha} [m(g, \Gamma g)]^{\beta}$$

for all $h, g \in \Upsilon$ *with* $h \ne \Gamma h, g \ne \Gamma g$ *and* $m(h, \Gamma h) \ge 1, m(g, \Gamma g) \ne 0$.

Theorem 3.6. *Let* (Υ, m) *be a complete* m*-metric space and* $\Gamma : \Upsilon \to \Upsilon$ *be a* (λ, α, β)*-*m*-interpolative Kannan type contraction. Then* Γ *has a fixed point.*

Proof. Starting from $h_0 \in \Upsilon$, construct a sequence (h_n) for all $n \in N$ by $h_{n+1} = \Gamma h_n$. As in the previous theorem, without any loss of generality, we assume $h_n \ne h_{n+1}$ for each non negative integer n.

Next,

$$\begin{aligned} m(h_n, h_{n+1}) &= m(\Gamma h_{n-1}, \Gamma h_n) \\ &\le \lambda [m(h_{n-1}, \Gamma h_{n-1})]^{\alpha} [m(h_n, \Gamma h_n)]^{\beta}. \end{aligned}$$

Hence

$$m(h_n, h_{n+1}) \le \lambda [m(h_{n-1}, h_n)]^{\alpha} [m(h_n, h_{n+1})]^{\beta}$$
$$[m(h_n, h_{n+1})]^{1-\beta} \le \lambda [m(h_{n-1}, h_n)]^{\alpha} \le \lambda [m(h_{n-1}, h_n)]^{1-\beta}$$

because $\alpha < 1 - \beta$ and $m(h_{n-1}, h_n) \ge 1$. Thus

$$m(h_n, h_{n+1}) \le \lambda^{1/(1-\beta)} m(h_{n-1}, h_n) \le \lambda m(h_{n-1}, h_n).$$

The rest of the proof follows the similar procedure as in Theorem 3.2. To avoid the repetition, we leave it for the interested reader to dig out the details. □

In order to validate our above result, we give the following example.

Example 3.7. *Let* $\Upsilon = [2.7, \infty)$ *and* m*-metric on* Υ *be defined as follows (as in the previous Example)*

$$m(h, g) = \begin{cases} h & ; \ h = g, \\ h + g & ; \ h \ne g. \end{cases}$$

Define a self mapping $\Gamma : \Upsilon \to \Upsilon$ *as follows*

$$\Gamma h = \begin{cases} 3 & ; \ h \in [2.7, 27] \\ h & ; \ h \in (27, \infty). \end{cases}$$

We discuss the following four case to confirm that Γ *is* $(7/8, 1/2, 1/4)$*-*m*-interpolative contraction used in Theorem 3.6. If* $h, g \in [2.7, 27]$*, then* $m(\Gamma h, \Gamma g) = 3$ *and for* $h \neq 3,\ g \neq 3$*, we have*

$$\lambda m(h, \Gamma h)^{1/2} m(g, \Gamma g)^{1/4} = \lambda (h+3)^{1/2}(g+3)^{1/4} \geq 3.688\lambda = 3.227$$

Consequently, Γ *satisfies all the required interpolative conditions of Theorem 3.6, so* $h = 3$ *and* $h \in (27, \infty)$ *are the fixed points of* Γ*.*

Finally, we deal with the condition when the sum of interpolative exponents exceeds unity.

Theorem 3.8. *Let* (Υ, m) *be a* m*-complete metric space and* $\Gamma : \Upsilon \to \Upsilon$ *be a self mapping such that*

$$m(\Gamma h, \Gamma g) \leq \lambda [m(h, \Gamma h)]^{\alpha}.[m(g, \Gamma g)]^{\beta}$$

for all $h, g \in \Upsilon$ *with* $h \neq \Gamma h, g \neq \Gamma g, m(h, \Gamma h) \neq 0, m(g, \Gamma g) \neq 0, \lambda \in (0,1)$ *and* $\alpha, \beta \in (0,1)$ *such that* $\alpha + \beta > 1$*. If there exist* $h_0 \in \Upsilon$ *such that* $m(h_0, \Gamma h_0) \leq 1$*. Then* Γ *has a fixed point.*

Proof. As usual, set $h_{n+1} = \Gamma h_n$ for all non negative integers n with $h_n \neq \Gamma h_n$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} m(h_1, h_2) &= m(\Gamma h_0, \Gamma h_1) \\ &\leq \lambda [m(h_0, \Gamma h_0)]^{\alpha}.[m(h_1, \Gamma h_1)]^{\beta} \end{aligned}$$

implies

$$[m(h_1, h_2)]^{1-\beta} \leq \lambda [m(h_0, h_1)]^{\alpha}.$$

Thus

$$m(h_1, h_2) \leq \lambda^{1/(1-\beta)} [m(h_0, h_1)]^{\alpha/(1-\beta)} \leq \lambda,$$

because $\alpha/(1-\beta) > 1$ and $m(h_0, h_1) \leq 1$.

