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FI_{ss} −Lifting Modules

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Abstract

The purpose of this note is to show some key features of FI_{ss} −lifting and strongly FI_{ss} −lifting modules. We examine that under whose condition for direct summands, direct sums and submodules of (strongly) FI_{ss} −lifting modules are (strongly) FI_{ss} −lifting. We give an example to exhibit that an FI −lifting module needs not to be FI_{ss} −lifting. We provide that the property of being strongly FI_{ss} −lifting module is inherited by direct summands.

Keywords: ss −supplement submodule, FI_{ss} −lifting module, strongly FI_{ss} −lifting module.

FI_{ss} −Yükseltilebilir Modüller

Öz

Bu çalışmanın amacı FI_{ss} −yükseltilebilir ve güçlü FI_{ss} −yükseltilebilir modüllerin bazı temel özelliklerini göstermektir. (Güçlü) FI_{ss} −yükseltilebilir modüllerin direkt toplam terimlerinin, direkt toplamlarının ve alt modüllerinin hangi koşullar altında altında (güçlü) FI_{ss} −yükseltilebilir modül olduğunu inceliyoruz. FI −yükseltilebilir bir modülün FI_{ss} −yükseltilebilir olmak zorunda olmadığını gösteren bir örnek veriyoruz. Güçlü FI_{ss} −yükseltilebilir modül olma özelliğinin direkt toplam terimleri tarafından aktarıldığını ispatlıyoruz.

Anahtar Kelimeler: ss −tümleyen alt modül, FI_{ss} −yükseltilebilir modül, güçlü FI_{ss} −yükseltilebilir modül.

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1. Introduction

Along this article, whole modules are keep in view as unital left R −modules where R signifies an arbitrary ring that has unit element. Let M be such a module. The impressions $L \leq M$, $L \leq_{\oplus} M$ mean that L is a submodule of M, L is a direct summand of M, respectively. A submodule L of M is *small* in M, notated with $L \ll M$, if $M \neq L + A$ for each proper submodule A of M. By $E(M), Rad(M)$ and $Soc(M)$, we imply the injective hull of M, the radical of M and the socle of M, respectively (Wisbauer, 1991). $Soc_s(M)$ stands for the sum of whole simple submodules of M that are small in M (Zhou and Zhang, 2011). A submodule L of a module M is *fully invariant* if $\gamma(L)$ is contained in L for each endomorphism γ of M (Wisbauer, 1991). Note that $Soc_s(M)$ is a fully invariant submodule of M. A submodule L of M is *coclosed* in M, if L has no proper submodule L' for which $L' \subset L$ is cosmall in M, that is, $L/L' \ll M/L'$. For example, each direct summand is coclosed in M (see 3.6 in (Clark et al., 2006)).

Let M be a module and $L, V \leq M$. L is a *supplement submodule* in M of V, if $M = L + V$ and $L \cap V \ll L$ (Wisbauer, 1991). M is a *supplemented module* if each submodule possesses a supplement in M. M is an *amply supplemented module,* if each submodule T of M possesses ample supplements in M, that is, for any $X \leq M$ providing $M = T + X$ includes a supplement of T in M (Wisbauer, 1991). As a special concept of supplements, in (Kaynar et al., 2020), the authors introduced that T is an ss −*supplement* of V in M, if $M = T + V$ and $T \cap V \leq Soc_s(T)$, where $Soc_s(T) = \sum\{T' \ll T \mid T'$ is simple$\} = Rad(T) \cap Soc(T)$ (see Lemma 2 and Lemma 3 in (Kaynar et al., 2020)). Clearly, the notion of ss −supplement is a proper generalization of the notion of direct summands. M is ss −*supplemented* if each submodule L of M possesses an ss −supplement in M. M is *amply* ss −*supplemented,* if each submodule L of M possesses ample ss −supplements in M (Kaynar et al., 2020).

In (Clark et al., 2006), it is defined that M is a *lifting module* if each submodule K includes $D \leq_{\oplus} M$ providing $K/D \ll M/D$. This definition is called *lifting condition* of a module. It is proved in 22.3 in (Clark et al., 2006) that a module is lifting if and only if supplement submodules are direct summands and the module is amply supplemented .

In (Eryılmaz, 2021), the author said a module M ss −*lifting* if for each submodule L of M, M possesses the decomposition $M = M_1 \oplus M_2$ providing $M_1 \leq L$ and $L \cap M_2 \leq Soc_s(M_2)$. It is obvious that each ss −lifting module is lifting.

