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New Conservative Schemes for Zakharov Equation

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Abstract. New first-order and second-order energy preserving schemes are proposed for the Zakharov system. The methods are fully implicit and semi-explicit. It has been found that the first order method is also mass-conserving. Concrete schemes have been applied to simulate the soliton evolution of the Zakharov system. Numerical results show that the proposed methods capture the remarkable features of the Zakharov equation. We have obtained that the semi-explicit methods are more efficient than the fully implicit methods. Numerical results also demonstrate that the new energy-preserving schemes accurately simulate the soliton evolution of the Zakharov system.

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1. Introduction

Partial differential equations (PDEs) frequently arise as a mathematical model of fundamental laws of nature. Many problems in applied mathematics, mathematical physics, engineering, and health science are described by a PDE with appropriate initial and/or boundary conditions (see [3] and reference therein). Over the last several decades, numerous mathematical models have been proposed to understand the wave motion theory such as the well-known Korteweg-de Vries (KdV) equation [1,5,6,9,32,33,37], nonlinear Schrödinger (NLS) equation [7,10–12,44], regularized long wave equation [8], Burger equation [34], Boussinesq equation [38], and nonlinear fractional equations [28,36].

The Zakharov equation is one of the important model describing the interaction between high-frequency Langmuir waves and low-frequency ion-acoustic waves in a plasma. It has been the focus of many researchers due to the fact that the system is a classic nonlinear mathematical model in deep-water wave theory [31], communication [24], nonlinear pulse propagation in fibers [4], optical systems [26,35,41], superfluid [21] and plasma [13].

In the present work, we consider the following initial-boundary value problem proposed by Zakharov [48].

$$\begin{aligned} i\partial_t\phi + \partial_{xx}\phi + 2\phi\psi &= 0, \quad (x,t) \in (x_L, x_R)\times(0,T], \\ \partial_{tt}\psi - \partial_{xx}\psi + \partial_{xx}(|\phi|^2) &= 0, \quad (x,t) \in (x_L, x_R)\times(0,T], \end{aligned} \tag{1.1}$$

with the initial condition

$$\phi(x,0) = \phi_0(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(0), \qquad x \in (x_L, x_R), \tag{1.2}$$

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and the periodic boundary condition

$$\phi(x_L,t)=\phi(x_R,t),\ \psi(X_L,t)=\psi(x_R,t),\quad t\in(0,T], \tag{1.3}$$

where the complex-valued function $\phi(x,t)$ is the slowly varying envelope of the highly oscillatory electric field and $\psi(x,t)$ is the real-valued function that represents the fluctuation of an ion density about its equilibrium value. The exact Langmuir soliton solution of the equation (1.1) is given by [45]

$$\begin{aligned}\phi(x,t) &= Bi\sqrt{1-c^2}sech(2B\xi)\times exp\left(i\left(\frac{c(x-x_0)}{2}-(c^2/4-4B^2)t\right)\right),\\ \psi(x,t) &= 4B^2sech^2(2B\xi),\end{aligned} \tag{1.4}$$

where $\xi=x-x_0-ct$, B, c and x_0 are constants and $i^2=-1$. These are the solitary waves initially located at the spatial position x_0, and moving to the right with the velocity c and amplitudes $B\sqrt{1-c^2}$ and $4B^2$ for ϕ and ψ, respectively. Due to the square root $\sqrt{1-c^2}$, the solitary wave exists only for a finite range of velocity.