Next,

$$m(h_2, h_3) \leq \lambda (m(h_1, h_2))^{\alpha} (m(h_2, h_3))^{\beta}$$

gives

$$m(h_2, h_3)^{1-\beta} \leq \lambda (m(h_1, h_2))^{\alpha}$$

and so

$$m(h_2, h_3) \leq \lambda^{1/(1-\beta)} (m(h_1, h_2))^{\alpha/(1-\beta)} \leq \lambda.\lambda^{\alpha/(1-\beta)}.$$

In effect,

$$m(h_2, h_3) \leq \lambda^2.$$

By using mathematical induction and interpolative condition, the following relation holds for all natural numbers n.

$$m(h_n, h_{n+1}) \leq \lambda^n.$$

This means

$$\lim_{n \to \infty} m(h_n, h_{n+1}) = 0.$$

Also by the condition (m_2) of m-metric, we have

$$m_{h_n, h_{n+1}} \leq m(h_n, h_{n+1})$$

giving

$$\lim_{n \to \infty} m_{h_n, h_{n+1}} = 0.$$

Since

$$\begin{aligned} m(\Gamma h_{n-1}, \Gamma h_{n-1}) &= m(h_n, h_n) \\ &\leq \lambda [m(h_{n-1}, \Gamma h_{n-1})]^{\alpha}.[m(h_{n-1}, \Gamma h_{h-1})]^{\beta} \end{aligned}$$

therefore

$$m(\Gamma h_{n-1}, \Gamma h_{n-1}) \leq [m(h_{n-1}, h_n)]^{\alpha}.[m(h_{n-1}, h_n)]^{\beta}$$
$$m(\Gamma h_{n-1}, \Gamma h_{n-1}) \leq \lambda[m(h_{n-1}, h_n)]^{\alpha+\beta}.$$

Since $\alpha + \beta > 1$ and $\alpha, \beta \in (0,1)$, the sum of α and β is not greater than two. So we have

$$\lim_{n\to\infty} m(h_n, h_n) \leq \lambda \lim_{n\to\infty} \lambda^{(n-1)(\alpha+\beta)} = 0.$$

That is,

$$\lim_{n\to\infty} m(h_n, h_n) = 0. \tag{3}$$

Equivalently,

$$\lim_{n\to\infty} m(h_{n+1}, h_{n+1}) = 0.$$

Hence, we have

$$\lim_{n\to\infty} \min(m(h_n, h_n), m(h_{n+1}, h_{n+1})) = \lim_{n\to\infty} m_{h_n,h_{n+1}} = 0.$$

And

$$\lim_{n,m\to\infty} (m(h_n, h_n), m(h_m, h_m)) = \lim_{n,m\to\infty} m_{h_n,h_m} = 0. \tag{4}$$

Thus by using the property (m_4) of m-metric together with the expressions (3) and (4), we get

$$\lim_{n,j\to\infty} (m(h_n, h_j) - m_{h_n,h_j}) = 0.$$

Likewise,

$$\lim_{n,j\to\infty} M_{h_n,h_j} = \lim_{n,j\to\infty} \max(m(h_n, h_n), m(h_j, h_j)) = 0$$

allows

$$\lim_{n,j\to\infty} (M_{h_n,h_j} - m(h_n, h_j)) = 0.$$

Hence (h_n) is an m-Cauchy sequence in Υ. Since (Υ, m) is complete, so (h_n) converges to a point, say h, in Υ with respect to the convergence in m-metric.

Also we have

$$m(\Gamma h_n, \Gamma h) \leq \lambda[m(h_n, h_{n+1})]^{\alpha}.[m(h, \Gamma h)]^{\beta}$$

Since $m(h_n, h_{n+1}) \leq \lambda^n$ and $m(h, \Gamma h) \in [0, \infty)$ and $\beta \in (0,1)$ so $m(h, \Gamma h)$ is finite, thus we have

$$\lim_{n\to\infty} m(\Gamma h_n, \Gamma h) \leq \lim_{n\to\infty} \lambda^{1+\alpha n}[m(h, \Gamma h)]^{\beta} = 0.$$

This also shows $\lim_{n\to\infty} m_{\Gamma h_n,\Gamma h} = 0$ and, in turn, we can write

$$\lim_{n\to\infty} (m(\Gamma h_n, \Gamma h) - m_{\Gamma h_n,\Gamma h}) = 0.$$

Thus by definition of convergence in m-metric Γh_n converges to Γh w.r.t m-metric.