In (Koşan, 2005), it is investigated that the lifting condition of a module and its fully invariant submodules. It is defined that if the whole submodules of M which are fully invariant possess the lifting condition, then M is FI −*lifting*. On the other side, Talebi and Amoozegar generalized the concept of FI −lifting modules to strongly FI −lifting modules in (Talebi and Amoozegar, 2008). If each fully invariant submodule F of a module M includes $X \leq_{\oplus} M$ that is fully invariant providing $F/X \ll M/X$, then the authors said M *strongly* FI *−lifting*. In the same paper, the authors showed several properties of these modules.

Inspired by these concepts, we generalize the notion of ss −lifting and FI −lifting modules to FI_{ss} −lifting and strongly FI_{ss} −lifting modules by taking into account each fully invariant submodule of a module instead of each submodule of a module. We give an example to demonstrate that an FI −lifting module does not need to be FI_{ss} −lifting. It is proved that a finite direct sum of FI_{ss} −lifting modules is FI_{ss} −lifting. We give an example indicating that any infinite direct sum of FI_{ss} −lifting modules is not FI_{ss} −lifting. We verify that the class of FI_{ss} −lifting modules is closed under fully invariant quotients. It is showed that a fully invariant which is coclosed submodule of an FI_{ss} −lifting module is FI_{ss} −lifting. We show that the property being strongly FI_{ss} −lifting module is transferred by each direct summand. We give the necessary and sufficient condition for a module which is a finite direct sum of fully invariant submodules to be strongly FI_{ss} −lifting.

2. Materials and Methods

In this section, we define FI_{ss} −lifting and strongly FI_{ss} −lifting modules. But firstly, we give some facts about ss −lifting modules and some properties of fully invariant submodules.

Lemma 2.1. For a module M, we have;

1. Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M.
2. If $K \leq L \leq M$ such that K is a fully invariant submodule of L and L is a fully invariant submodule of M , then K is fully invariant submodule of M.
3. If $M = \bigoplus_{i \in I} M_i$ and N is a fully invariant submodule of M, then $N = \bigoplus_{i \in I} (N \cap M_i)$.
4. If $K \leq L \leq M$ such that K is a fully invariant submodule of M and L/K is a fully invariant submodule of M/K , then L is a fully invariant submodule of M.

Proof. See Lemma 1.1 in (Birkenmeir et al., 2002) and Lemma 3.2 in (Koşan, 2005).

Lemma 2.2. Let M be a module and $U \leq M$. The following conditions are equivalent:

1. There is a direct summand K of M such that $K \leq U$ and $U/K \leq Soc_s(M/K)$.
2. There are a direct summand K of M and a submodule L of M such that $K \leq U$, $U = K + L$ and $L \leq Soc_s(M)$.

3. There is a decomposition $M = K \oplus K'$ with $K \leq U$ and $K' \cap U \leq Soc_s(M)$.
4. U has an ss −supplement K' in M such that $K' \cap U$ is a direct summand of U.
5. There is a homomorphism $f: M \longrightarrow M$ with $f^2 = f$ such that $f(M) \leq U$ and $(1-f)(U) \leq Soc_s(1-f)(M)$.

Proof. See Lemma 2 in (Eryılmaz, 2021).

Theorem 2.3. For a module M, the following conditions are equivalent:
1. M is ss −lifting.
2. Every submodule U of M can be written as $U = K \oplus S$ where K is a direct summand of M and $S \leq Soc_S(M)$.
3. M is amply ss −supplemented and every ss −supplement submodule of M is a direct summand of M.

Proof. By Theorem 1 in (Eryılmaz, 2021).

Definition 2.4. We call a module M *FI_{ss} −lifting*, if each fully invariant submodule U of M includes $X \leq_{\oplus} M$ providing semisimple $U/X \ll M/X$.

Clearly, ss −lifting modules are FI_{ss} −lifting. But, each FI_{ss} −lifting module does not need to be ss −lifting as can be seen from the following example.

Example 2.5. The $\mathbb{Z}$ −module $\mathbb{Q}$, the set of all rational numbers, is not ss −lifting as it is not lifting module by Example 2.8 in (Koşan, 2005). However, ${}_{\mathbb{Z}}\mathbb{Q}$ is an FI_{ss} −lifting module as its only fully invariant submodules are 0 and ${}_{\mathbb{Z}}\mathbb{Q}$.

