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Araştırma Makalesi - Research Article

On Some Class of Normal Differential Operators for First Order

Birinci Mertebeden Normal Diferansiyel Operatörlerin Bazı Sınıfları

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ABSTRACT

In this work, we construct the minimal and maximal operators generated by linear differential-operator expression for first order in the Hilbert space of vector-functions on finite symmetric interval. Then, deficiency indices of the minimal operator will be calculated and the space of boundary values of this operator will be constructed. By using of Calkin-Gorbachuk method, the general representation of all normal extensions of the formally normal minimal operator in terms of boundary values will also be established. Moreover, we explore the spectrum structure of these extensions.

Keywords- Normal Differential Operator, Deficiency Indices, Space of Boundary Value, Spectrum

ÖZ

Bu çalışmada, sonlu simetrik aralıktaki vektör fonksiyonların Hilbert uzayında, birinci mertebeden lineer diferansiyel-operatör ifadesi tarafından doğrulan minimal ve maksimal operatörleri oluşturulmuştur. Daha sonra, bu minimal operatörün defekt sayıları hesaplanmış ve sınır değer uzayı oluşturulmuştur. Calkin-Gorbachuk yöntemi kullanılarak, formal normal minimal operatörün tüm normal genişlemelerinin sınır değerler dilinde genel formu oluşturulmuştur. Son olarak, bu genişlemelerin spektrum yapısı araştırılmıştır.

Anahtar Kelimeler- Normal Diferansiyel Operatör, Defekt Sayıları, Sınır Değer Uzayı, Spektrum

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I. NTRODUCTION AND NOTATIONS

The operator theory has a key role in multi-particle quantum calculus, quantum field theory, and differential equations with multi boundary values [1,2]. Selfadjoint extensions of linear densely defined closed symmetric operators have been introduced by von Neumann and Stone [3,4]. Then applications of these operators have been examined by Glazman [5] and Naimark [6]. It is noteworthy to mention that Glazman-Krein-Naimark Theorem and the Calkin-Gorbachuk method are really important in the existing literature (see [7, 8]). Everitt, Markus, O'Regan, Agarwal, Zettl and Sun [9-12] have obtained some results in scalar cases and these results motivate us to investigate them in vector cases.

Let L^* be Hilbert adjoint of L. Note that if $D(L) \subset D(L^*)$ and $\|Lu\| = \|L^*u\|$ for all $u \in D(L)$ then a densely-defined closed operator L is said to be formally normal. Also, if there is no formally normal extension for formally operator L then it is said that L is maximal formally normal. If $D(L) = D(L^*)$ holds for a formally normal operator L then L is said to be normal (see [13]).

In [13] Coddington has presented the generalized versions of the results given by von Neumann for normal extensions of formally normal operators in Hilbert space, and analogous results in unbounded cases have been studied by Kilpi [14-16] and Davis [17]. Note that these results have important applications in differential operators [18-23].

Ismailov and et. all have investigated some spectral problems associated with functional type linear singular differential operator of first order in the Hilbert space of vector functions [24-34].

This study, we examine the differential expression

$$k(u) = iu'(-t) + Au(t)$$

in $L^2(H,(-1,1))$, where H is a separable Hilbert space, $A: D(A) \subset H \to H$ densely defined selfadjoint operator, $A \geq 0$, and $L^2(H,(-1,1))$ the Hilbert space consisting vector functions.

Observe $u, v \in C_0^\infty(H,(-1,1))$,

$$< k(u), v >_{L^2(H,(-1,1))} = \int_{-1}^{1} < iu'(-t), v(t) >_H dt + < Au, v >_{L^2(H,(-1,1))}$$

$$= -\int_{-1}^{1} < iu'(-t), v(t) >'_H dt + \int_{-1}^{1} < iu(-t), v'(t) >_H dt + < Au, v >_{L^2(H,(-1,1))}$$

$$= < iu(1), v(-1) >_H - < iu(-1), v(1) >_H + \int_{-1}^{1} < iu(-t), v'(t) >_H dt$$