Introducing the real-valued auxiliary variables u,v,p and q such that $\phi=u+iv$,$\partial_t\phi=p+iq$ and $\partial_{xx}f=\partial_t\psi$, the system (1.1) can be written as

$$\begin{aligned}\partial_t u &= -\partial_{xx}v-2v\psi,\\ \partial_t v &= \partial_{xx}u+2u\psi,\\ \partial_t\psi &= \partial_{xx}f,\\ \partial_t f &= \psi-(u^2+v^2).\end{aligned} \tag{1.5}$$

It can be expressed in the infinite-dimensional Hamiltonian form

$$\partial_t z=J\frac{\delta\mathcal{H}}{\delta z},$$

where

$$J=\begin{pmatrix}J_1 & 0\\ 0 & J_1\end{pmatrix},J_1=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

$z=(u,v,\psi,f)^T$, and the related Hamiltonian function is [45]

$$\mathcal{H}(z)=\int_{x_L}^{x_R}\frac{1}{2}[(\partial_x u)^2+(\partial_x v)^2+\psi^2+(\partial_x f)^2-2\psi(u^2+v^2)]\,\mathrm{d}x.$$

In addition to the energy conservation $\mathcal{H}(z(t))=\mathcal{H}(z(0))$, the coupled system (1.1) possess the mass conservation

$$M(u(t),v(t)):=\int_{x_L}^{x_R}\left(u^2+v^2\right)\,dx. \tag{1.6}$$

The Zakharov system (1.1) was first introduced by Zakharov in 1972 [48]. Later on, Degtyarev, et al. [19] have discovered many interesting properties of Langmuir soliton by solving it numerically . The formation and interaction of solitons by the inverse scattering technique has been shown in [47]. Recently, different methods are applied to different types of solutions for the system (1.1). An extended trial equation method (ETEM) is used to obtain different types of solutions for the system (1.1) [20]. Two new analytical methods are used by Triphaty, et al. [43]. The results have shown various wave natures like hyperbolic function, exponential function, trigonometric function, rational function, periodic, anti-kink, dark, bright, and singular soliton solutions in various conditions.

Several numerical studies have been proposed for the numerical solution of the Zakharov system (1.1). In [39], an explicit and second-order accurate spectral method is designed. Bao, et al. [14] derived a time-splitting spectral scheme to solve the generalized Zakharov system (1.1), but it is not conservative. Chang et al. [18] presented a conservative finite difference scheme. Wang [45] introduced a multi-symplectic pseudo-spectral scheme for the multi-symplectic structure of the Zakharov system but ignores energy and mass conservations. Hong et al. [27] proposed a local energy-preserving scheme. An energy-preserving three-level finite difference scheme is proposed and its convergence is proven in [22, 23]. However, three-level schematics have problems depending on the initial condition. A three-level scheme cannot start the iteration without two initial approximations. It is possible to follow different approaches to derive energy-preserving schemes. Recently, a new class of energy-preserving methods called the average vector field (AVF)

method was introduced by Quispel, et al. [40]. For ordinary differential equation $dy/dt = f(y)$, the second-order AVF method is defined by

$$\frac{y_{n+1} - y_n}{\tau} = \int_0^1 f(\xi y_{n+1} + (1-\xi)y_n)\, d\xi,$$

where τ represents the time step. One of the most important features of the AVF method is that it only requires the evaluation of vector fields. Furthermore, if the vector field is Hamiltonian, i.e. $f(y) = J\nabla H(y)$, then using the fundamental theorem calculus and the skew-symmetry property of J, the Hamiltonian in preseved at every time step, that is $H(y_{n+1}) = H(y_n)$. In this respect, we refer the reader to [2,15,17,25,29,46] and reference therein. For illustration, consider the Hamiltonian system

$$\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix} = J\begin{pmatrix} \nabla_p H(p,q) \\ \nabla_q H(p,q) \end{pmatrix}, \quad p, q \in \mathbb{R}^n.$$

The AVF method is defined by [40]

$$\frac{1}{\tau}\begin{pmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{pmatrix} = J\begin{pmatrix} \int_0^1 \nabla_p(\xi p_{n+1} + (1-\xi)p_n, \xi q_{n+1} + (1-\xi)q_n)\, d\xi, \\ \int_0^1 \nabla_q(\xi p_{n+1} + (1-\xi)p_n, \xi q_{n+1} + (1-\xi)q_n)\, d\xi, \end{pmatrix}. \tag{1.7}$$