Definition 2.6. We call a module M *strongly FI_{ss} −lifting*, if each fully invariant submodule U of M includes $X \leq_{\oplus} M$ that is fully invariant providing semisimple $U/X \ll M/X$.

3. Findings and Discussion

Recall from (Talebi and Vanaja, 2002) that a module M is *cosingular* (*non-cosingular*) *module* if $\bar{Z}(M) = \cap \{Ker(\varphi) \mid \varphi: M \longrightarrow X \text{ is a homomorphism and } X \ll E(X)\} = 0$ $(\bar{Z}(M) = M)$.

Proposition 3.1. Let M be a module. Let $A \leq M$ and $L \leq_{\oplus} M$. Assume that M/L is ss −lifting. If $A/A \cap L$ is non-cosingular, then $A + L \leq_{\oplus} M$.

Proof. By the assumption, there is $X/L \leq_{\oplus} M/L$ such that $X/L \leq (A+L)/L$, $(A+L)/X \ll M/X$ and $(A+L)/X$ is semisimple, as M/L is $ss-$lifting. Then $(A+L)/X$ is cosingular. Since $(A+L)/L \cong A/A \cap L$, we get $(A+L)/L$ is non-cosingular, by the assumption. By Proposition 2.4 in (Talebi and Vanaja, 2002), $(A+L)/X$ is non-cosingular. Hence $X = A+L$.

Proposition 3.2. If M is non-cosingular module and M/T is $ss-$lifting where $T \leq_{\oplus} M$, then $(T+L)/T \leq_{\oplus} M/T$ for whole direct summands L of M.

Proof. Let M/T be $ss-$lifting where $T \leq_{\oplus} M$. Let $L \leq_{\oplus} M$. Then $L/L \cap T$ is non-cosingular by Proposition 2.4 in (Talebi and Vanaja, 2002). Thus by Proposition 3.1, $L+T \leq_{\oplus} M$. Hence $(L+T)/T \leq_{\oplus} M/T$.

Recall from (Garcia, 1989) that a module M has *Summand Sum Property* if $M_1 + M_2 \leq_{\oplus} M$ for any $M_1, M_2 \leq_{\oplus} M$.

Corollary 3.3. Each non-cosingular $ss-$lifting module has the Summand Sum Property.

Proof. Let M be $ss-$lifting non-cosingular module and $A, B \leq_{\oplus} M$. Then $M = A \oplus A' = B \oplus B'$ for some submodules A', B' of M. By Theorem 3 in (Eryılmaz, 2021) A' and B' are $ss-$lifting modules. Since $M/A \cong A'$ and $M/B \cong B'$, $(A+B)/A \leq_{\oplus} M/A$ and $(A+B)/B \leq_{\oplus} M/B$ by Proposition 3.2. Hence $A+B \leq_{\oplus} M$.

Theorem 3.4. Let M be a module. Then the following statements are equivalent:

1. M is $FI_{ss}-$lifting.
2. For each fully invariant submodule U of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq U$, $M_2 \cap U \ll M_2$ and $M_2 \cap U$ is semisimple.
3. Each fully invariant submodule U of M possesses a decomposition $U = D \oplus S$ where $D \leq_{\oplus} M$, $S \ll M$ and S is semisimple.

Proof. $(1) \Longrightarrow (2)$ Let $U \leq M$ be fully invariant. Since M is $FI_{ss}-$lifting, then M has the decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq U$, $U/M_1 \ll M/M_1$ and U/M_1 is semisimple. Consider the isomorphism $f: M/M_1 \longrightarrow M_2$. By the way, since $U/M_1 \ll M/M_1$, then $M_2 \cap U = f(U/M_1) \ll f(M/M_1) = M_2$ by 19.3(4) in (Wisbauer, 1991). Moreover, $M_2 \cap U$ is semisimple by 8.1.5 in (Kasch, 1982).

$(2) \Longrightarrow (3)$ Let $U \leq M$ be fully invariant. Then by the assumption, there is a decomposition of M such that $M = M_1 \oplus M_2$, $M_1 \leq U$, $M_2 \cap U \ll M_2$ and $M_2 \cap U$ is semisimple. Note that $M_2 \cap U \ll$

M by 19.3(5) in (Wisbauer, 1991). We have that $U = M_1 \oplus (M_2 \cap U)$, by the modular law. Say $D = M_1$ and $S = M_2 \cap U$. Therefore, $U = D \oplus S$ where $D \leq_{\oplus} M$, $S \ll M$ and S is semisimple.