$$+ < Au, v >_{L^2(H,(-1,1))}$$

$$= \int_{-1}^{1} < u(t), -iv'(-t) >_H dt + < Au, v >_{L^2(H,(-1,1))}$$

$$= < u, k^+(v) >_{L^2(H,(-1,1))}.$$

If so, formally adjoint expression of $k(\cdot)$ in $L^2(H,(-1,1))$ is found as

$$k^+(v) = -iv'(-t) + Av(t).$$

The minimal K_0 and maximal K operators associated with differential-operator expression $k(\cdot)$ in $L^2(H,(-1,1))$ can be constructed with the use of same method as in [18, 35]. The operator K_0 is formally normal in $L^2(H,(-1,1))$. One can easily observe that K_0 is not maximal. Furthermore, the differential-operator expression $k(\cdot)$ with the boundary condition $u(1) = 0$ generates a normal extension of K_0 in $L^2(H,(-1,1))$.

Here, our aim is to present the general representation of all normal extensions of K_0 in $L^2(H,(-1,1))$ and examine the spectrum of these extensions.

II. MAIN RESULTS

In Section 2, the general representation of all normal extensions of K_0 will be investigated.

The real and imaginary parts of $k(\cdot)$ can be represented as

$$k_r(u) = \frac{k(u)+k^+(u)}{2} = A(t)u(t)$$

and

$$k_i(u) = \frac{k(u)-k^+(u)}{2i} = u'(-t).$$

It is clear that, the complex part of $k(\cdot)$ is a formally symmetric operator. Now, we will examine the general representation of all normal extensions of K_0 in $L^2(H,(-1,1))$ via Calkin-Gorbachuk method. To describe all normal extensions of K_0, it is sufficient to represent all selfadjoint extensions of the minimal operator K_{i_0} which is generated by the differential expression $k_i(\cdot)$ in $L^2(H,(-1,1))$. Then, in a Hilbert space, the deficiency indices of any symmetric operator are defined at [6].

First of all, we demonstrate the following lemma which we will require later.

Lemma 1.

$$\left(n_-(K_{i_0}), n_+(K_{i_0})\right) = (\dim H, \dim H)$$

holds for the deficiency indices of K_{i_0}.

Proof. Since M_0 is closed symmetric and A is selfadjoint, then the operators M_0 and $M_0 + A$ have equal deficiency indices [6], where M_0 is the minimal operator generated by following expression

$$m(u) = u'(-t)$$

in $L^2(H,(-1,1))$.

In order to find the deficiency indices of M_0, we have to solve the following differential equations

$$u'(-t) \pm iu(t) = 0,$$

in $L^2(H,(-1,1))$.

Let us define the operator J as follows:

$$J: L^2(H,(-1,1)) \to L^2(H,(-1,1)),$$

$$Ju(t) := u(-t).$$

Then $J^2 = I, J^* = J$ and $\|J\| = 1$ hold.

Therefore, we can rewrite the above equations as follows:

$$-(Ju)'(t) \pm iJ(Ju)(t) = 0$$

in $L^2(H,(-1,1))$.

If we take $Ju = v$, then we get

$$-v'(t) \pm Jv(t) = 0$$

in $L^2(H,(-1,1))$.

The solutions of the last equations have the form

$$v_\pm(t) = e^{\mp iJt} f, f \in H.$$

From these representations, we find

$$\|v_+\|^2_{L^2(H,(-1,1))} = \int_{-1}^{1} \|v_+\|^2_H dt = \int_0^1 \left\|e^{-iJt} f\right\|^2_H dt = \|f\|^2_H < \infty.$$

Consequently, $n_-(M_0) = \dim \ker(M + iI) = \dim H$.

Besides them,

$$\|v_-\|^2_{L^2(H,(-1,1))} = \int_{-1}^{1} \|v_-\|^2_H dt = \int_{-1}^{1} \left\|e^{iJt} f\right\|^2_H dt = \|f\|^2_H < \infty$$

holds for any $f \in H$.

Therefore $n_+(M_0) = \dim \ker(M - iI) = \dim H$.

Hence, we get $\left(n_-(K_{i_0}), n_+(K_{i_0})\right) = (\dim H, \dim H)$.

Lemma 2. The triplet (H, γ_1, γ_2)

$$\gamma_1: D(K_i) \to H, \gamma_1(u) = u(1),$$

$$\gamma_2: D(K_i) \to H, \gamma_2(u) = u(-1), u \in D(K)$$

is a boundary values space of K_{i_0} in $L^2(H,(-1,1))$.

