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Research Article

On Carnot's Theorem in the Plane $\mathbb{R}^2_{\pi 3}$

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Abstract
In this paper, we consider the relationship between iso-taxicab distance and Euclidean distance and give Carnot's theorem in the plane $\mathbb{R}^2_{\pi 3}$, the theorem can also be thought of as a generalization of the Pythagorean theorem.

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1. Introduction

Pythagorean theorem, the well-known geometric theorem that the sum of the squares on the legs of a right triangle is equal to the square on the hypotenuse (the side opposite the right angle) or, in familiar algebraic notation, $a^2+b^2=c^2$. Although the theorem has long been associated with Greek mathematician-philosopher Pythagoras, it is actually far older. Carnot's theorem (named after Lazare Carnot) describes a necessary and sufficient condition for three lines that are perpendicular to the (extended) sides of a triangle having a common point of intersection. The theorem can also be thought of as a generalization of the Pythagorean theorem.

Iso-taxicab geometry is a non-Euclidean geometry defined by K. O. Sowell in 1989 in [1]. In this geometry presented by Sowell three distance functions arise depending upon the relative positions of the points A and B. There are three axes at the origin; the x-axis, the y-axis and the y' -axis, having $60°$ angle which each other. These three axes separate the plane into six regions. The iso-taxicab trigonometric functions in iso-taxicab plane with three axes were given in [2,3]. A family of distances, $d_{\pi n}$, that includes Taxicab, Chinese-Checker and Iso-taxi distances, as special cases introduced and the group of isometries of the plane with $d_{\pi n}$ metric that is the semi-direct product of D_{2n} and $T(2)$ was shown in [4]. The trigonometric functions and cosine and sine rules in taxicab plane are given in [5]. Also, taxicab lengths under rotations introduced in [6]. The shortest distance of a point to the line in $\mathbb{R}^2_{\pi 3}$ and the versions of some Euclidean theorems in the plane $\mathbb{R}^2_{\pi 3}$ were given in [7,8].

2. Distance in the plane $\mathbb{R}^2_{\pi 3}$

Definition 1. *(See [4]) Let $A=(x_1,y_1)$ and $B=(x_2,y_2)$ be any two points in $\mathbb{R}^2$, a family of $d_{\pi n}$ distances is defined by;*

$$d_{\pi n}(A,B)=\frac{1}{\sin\frac{\pi}{n}}\left(\left|\sin\frac{k\pi}{n}-\sin\frac{(k-1)\pi}{n}\right||x_1-x_2|+\left|\cos\frac{(k-1)\pi}{n}-\cos\frac{k\pi}{n}\right||y_1-y_2|\right),$$

where

$$\begin{cases} 1\le k\le\left[\frac{n-1}{2}\right],\, k\in\mathbb{Z} & , \quad \tan\frac{(k-1)\pi}{n}\le\left|\frac{y_2-y_1}{x_2-x_1}\right|\le\tan\frac{k\pi}{n}, \\ k=\left[\frac{n+1}{2}\right] & , \quad \tan\frac{\left[\frac{n-1}{2}\right]\pi}{n}\le\left|\frac{y_2-y_1}{x_2-x_1}\right|<\infty \text{ or } x_1=x_2. \end{cases}$$

For $n = 3$ and accordingly $k = 1$, $k = 2$, we obtain the formula of $d_{\pi 3}$-distance between the points A and B according to the inclination in the plane $\mathbb{R}^2_{\pi 3}$ as

$$d_{\pi 3}(A,B) = \frac{1}{\sin\frac{\pi}{3}}\left(\left|\sin\frac{k\pi}{3} - \sin\frac{(k-1)\pi}{3}\right||x_1-x_2| + \left|\cos\frac{(k-1)\pi}{3} - \cos\frac{k\pi}{3}\right||y_1-y_2|\right),$$

where

$$\begin{cases} k=1 & , \quad 0 \le \left|\frac{y_2-y_1}{x_2-x_1}\right| \le \tan\frac{\pi}{3} \\ k=2 & , \quad \tan\frac{\pi}{3} \le \left|\frac{y_2-y_1}{x_2-x_1}\right| < \infty \ \text{ or } x_1 = x_2 \end{cases}$$

or

$$d_{\pi 3}(A,B) = \begin{cases} |x_1-x_2| + \frac{1}{\sqrt{3}}|y_1-y_2| & , \quad 0 \le \left|\frac{y_2-y_1}{x_2-x_1}\right| \le \sqrt{3} \\ \frac{2}{\sqrt{3}}|y_1-y_2| & , \quad \sqrt{3} \le \left|\frac{y_2-y_1}{x_2-x_1}\right| < \infty \ \text{or } x_1 = x_2. \end{cases}$$

Theorem 2. *(See [8]) The shortest distance of a point $P_0 = (x_0, y_0)$ to the line l given by*

$$ax + by + c = 0$$

in the plane $\mathbb{R}^2_{\pi 3}$ is

$$d_{\pi 3}(P_0, l) = \rho(\frac{-1}{m})d_E(P_0, l).$$

Carnot's theorem (named after Lazare Carnot) describes a necessary and sufficient condition for three lines that are perpendicular to the (extended) sides of a triangle having a common point of intersection. The theorem can also be thought of as a generalization of the Pythagorean theorem.