(3) $\Longrightarrow$ (1) Let $U \leq M$ be fully invariant. By the assumption, there is a decomposition $U = D \oplus S$ where $D \leq_{\oplus} M$, $S \ll M$ and S is semisimple. So there is $D \leq_{\oplus} M$ such that $D \leq U$. It is obvious that $S \cong U/D$ is semisimple. Furthermore, suppose that $M/D = (U/D) + (V/D)$ for some $V \leq M$ such that $D \leq V$. Thus we have $M = U + V = D + S + V = S + V$. Since $S \ll M$, then we get that $M = V$. Hence $M/D = V/D$, and so $U/D \ll M/D$. Consequently, M is an FI_{ss} $-$lifting module.

Proposition 3.5. Let M be an FI_{ss} $-$lifting module and F be a fully invariant submodule of M. Then the factor module M/F of M is FI_{ss} $-$lifting.

Proof. Let $L/F \leq M/F$ be fully invariant. Then by Lemma 2.1, L is fully invariant in M. As M is FI_{ss} $-$lifting, then M has the decomposition $M = A \oplus A'$ such that $A \leq L$, $L/A \ll M/A$ and L/A is semisimple. Consider the canonical projection $\pi: M \longrightarrow M/A$ and the canonical injection $\iota: A' \longrightarrow M$. $f = \iota\pi: M \longrightarrow M$ be an endomorphism of M. Since F and L are fully invariant submodules of M, $f(F) \leq F$ and $f(L) \leq L$. It can be seen that $L = f^{-1}(L)$. Note that $f^{-1}(F) \leq L = f^{-1}(L)$. Assume that T be a submodule of M such that $f^{-1}(F) \leq T$ and $M/(f^{-1}(F)) = (L/(f^{-1}(F))) + (T/(f^{-1}(F)))$. Then we get that $M = L + T$, and since $L/A \ll M/A$, $M = T$. Therefore, $L/(f^{-1}(F)) \ll M/(f^{-1}(F))$, that is, $(L/F)/((f^{-1}(F))/F) \ll (M/F)/((f^{-1}(F))/F)$. Moreover, since L/A is semisimple, then $L/(F + A)$ is semisimple as a factor module of L/A by 8.1.5 in (Kasch, 1982). Note that

$$L/(f^{-1}(F)) = (L/F)/((f^{-1}(F))/F) = [(L/(F + A))/((F + A)/F)]/[(f^{-1}(F)/F)].$$

Then $L/f^{-1}(F)$ is semisimple by 8.1.5 in (Kasch, 1982). Now we want to show that $f^{-1}(F)/F \leq_{\oplus} M/F$. Since $M = A \oplus A'$, then $M = f^{-1}(F) + A'$. Thus $M/F = ((f^{-1}(F))/F) + ((A' + F)/F)$. Since $f^{-1}(F) \cap (A' + F) = F + (f^{-1}(F) \cap A') = F$, then $(f^{-1}(F))/F \leq_{\oplus} M/F$. Hence M/F is FI_{ss} $-$lifting.

Theorem 3.6. Let M_i be FI_{ss} $-$lifting module for each $1 \leq i \leq n$ and $M = \bigoplus_{i=1}^{n} M_i$. Then M is FI_{ss} $-$lifting.

Proof. Let $F \leq M$ be fully invariant. Then $F = \bigoplus_{i=1}^{n} (F \cap M_i)$ by Lemma 2.1. Note that $F \cap M_i$ is fully invariant in M_i for each $1 \leq i \leq n$. Since each M_i is FI_{ss} $-$lifting, then $F \cap M_i = D_i \oplus S_i$ where $D_i \leq_{\oplus} M_i$, $S_i \ll M_i$ and S_i is semisimple for each $1 \leq i \leq n$ by Theorem 3.4. Say $D = \bigoplus_{i=1}^{n} D_i$ and

$S = \bigoplus_{i=1}^{n} S_i$. Then $F = D \oplus S$, where $D \leq_{\oplus} M$, $S \ll M$ and S is semisimple by 19.3(3) in (Wisbauer, 1991) and 8.1.5 in (Kasch, 1982). Hence by Theorem 3.4, M is FI_{ss} −lifting.