Moreover,

$$\lim_{n\to\infty} (m(h_n, \Gamma h_n) - m_{h_n,\Gamma h_n}) = 0.$$

Since h_n and Γh_n converges to h and Γh, so by using Lemma 2.4, we get

$$0 = \lim_{n\to\infty} (m(h_n, \Gamma h_n) - m_{h_n,\Gamma h_n}) = m(h, \Gamma h) - m_{h,\Gamma h}$$

or

$$m(h, \Gamma h) = m_{h,\Gamma h}.$$

Similarly, we have $m(h,h) = m_{h,\Gamma h}$ and $m(\Gamma h,\Gamma h) = m_{h,\Gamma h}$. Thus

$$m(\Gamma h,\Gamma h) = m(h,\Gamma h) = m(h,h) = m_{h,\Gamma h}.$$

Finally, by the condition (m_1) of m-metric, we have

$$m(h,\Gamma h) = m(h,h) = m(\Gamma h,\Gamma h) \Longleftrightarrow h = \Gamma h.$$

Hence h is the fixed point of Γ. □

In order to validate our above result, we give the following example.

Example 3.9. *Let $\Upsilon = [1/2, 2]$ and m-metric on Υ is defined as follows*

$$m(h,g) = \begin{cases} h & ; h = g, \\ h+g & ; h \neq g. \end{cases}$$

Also a self mapping $\Gamma : \Upsilon \to \Upsilon$ on Υ is defined as follows

$$\Gamma h = \begin{cases} h & ; h \in [1/2, 1) \\ 1/h & ; h \in [1, 2], \end{cases}$$

Now we prove that Γ satisfies the interpolative condition used in Theorem 3.8, for $\alpha = 1/2, \beta = 3/4$ and $\lambda = 98/101$.
If $h, g \in [1,2]$ then we have $m(\Gamma h,\Gamma g) = 1/h + 1/g \leq 2$ and for $h \neq 1, g \neq 1$ we have

$$\lambda m(h,\Gamma h)^{1/2} m(g,\Gamma g)^{3/4} = \lambda(h+1/h)^{1/2}(g+1/g)^{3/4} \geq \lambda(2)^{1/2}(2)^{3/4} > 2.$$

Hence Γ satisfies the required interpolative condition, so by Theorem 3.8, Γ have infinite fixed points for all $h \in [1/2, 1]$.

3.1. Interpolative results for p-metric spaces

In this section, we discuss interpolative results in the setting of p-metric spaces. By Lemma 1.1, every p-metric is m-metric so our results for p-metric will be the special cases of our corresponding m-metric. Here also, we discusses all the three cases: the sum of the interpolative exponents equal to 1, less than 1 and greater than 1.

Corollary 3.10. *Let (Υ, p) be a p-metric space and $\Gamma : \Upsilon \to \Upsilon$ be a self mapping. If there exist constants $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that*

$$p(\Gamma h,\Gamma g) \leq \lambda[p(h,\Gamma h)]^{\alpha}.[p(g,\Gamma g)]^{1-\alpha}$$

for all $h, g \in \Upsilon$ with $h \neq \Gamma h, g \neq \Gamma g$ and $p(h,\Gamma h) \neq 0$, $p(g,\Gamma g) \neq 0$, then Γ has a fixed point.

Proof. Since by Lemma 1.1, every p-metric is an m-metric, the result follows from Theorem 3.2. □

Using the similar argument as in the proof of the above Theorem, we can prove the following results by Theorem 3.6 and Theorem 3.8 respectively.

Corollary 3.11. *Let (Υ, p) be a complete p-metric space and $\Gamma : \Upsilon \to \Upsilon$ be a a self mapping. If there exist $\lambda \in [0,1)$ and $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$ such that*

$$p(\Gamma h,\Gamma g) \leq \lambda[p(h,\Gamma h)]^{\alpha}[p(g,\Gamma g)]^{\beta}$$

for all $h, g \in \Upsilon$ with $h \neq \Gamma h, g \neq \Gamma g$ and $p(h,\Gamma h) \geq 1$, $p(g,\Gamma g) \neq 0$, then Γ has a fixed point.

Corollary 3.12. *Let (Υ, p) be a complete p-metric space and $\Gamma : \Upsilon \to \Upsilon$ be a self mapping such that*

$$p(\Gamma h,\Gamma g) \leq \lambda[p(h,\Gamma h)]^{\alpha}.[p(g,\Gamma g)]^{\beta}$$

for all $h, g \in \Upsilon$ with $h \neq \Gamma h, g \neq \Gamma g, p(h,\Gamma h) \neq 0, p(g,\Gamma g) \neq 0, \lambda \in (0,1)$ and $\alpha, \beta \in (0,1)$ such that $\alpha + \beta > 1$. If there exist $h_0 \in \Upsilon$ such that $p(h_0,\Gamma h_0) \leq 1$, then Γ has a fixed point.

Remark 3.13. *Since every ordinary metric d is a p-metric, our Theorems 3.10,3.11 and 3.12 generalize the corresponding results of [5], [4] and [3] respectively.*

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