3. Carnot's theorem in the plane $\mathbb{R}^2_{\pi 3}$

Now we consider Carnot's theorem in any triangle in the Euclidean plane.

Theorem 3. *For a triangle ABC with sides a, b, c consider three lines that are perpendicular to the triangle sides and intersect in a common point P. If P_a, P_b, P_c are the pedal points of those three perpendiculars on the sides a, b, c, then the following equation holds:*

$$|AP_c|^2 + |BP_a|^2 + |CP_b|^2 = |BP_c|^2 + |CP_a|^2 + |AP_b|^2.$$

The converse of the statement above is true as well, that is if the equation holds for the pedal points of three perpendiculars on the three triangle sides then they intersect in a common point. Therefore, the equation provides a necessary and sufficient condition.

We give Carnot's theorem in any triangle in the plane $\mathbb{R}^2_{\pi 3}$.

Theorem 4. *For a triangle ABC in the plane $\mathbb{R}^2_{\pi 3}$, let P be a fixed point in this triangle, the feet of the perpendiculars descending from the point P to the sides of the triangle ABC be the points K, L and M, the line segments that these perpendiculars separate on the sides be $a_{\pi 3} = d_{\pi 3}(B,K)$, $x_{\pi 3} = d_{\pi 3}(K,C)$, $b_{\pi 3} = d_{\pi 3}(C,L)$, $y_{\pi 3} = d_{\pi 3}(L,A)$, $c_{\pi 3} = d_{\pi 3}(A,M)$, $z_{\pi 3} = d_{\pi 3}(M,B)$. If the slopes of the line segments BK and KC are m_a, and the slopes of the line segments CL and LA are m_b and the slopes of the line segments MA and MB are m_c, then the relationship among the lengths of the line segments separated by the perpendiculars descended from the point P to the sides of the triangle is*

$$\left(\frac{1}{\rho(m_a)}a_{\pi 3}\right)^2 + \left(\frac{1}{\rho(m_b)}b_{\pi 3}\right)^2 + \left(\frac{1}{\rho(m_c)}c_{\pi 3}\right)^2 = \left(\frac{1}{\rho(m_a)}x_{\pi 3}\right)^2 + \left(\frac{1}{\rho(m_b)}y_{\pi 3}\right)^2 + \left(\frac{1}{\rho(m_c)}z_{\pi 3}\right)^2.$$

Proof. The relationship between the $d_{\pi 3}$-distance and the Euclidean distance in the plane is given by the equation

$$d_{\pi 3}(A,B) = \rho(m)d_E(A,B).$$

From here

$$d_E(A,B) = \frac{1}{\rho(m)} d_{\pi 3}(A,B)$$

can be written. From the previously found values $\rho(m)$

$$\frac{1}{\rho(m)} = \begin{cases} \frac{\sqrt{1+m^2}}{1+\frac{1}{\sqrt{3}}|m|}, & \text{If } 0 \leq m \leq \sqrt{3} \\ \frac{\sqrt{3}\sqrt{1+m^2}}{2|m|}, & \text{If } \sqrt{3} \leq m \\ \frac{\sqrt{3}}{2}, & \text{If } m \to \infty \end{cases}$$

can be written. To prove the theorem, we assume that the slopes of each side of any triangle ABC are m_a, m_b and m_c on the plane $\mathbb{R}^2_{\pi 3}$.