Corollary 3.7. Let M be a finite direct sum of ss −lifting modules, then M is FI_{ss} −lifting.

Now we shall give an example to exhibit any infinite direct sum of FI_{ss} −lifting modules needs not to be FI_{ss} −lifting.

A projective module M together with an epimorphism $f: M \longrightarrow N$ such that $Ker(f) \ll M$ is called *projective cover* of N (see 19.4 in (Wisbauer, 1991)). Recall from 43.9 in (Wisbauer, 1991) that a ring R is *left perfect* if each left R −module has a projective cover.

A ring R is *semiperfect* if $R/Rad(R)$ is left semisimple and idempotents in $R/Rad(R)$ can be lifted to R (see 42.6 in (Wisbauer, 1991)). It is proved in Theorem 5 in (Eryılmaz, 2021) that R is semiperfect ring with $Rad(R) \leq Soc({}_R R)$ if and only if ${}_R R$ is ss −lifting.

Example 3.8. Let R be semiperfect ring with $Rad(R) \leq Soc({}_R R)$ which is not left perfect and M be countably generated free R −module. By Theorem 5 in (Eryılmaz, 2021) each direct summand of M is ss −lifting, and so FI_{ss} −lifting. Note that $Rad(M)$ is not small in M and $Rad(M)$ is fully invariant in M. Here, $Rad(M)$ can not include a nonzero direct summand of M. If $Rad(M)$ included a direct summand D of M, then it would be $D = Rad(D)$. But this contradicts with the fact that for any projective R −module N, $N \neq Rad(N)$. Hence, M is not an FI_{ss} −lifting module when considering its fully invariant submodule $Rad(M)$, although each direct summand of M is an FI_{ss} −lifting module.

Proposition 3.9. Let M be a projective module. Then M is FI_{ss} −lifting if and only if M/U has a projective cover for each fully invariant submodule U of M such that U/X is semisimple for any $X \leq_{\oplus} M$.

Proof. ($\Longrightarrow$) Suppose that the projective module M is FI_{ss} −lifting and $U \leq M$ be fully invariant. Then by Theorem 3.4, $U = A \oplus B$ where $A \leq_{\oplus} M$, $B \ll M$ and B is semisimple. Note that $B \cong U/A$. Since $B \ll M$, then $(B + A)/A \ll M/A$ by 19.3(4) in (Wisbauer, 1991). Hence the canonical projection $\pi: M/A \longrightarrow M/(A + B) = M/U$ is a projective cover of M/U, as desired.
($\Longleftarrow$) Suppose that a projective module P with $f: P \longrightarrow M/U$ is a projective cover of the factor module M/U. Then there is a homomorphism $g: M \longrightarrow P$ providing $fg = \pi$ where $\pi: M \longrightarrow M/U$ is the canonical projection. Since $Ker(f) \ll P$ and π is an epimorphism, then g is an epimorphism, and

hence g splits. Thus $M = Ker(g) \oplus T$ for some submodule T of M. Then $U = Ker(g) \oplus (U \cap T)$, and also $U \cap T \ll M$. Moreover, by the assumption, $U \cap T$ is semisimple.

Corollary 3.10. Let R be a ring. The module ${}_R R$ is FI_{ss} −lifting if and only if for each two sided ideal I of R such that I/J is semisimple for any $J \leq_\oplus R$, ${}_R R/I$ has a projective cover.

Proposition 3.11. Let M be an FI_{ss} −lifting module and $U \leq M$ be coclosed and fully invariant. Then U is FI_{ss} −lifting.

Proof. Let $F \leq U$ be fully invariant. Then by Lemma 2.1, F is fully invariant in M. As M is FI_{ss} −lifting, there is $K \leq_\oplus M$ such that $K \leq F$, $F/K \ll M/K$ and F/K is semisimple. As U is coclosed in M, then U/K is coclosed in M/K by 3.7(1) in (Clark et al., 2006). Thus $F/K \ll U/K$ by 3.7(3) in (Clark et al., 2006). Also $K \leq_\oplus U$. Hence U is FI_{ss} −lifting.

Proposition 3.12. If M is an FI_{ss} −lifting module, then each fully invariant submodule of $M/Soc_s(M)$ is a direct summand.