Proof. For any $u,v \in D(K_{i_0})$, direct calculations show that

$$< K_i u, v >_{L^2(H,(-1,1))} - < u, K_i v >_{L^2(H,(-1,1))}$$
$$=< u'(-t) + Au(t), v(t) >_{L^2(H,(-1,1))} - < u(t), v'(-t) + Av(t) >_{L^2(H,(-1,1))}$$
$$=< u'(-t), v(t) >_{L^2(H,(-1,1))} - < u(t), v'(-t) >_{L^2(H,(-1,1))}$$
$$= \int_{-1}^{1} < u'(-t), v(t) >_H dt - \int_{-1}^{1} < u(t), v'(-t) >_H dt$$
$$= -\int_{-1}^{1} < u(-t), v(t) >'_H dt$$
$$=< u(1), v(-1) >_H - < u(-1), v(1) >_H$$
$$=< \gamma_1(u), \gamma_2(v) >_H - < \gamma_2(u), \gamma_1(v) >_H.$$

Now let $f,g \in H$. We can find a function $u \in D(K_i)$ such that

$$\gamma_1(u) = u(1) = f \text{ and } \gamma_2(u) = u(-1) = g.$$

If one picks u as follows

$$u(t) = \frac{1}{2}((1-t)g + (1+t)f), u \in D(K),$$

then

$$\gamma_1(u) = f \text{ and } \gamma_2(u) = g$$

hold.

As a consequence of the technical used in [7], one can immediately have the following.

Theorem 1. If $\widetilde{K}_\iota$ is a selfadjoint extension of K_{i_0} in $L^2(H,(-1,1))$, then it is generated by the differential-operator expression $k_i(\cdot)$ and the boundary condition

$$(U - I)u(1) + i(U + I)u(-1) = 0,$$

where U is unitary, I is identity operator on H. Furthermore, the unitary operator U is determined uniquely by the extension $\widetilde{K}_\iota$, i.e., $\widetilde{K}_\iota = K_{i_U}$ and vice versa.

Proof. Each selfadjoint extensions of K_{i_0} are defined by the differential-operator expression $k(\cdot)$ with boundary condition

$$(U - I)\gamma_1(u) + i(U + I)\gamma_2(u) = 0,$$

where $U: H \to H$ is a unitary operator. By using last lemma, we obtain

$$(U - I)u(1) + i(U + I)u(-1) = 0.$$

It completes the proof.

Now, let us describe the characterization of all normal extensions of K_0 in $L^2(H,(-1,1))$.

Theorem 2. Let $A^{1/2}(D(\tilde{L})) \subset W_2^1(H,(-1,1))$ and $AJ = JA$. Then $k(\cdot)$ generates each normal extension $\widetilde{K}$ of K_0 in $L^2(H,(-1,1))$ together with the boundary condition

$$(U - I)u(1) + i(U + I)u(-1) = 0,$$

where U is a unitary operator on H. Moreover, U is uniquely determined by $\widetilde{K}$ and vice versa.

Proof. If $\widetilde{K}$ is any normal extension of K_0 in $L^2(H,(-1,1))$, then

$$Re(\widetilde{K}) = \overline{A \otimes I}, Re(\widetilde{K}): D(\widetilde{K}) \subset L^2(H,(-1,1)) \to L^2(H,(-1,1)),$$

$$Im(\widetilde{K}) = \overline{I \otimes \frac{d}{dt}J}, Im(\widetilde{K}): D(\widetilde{K}) \subset L^2(H,(-1,1)) \to L^2(H,(-1,1)),$$

where the symbol $\otimes$ denotes a tensor product, are selfadjoint extensions of $Re(K_0)$ and $Im(K_0)$ in $L^2(H,(-1,1))$, respectively. By Theorem 1, the extension $Im(\widetilde{K})$ is generated by $k_i(\cdot)$ and the boundary condition

$$(U - I)u(1) + i(U + I)u(-1) = 0,$$

where U is a unitary operator on H. Furthermore, U is uniquely determined by $\widetilde{K}$.