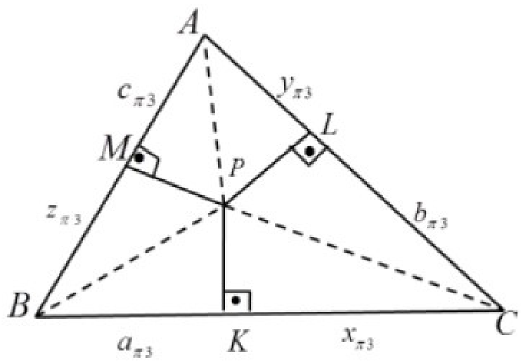

Fig. 1. Three perpendiculars on triangle sides intersect in a common point P

i. If $m_a, m_b, m_c \neq \infty$, then we have

$$\begin{aligned}
d_E(B,K) &= a = \left(\frac{\sqrt{1+m_a^2}}{1+\frac{1}{\sqrt{3}}|m_a|}\right) a_{\pi 3} \text{ or } d_E(B,K) = a = \left(\frac{\sqrt{3}\sqrt{1+m_a^2}}{2|m_a|}\right) a_{\pi 3}, \\
d_E(C,L) &= b = \left(\frac{\sqrt{1+m_b^2}}{1+\frac{1}{\sqrt{3}}|m_b|}\right) b_{\pi 3} \text{ or } d_E(C,L) = b = \left(\frac{\sqrt{3}\sqrt{1+m_b^2}}{2|m_b|}\right) b_{\pi 3}, \\
d_E(A,M) &= c = \left(\frac{\sqrt{1+m_c^2}}{1+\frac{1}{\sqrt{3}}|mc|}\right) c_{\pi 3} \text{ or } d_E(A,M) = c = \left(\frac{\sqrt{3}\sqrt{1+m_c^2}}{2|m_c|}\right) c_{\pi_3}, \\
d_E(K,C) &= x = \left(\frac{\sqrt{1+m_a^2}}{1+\frac{1}{\sqrt{3}}|m_a|}\right) x_{\pi 3} \text{ or } d_E(K,C) = x = \left(\frac{\sqrt{3}\sqrt{1+m_a^2}}{2|m_a|}\right) x_{\pi 3}, \\
d_E(L,A) &= y = \left(\frac{\sqrt{1+m_b^2}}{1+\frac{1}{\sqrt{3}}|m_b|}\right) y_{\pi 3} \text{ or } d_E(L,A) = y = \left(\frac{\sqrt{3}\sqrt{1+m_b^2}}{2|m_b|}\right) y_{\pi 3}, \\
d_E(M,B) &= z = \left(\frac{\sqrt{1+m_c^2}}{1+\frac{1}{\sqrt{3}}|m_c|}\right) z_{\pi 3} \text{ or } d_E(M,B) = z = \left(\frac{\sqrt{3}\sqrt{1+m_c^2}}{2|m_c|}\right) z_{\pi 3}.
\end{aligned}$$

If each situation here is examined separately;

Case I If $0 \leq |m_a|, |m_b|, |m_c| \leq \sqrt{3}$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\begin{aligned}
&\left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) a_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) b_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) c_{\pi 3}^2 \\
&= \left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) x_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) y_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) z_{\pi 3}^2
\end{aligned}$$

is found.

Case II If $0 \leq |m_a|, |m_b| \leq \sqrt{3}$ and $\sqrt{3} \leq |m_c|$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) a_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) b_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) c_{\pi 3}^2 = \left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) x_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) y_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) z_{\pi 3}^2$$

is obtained.

Case III If $0 \leq |m_b| \leq \sqrt{3}$ and $\sqrt{3} \leq |m_a|, |m_c|$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) a_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) b_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) c_{\pi 3}^2 = \left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) x_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) y_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) z_{\pi 3}^2$$

is found.

Case IV If $0 \leq |m_b|, |m_c| \leq \sqrt{3}$ and $\sqrt{3} \leq |m_a|$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) a_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) b_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) c_{\pi 3}^2 = \left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) x_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) y_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) z_{\pi 3}^2$$

is found.

Case V If $\sqrt{3} \leq |m_a|, |m_b|, |m_c|$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) a_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) b_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) c_{\pi 3}^2 = \left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) x_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) y_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) z_{\pi 3}^2$$

is found.

Case VI If $0 \leq |m_c| \leq \sqrt{3}$ and $\sqrt{3} \leq |m_a|, |m_b|$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) a_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) b_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) c_{\pi 3}^2 = \left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) x_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) y_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) z_{\pi 3}^2$$

is found.

Case VII If $0 \leq |m_a|, |m_c| \leq \sqrt{3}$ and $\sqrt{3} \leq |m_b|$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) a_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) b_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) c_{\pi 3}^2 = \left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) x_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) y_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) z_{\pi 3}^2$$

is found.