Proof. Let $N/Soc_s(M) \leq M/Soc_s(M)$ be fully invariant, and hence N is too in M by Lemma 2.1. By the hypothesis, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$, $M_2 \cap N \ll M_2$ and $M_2 \cap N$ is semisimple. Note here that $M_2 \cap N \leq Soc_s(M)$. Thus we get that $M/Soc_s(M) = (N/Soc_s(M)) \oplus ((M_2 + Soc_s(M))/Soc_s(M))$, as desired.

4. Strongly FI_{ss} −Lifting Modules

Now the following theorem can be easily proved as in Theorem 3.4.

Theorem 4.1. Let M be a module. Then the following statements are equivalent:

1. M is strongly FI_{ss} −lifting.
2. For each fully invariant submodule U of M, there are submodules D and S of M such that $U = D + S$ where $D \leq_\oplus M$ is fully invariant , $S \ll M$ and S is semisimple.
3. Each fully invariant submodule U of M possesses a decomposition $U = D \oplus S$ where $D \leq_\oplus M$ is fully invariant, $S \ll M$ and S is semisimple.

Proposition 4.2. Let M be an FI_{ss} −lifting module with $Soc_s(M) = 0$ and $U \leq M$ be fully invariant. Then U is strongly FI_{ss} −lifting.

Proof. Let $X \leq U$ be fully invariant. Thus by Lemma 2.1, X is fully invariant in M. Since M is FI_{ss} −lifting, then $X = D \oplus S$ where $D \leq_{\oplus} M$, $S \ll M$ and S is semisimple by Theorem 3.4. Note that $S \leq Soc_s(M)$. Since $Soc_s(M) = 0$, then $X \leq_{\oplus} M$. Hence $X \leq_{\oplus} U$.

Theorem 4.3. Strongly FI_{ss} −lifting modules are transferred by direct summands.

Proof. Suppose that M be a strongly FI_{ss} −lifting module and $M_1 \leq_{\oplus} M$. Thus $M = M_1 \oplus M_2$ for some submodule M_2 of M. Let the module U_1 be fully invariant in M_1. Therefore, there is a fully invariant submodule U_2 of M_2 providing that $U_1 \oplus U_2$ is fully invariant in M by Lemma 1.11 in (Rizvi and Cosmin, 2004). Since M is strongly FI_{ss} −lifting, then $U_1 \oplus U_2 = K \oplus T$ where $K \leq_{\oplus} M$ is fully invariant, $T \ll M$ and T is semisimple. $K = (K \cap M_1) \oplus (K \cap M_2)$ by Lemma 2.1 and $K \cap M_1$ is fully invariant in M_1. Also $K \cap M_1 \leq_{\oplus} M$. Thus we have $U_1 = \pi_{M_1}(K) + \pi_{M_1}(T)$=$(K \cap M_1)$+$\pi_{M_1}(T)$ where $\pi_{M_1}: M \longrightarrow M / M_2$ is the canonical projection. Since $T \ll M$, $\pi_{M_1}(T) \ll M_1$ by 19.3(4) in (Wisbauer, 1991). Moreover, since T is semisimple, then $\pi_{M_1}(T)$ is semisimple by 8.1.5 in (Kasch, 1982). Hence M_1 is strongly FI_{ss} −lifting by Theorem 4.1.

Proposition 4.4. Let $M = \bigoplus_{i=1}^{n} M_i$ and M_i be fully invariant in M for each $1 \leq i \leq n$. Then M is strongly FI_{ss} −lifting module if and only if M_i is strongly FI_{ss} −lifting module for each $1 \leq i \leq n$.

Proof. $(\Longrightarrow)$ The proof follows from Theorem 4.3.

$(\Longleftarrow)$ Let U be a fully invariant submodule of M. Then $U = \bigoplus_{i=1}^{n} (U \cap M_i)$ by Lemma 2.1. Also $U \cap M_i$ is fully invariant in M_i for each $1 \leq i \leq n$. Since M_i is strongly FI_{ss} −lifting, then $U \cap M_i = A_i \oplus B_i$ where $A_i \leq_{\oplus} M_i$ is fully invariant, $B_i \ll M_i$ and B_i is semisimple for each $1 \leq i \leq n$ by Theorem 4.1. Put $A = \bigoplus_{i=1}^{n} A_i$ and $B = \bigoplus_{i=1}^{n} B_i$. Note that $B \ll M$ by 19.3(3) in (Wisbauer, 1991) and B is semisimple by 8.1.5 in (Kasch, 1982). Then we have $U = A \oplus B$, where $A \leq_{\oplus} M$. Moreover, since A_i is fully invariant in M_i and M_i is fully invariant in M for each $1 \leq i \leq n$, then A_i is a fully invariant submodule of M for each $1 \leq i \leq n$ by Lemma 2.1. By using Lemma 2.1, A is fully invariant in M. Hence M is strongly FI_{ss} −lifting.