On the contrary, assume that K_U be an operator which is generated by $k(\cdot)$ and the boundary condition

$(U-I)u(1)+i(U+I)u(-1)=0,$

with any unitary operator $U: H \rightarrow H$. In this case, it is obvious that

$$Re(\widetilde{K})=\overline{A \otimes I}, Re(\widetilde{K}): D(\widetilde{K}) \subset L^2(H,(-1,1)) \rightarrow L^2(H,(-1,1)),$$

$$Im(\widetilde{K})=\overline{I \otimes \tfrac{d}{dt} J}, Im(\widetilde{K}): D(\widetilde{K}) \subset L^2(H,(-1,1)) \rightarrow L^2(H,(-1,1))$$

are selfadjoint operators. Additionally, since $AJ=JA$, we have for every $u \in D(K_U)$

$$< Re(\widetilde{K})u, Im(\widetilde{K})u >_{L^2(H,(-1,1))} - < Im(\widetilde{K})u, Re(\widetilde{K})u >_{L^2(H,(-1,1))}$$

$$=< u'(-t), Au(t) >_{L^2(H,(-1,1))} - < Au(t), u'(-t) >_{L^2(H,(-1,1))}$$

$$=< A^{1/2}u'(-t), A^{1/2}u(t) >_{L^2(H,(-1,1))} - < A^{1/2}u(t), A^{1/2}u'(-t) >_{L^2(H,(-1,1))}$$

$$=< A^{1/2}u(-t), A^{1/2}u(t) >'_{L^2(H,(-1,1))}$$

$$=< A^{1/2}u(-1), A^{1/2}u(1) >_H - < A^{1/2}u(1), A^{1/2}u(-1) >_H$$

$$=< AJu(1), u(1) >_H - < JAu(1), u(1) >_H = 0.$$

Then, we complete the proof.

III. SPECTRAL ANALYSIS OF NORMAL EXTENSIONS

Here, we examine the spectral properties of any normal extensions K_U of K_0 in $L^2(H,(-1,1)$.

We first prove the following theorem which deals with the structure of the spectrum.

Theorem 3. The followings are equivalent:

i. $\mu \in \sigma(K_U)$.
ii. $0 \in \sigma((U-I)e^{-i(A-\mu)J}+i(U+I)e^{i(A-\mu)J})$.

Proof. Let the proof begin with the spectrum problem. We have the following problem

$K_U(u)=\mu u+f, f \in L^2(H,(-1,1)), \mu \in \mathbb{C}.$

Then we get

$iu'(-t)+Au(t)=\mu u(t)+f(t), f \in L^2(H,(-1,1)), \mu \in \mathbb{C},$

$(U-I)u(1)+i(U+I)u(-1)=0$

that is,

$-i(Ju)'(t)+AJ(Ju)(t)=\mu J(Ju)(t)+f(t), f \in L^2(H,(-1,1)), \mu \in \mathbb{C}.$

On the other side, a direct calculation shows that

$$u(t)=Je^{-i(A-\mu)Jt}f_\mu+iJ\int_{-1}^{t} e^{-i(A-\mu)J(t-s)}f(s)ds, f_\mu \in H, -1<t<1.$$

Taking the boundary conditions into account, we obtain that

$$\left[(U-I)e^{-i(A-\mu)J}+i(U+I)e^{i(A-\mu)J}\right]Jf_\mu=-i(U-I)J\int_{-1}^{1} e^{-i(A-\mu)J(1-s)}f(s)ds.$$

Therefore, $\mu \in \sigma(K_U)$ iff

$0 \in \sigma((U-I)e^{-i(A-\mu)J}+i(U+I)e^{i(A-\mu)J}).$

Corollary 1. If $U=I$ and $U=-I$, then $\sigma(K_I)=\emptyset$ and $\sigma\left(K_{(-I)}\right)=\emptyset$, respectively.

Proof. If $U=I$, then we have

$(U-I)e^{-i(A-\mu)J}+i(U+I)e^{i(A-\mu)J}=2ie^{i(A-\mu)J}.$

We also get $0 \notin \sigma(2ie^{i(A-\mu)J})$. Then $\sigma(K_I)=\emptyset$ by using Theorem 3.

In a similar manner, if $U=-I$, then we obtain

$$(U-I)e^{-i(A-\mu)J}+i(U+I)e^{i(A-\mu)J}=-2ie^{-i(A-\mu)J}.$$

We get $0 \notin \sigma(-2ie^{-i(A-\mu)J})$. It gives from Theorem 3; we again find that $\sigma\left(K_{(-I)}\right)=\emptyset$. This completes the proof.

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