Case VIII If $0 \leq |m_a| \leq \sqrt{3}$ ve $\sqrt{3} \leq |m_b|, |m_c|$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) a_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) b_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) c_{\pi 3}^2 = \left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) x_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) y_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) z_{\pi 3}^2$$

is found.

ii. If $m_a \to \infty$, then we obtain

$$d_E(B,K) = a = \left(\frac{\sqrt{3}}{2}\right) a_{\pi 3},$$
$$d_E(K,C) = x = \left(\frac{\sqrt{3}}{2}\right) x_{\pi 3},$$
$$d_E(C,L) = b = \left(\frac{\sqrt{1+m_b^2}}{1+\frac{1}{\sqrt{3}}|m_b|}\right) b_{\pi 3} \text{ or } d_E(C,L) = b = \left(\frac{\sqrt{3}\sqrt{1+m_b^2}}{2|m_b|}\right) b_{\pi 3},$$
$$d_E(A,M) = c = \left(\frac{\sqrt{1+m_c^2}}{1+\frac{1}{\sqrt{3}}|mc|}\right) c_{\pi 3} \text{ or } d_E(A,M) = c = \left(\frac{\sqrt{3}\sqrt{1+m_c^2}}{2|m_c|}\right) c_{\pi 3},$$
$$d_E(L,A) = y = \left(\frac{\sqrt{1+m_b^2}}{1+\frac{1}{\sqrt{3}}|m_b|}\right) y_{\pi 3} \text{ or } d_E(L,A) = y = \left(\frac{\sqrt{3}\sqrt{1+m_b^2}}{2|m_b|}\right) y_{\pi 3},$$
$$d_E(M,B) = z = \left(\frac{\sqrt{1+m_c^2}}{1+\frac{1}{\sqrt{3}}|m_c|}\right) z_{\pi 3} \text{ or } d_E(M,B) = z = \left(\frac{\sqrt{3}\sqrt{1+m_c^2}}{2|m_c|}\right) z_{\pi 3}.$$

If each situation here is examined separately (In this case, it is clear that there cannot be $m_b \to \infty$ and $m_c \to \infty$);

Case I If $|m_a| = \infty$, $\sqrt{3} \leq |m_b|$ and $0 \leq |m_c| \leq \sqrt{3}$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{\sqrt{3}}{2}\right) a_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) b_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) c_{\pi 3}^2 = \left(\frac{\sqrt{3}}{2}\right) x_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right) y_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) z_{\pi 3}^2$$

is obtained.

Case II If $|m_a| = \infty$, $\sqrt{3} \leq |m_c|$ and $0 \leq |m_b| \leq \sqrt{3}$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{\sqrt{3}}{2}\right) a_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) b_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) c_{\pi 3}^2 = \left(\frac{\sqrt{3}}{2}\right) x_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) y_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) z_{\pi 3}^2$$

is obtained.

iii. If $m_b \to \infty$, then we write

$$d_E(C,L) = b = \left(\frac{\sqrt{3}}{2}\right) b_{\pi 3},$$
$$d_E(L,A) = y = \left(\frac{\sqrt{3}}{2}\right) y_{\pi 3},$$
$$d_E(B,K) = a = \left(\frac{\sqrt{1+m_a^2}}{1+\frac{1}{\sqrt{3}}|m_a|}\right) a_{\pi 3} \text{ or } d_E(B,K) = a = \left(\frac{\sqrt{3}\sqrt{1+m_a^2}}{2|m_a|}\right) a_{\pi 3},$$
$$d_E(K,C) = x = \left(\frac{\sqrt{1+m_a^2}}{1+\frac{1}{\sqrt{3}}|m_a|}\right) x_{\pi 3} \text{ or } d_E(K,C) = x = \left(\frac{\sqrt{3}\sqrt{1+m_a^2}}{2|m_a|}\right) x_{\pi 3},$$
$$d_E(A,M) = c = \left(\frac{\sqrt{1+m_c^2}}{1+\frac{1}{\sqrt{3}}|mc|}\right) c_{\pi 3} \text{ or } d_E(A,M) = c = \left(\frac{\sqrt{3}\sqrt{1+m_c^2}}{2|m_c|}\right) c_{\pi 3},$$
$$d_E(M,B) = z = \left(\frac{\sqrt{1+m_c^2}}{1+\frac{1}{\sqrt{3}}|m_c|}\right) z_{\pi 3} \text{ or } d_E(M,B) = z = \left(\frac{\sqrt{3}\sqrt{1+m_c^2}}{2|m_c|}\right) z_{\pi 3}.$$

If each situation here is examined separately (In this case, it is clear that there cannot be $m_a \to \infty$ and $m_c \to \infty$);

Case I If $|m_b| = \infty$, $\sqrt{3} \leq |m_a|$ and $0 \leq |m_c| \leq \sqrt{3}$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) a_{\pi 3}^2 + \left(\frac{\sqrt{3}}{2}\right) b_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) c_{\pi 3}^2 = \left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) x_{\pi 3}^2 + \left(\frac{\sqrt{3}}{2}\right) y_{\pi 3}^2 + \left(\frac{1+m_c^2}{\left(1+\frac{1}{\sqrt{3}}|m_c|\right)^2}\right) z_{\pi 3}^2$$

is obtained.