Proposition 4.5. Let M be strongly FI_{ss} −lifting module and S be an ss −supplement submodule of M such that S is fully invariant in M. If any of the conditions below is verified, then S is strongly FI_{ss} −lifting.

1. S is indecomposable.
2. M is a self-injective module.

Proof. (1) Since S is an indecomposable fully invariant ss −supplement submodule of M, then $S \leq_{\oplus} M$. Thus S is strongly FI_{ss} −lifting by Theorem 4.3.

(2) Let $V \leq S$ be fully invariant. Thus V is a fully invariant submodule of M by Lemma 2.1, and hence $V = X \oplus Y$ where $X \leq_{\oplus} M$ is fully invariant, $Y \ll M$ and Y is semisimple by Theorem 4.1. Thus $X \leq_{\oplus} S$, and so $Y \ll S$ as S is a supplement submodule of M, by 20.2 in (Clark et al., 2006). Also, since M is self-injective, each homomorphism from S to S can be extended to a homomorphism from M to M. Hence X is a fully invariant submodule of S.

Recall from (Özcan et al., 2006) that M is *duo module* if each submodule of M is fully invariant.

Proposition 4.6. Let M be a duo module. If M is ss −lifting, then M is strongly FI_{ss} −lifting.

Proof. Suppose that $U \leq M$ be fully invariant. Since M is ss −lifting, then U includes $T \leq_{\oplus} M$ such that $U/T \ll M/T$ and U/T is semisimple by Lemma 2.2. Since M is duo module, then T is fully invariant in M. Hence M is a strongly FI_{ss} −lifting module.

Proposition 4.7. Let M be a module such that $Rad(M)$ is semisimple. Then M is an FI −lifting module if and only if M is an FI_{ss} −lifting module.

Proof. Let M be an FI −lifting module and $U \leq M$ be fully invariant. Then M possesses the decomposition $M = M_1 \oplus M_2$ providing $M_1 \leq U$ and $M_2 \cap U \ll M_2$. Note that $M_2 \cap U \leq Rad(M)$. By the assumption, $M_2 \cap U$ is semisimple. Hence M is an FI_{ss} −lifting module. The converse assertion is clear.

Example 4.8. Consider the left $\mathbb{Z}$ −module $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\,\mathbb{Z}$. Then M is FI −lifting module by Example 3.9 in (Koşan, 2005). But since $Rad(M)$ is not semisimple, M is not FI_{ss} −lifting module by Proposition 4.7.

Example 4.9. Consider the module M given in Example 4.8 and the submodule $T = \mathbb{Z}/2\mathbb{Z} \oplus 4\mathbb{Z}/8\,\mathbb{Z}$ of M. T is not small in M and does not include a nonzero fully invariant direct summand of M. Consequently, M is not strongly FI_{ss} −lifting module.

Proposition 4.10. Let M be a duo module. M is FI_{ss} −lifting if and only if M is ss −lifting.

Proof. Let M be an FI_{ss} −lifting module and $U \leq M$. As M is a duo module, U is fully invariant in M. By the assumption, M possesses the decomposition $M = M_1 \oplus M_2$ with the conditions $M_1 \leq U$, $M_2 \cap U \ll M_2$ and semisimple $M_2 \cap U$. Hence M is an ss −lifting module. The converse assertion is clear.

5. Conclusions and Recommendations

Here, we define a new concept of $ss-$ lifting modules. Instead of every submodule which satisfies $ss-$lifting property of a module, we consider every fully invariant submodule which satisfies $ss-$lifting property. The results in this paper can be generalized for weak $ss-$lifting modules that are defined in (Nişancı Türkmen, 2020), and also can be generalized for $\delta_\tau-$lifting and τ_e-lifting modules that are defined in (Tian et al., 2023) and (Öztürk Sözen, 2020), respectively.

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Statement of Conflicts of Interest

The author declares that there is no conflict of interest.

Statement of Research and Publication Ethics

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