Case II If $|m_b| = \infty$, $\sqrt{3} \leq |m_c|$ and $0 \leq |m_a| \leq \sqrt{3}$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$\left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) a_{\pi 3}^2 + (\frac{\sqrt{3}}{2}) b_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) c_{\pi 3}^2 = \left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right) x_{\pi 3}^2 + (\frac{\sqrt{3}}{2}) y_{\pi 3}^2 + \left(\frac{3.(1+m_c^2)}{4.|m_c|^2}\right) z_{\pi 3}^2$$

is obtained.

iv. If $m_c \to \infty$, then

$$
\begin{aligned}
d_E(A,M) &= c = \left(\frac{\sqrt{3}}{2}\right) c_{\pi 3},\\
d_E(M,B) &= z = \left(\frac{\sqrt{3}}{2}\right) z_{\pi 3},\\
d_E(B,K) &= a = \left(\frac{\sqrt{1+m_a^2}}{1+\frac{1}{\sqrt{3}}|m_a|}\right) a_{\pi_3} \text{ or } d_E(B,K) = a = \left(\frac{\sqrt{3}\sqrt{1+m_a^2}}{2|m_a|}\right) a_{\pi 3},\\
d_E(K,C) &= x = \left(\frac{\sqrt{1+m_a^2}}{1+\frac{1}{\sqrt{3}}|m_a|}\right) x_{\pi 3} \text{ or } d_E(K,C) = x = \left(\frac{\sqrt{3}\sqrt{1+m_a^2}}{2|m_a|}\right) x_{\pi 3},\\
d_E(C,L) &= b = \left(\frac{\sqrt{1+m_b^2}}{1+\frac{1}{\sqrt{3}}|m_b|}\right) b_{\pi 3} \text{ or } d_E(C,L) = b = \left(\frac{\sqrt{3}\sqrt{1+m_b^2}}{2|m_b|}\right) b_{\pi 3},\\
d_E(L,A) &= y = \left(\frac{\sqrt{1+m_b^2}}{1+\frac{1}{\sqrt{3}}|m_b|}\right) y_{\pi 3} \text{ or } d_E(L,A) = y = \left(\frac{\sqrt{3}\sqrt{1+m_b^2}}{2|m_b|}\right) y_{\pi 3}.
\end{aligned}
$$

If each situation here is examined separately (In this case, it is clear that there cannot be $m_a \to \infty$ and $m_b \to \infty$);

Case I If $|m_c| = \infty$, $\sqrt{3} \leq |m_b|$ and $0 \leq |m_a| \leq \sqrt{3}$, from Carnot's theorem in the plane $\mathbb{R}^2$,

$$
\left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right).a_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right).b_{\pi 3}^2 + \left(\frac{\sqrt{3}}{2}\right).c_{\pi 3}^2 = \left(\frac{1+m_a^2}{\left(1+\frac{1}{\sqrt{3}}|m_a|\right)^2}\right).x_{\pi 3}^2 + \left(\frac{3.(1+m_b^2)}{4.|m_b|^2}\right).y_{\pi 3}^2 + \left(\frac{\sqrt{3}}{2}\right).z_{\pi 3}^2
$$

is obtained.

Case II If $|m_c| = \infty$, $\sqrt{3} \leq |m_a|$ and $0 \leq |m_b| \leq \sqrt{3}$, then from Carnot's theorem in the plane $\mathbb{R}^2$,

$$
\left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) a_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) b_{\pi 3}^2 + \left(\frac{\sqrt{3}}{2}\right) c_{\pi 3}^2 = \left(\frac{3.(1+m_a^2)}{4.|m_a|^2}\right) x_{\pi 3}^2 + \left(\frac{1+m_b^2}{\left(1+\frac{1}{\sqrt{3}}|m_b|\right)^2}\right) y_{\pi 3}^2 + \left(\frac{\sqrt{3}}{2}\right) z_{\pi 3}^2
$$

is obtained. ■

4. Conclusions

The plane $\mathbb{R}^2_{\pi 3}$ is constructed by simply replacing the Euclidean distance function d_E by the distance function $d_{\pi 3}$. Therefore it seems to study the plane $\mathbb{R}^2_{\pi 3}$ analogues of the topics which include the concept of distance in the plane $\mathbb{R}^2$. We give Carnot's theorem in any triangle in the plane $\mathbb{R}^2_{\pi 3}$.

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