Recently, a more efficient AVF-based method called the partitioned AVF (PAVF) method

$$\frac{1}{\tau}\begin{pmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{pmatrix} = J\begin{pmatrix} \int_0^1 \nabla_p(\xi p_{n+1} + (1-\xi)p_n, q_n)\, d\xi, \\ \int_0^1 \nabla_q(p_{n+1}, \xi q_{n+1} + (1-\xi)q_n)\, d\xi \end{pmatrix}. \tag{1.8}$$

is introduced by Cai, et al. [16]. If the PAVF method (1.8) is denoted by $Z_{AVF}(\tau)$, then reversing its path order the adjoint method $Z_{APAVF}(\tau)$ can be defined by

$$\frac{1}{\tau}\begin{pmatrix} p_{n+1} - p_n \\ q_{n+1} - q_n \end{pmatrix} = J\begin{pmatrix} \int_0^1 \nabla_p(\xi p_{n+1} + (1-\xi)p_n, q_{n+1})\, d\xi, \\ \int_0^1 \nabla_q(p_n, \xi q_{n+1} + (1-\xi)q_n)\, d\xi, \end{pmatrix}. \tag{1.9}$$

Together with the PAVF method (1.8) and the adjoint method (1.9), the following PAVF composition (PAVF-C) method

$$Z_{PAVF-C}(\tau) := Z_{APAVF}(\frac{\tau}{2}) \circ Z_{PAVF}(\frac{\tau}{2}), \tag{1.10}$$

and the PAVF plus (PAVF-P) method

$$Z_{PAVF-P}(\tau) := \frac{1}{2}(Z_{APAVF}(\tau) + Z_{PAVF}(\tau)) \tag{1.11}$$

can be defined [16]. Both the PAVF-C method (1.10) and the PAVF-P method (1.11) are second-order and preserve the Hamiltonian. In this study, following the methods (1.7)-(1.11), we propose four energy preserving two-level schemes for the Zakharov system (1.1). Two of them are fully implicit and the other two schemes are linearly implicit. Besides the energy conservations, the linearly implicit schemes preserve the mass of the system (1.1).

The remainder of the paper is organized as follows: In Section 2, we present four energy-conserving difference scheme for the initial-boundary problem (1.1)–(1.3). In Section 3, various numerical results are provided to demonstrate the theoretical results. Finally, we draw a conclusion in Section 4.

2. Energy Preserving Schemes for Zakharov System

In this section, we present the second-order energy-preserving schemes for the Zakharov system (1.1). We start with the finite-dimensional Hamiltonian formulation of (1.1).

In order to process the numerical discretization, we introduce a uniform grid $(x_j, t_n) \in \mathbb{R} \times \mathbb{R}$ with mesh-length $\Delta x = h$ in the x-direction and mesh-length $\Delta t = \tau$ in the t-direction. The spatial interval $[x_L, x_R]$ is divided into N equal subintervals with grid spacing $h = (x_R - x_L)/N$. We denote by u_j^n, v_j^n, ψ_j^n, f_j^n the approximation to $u(x,t)$, $v(x,t)$, $\psi(x,t)$ $f(x,t)$ at (x_j, t_n), respectively. Define the finite difference operators

$$\delta_x^+ z_j^n = \frac{z_{j+1}^n - z_j^n}{h}, \quad \delta_x^- z_j^n = \frac{z_j^n - z_{j-1}^n}{h}, \quad \delta_x^2 z_j^n = \frac{z_{j+1}^n - 2z_j^n + z_{j-1}}{h^2},$$
$$\delta_t^+ z_j^n = \frac{z_j^{n+1} - z_j^n}{\tau}, \quad \delta_t^- z_j^n = \frac{z_j^n - z_j^{n-1}}{\tau}.$$

We discretized the second-order spatial derivatives ∂_{xx} in (1.5) by using the second-order central difference approximation and get the semi-discrete system

$$\begin{aligned}
\frac{d}{dt}u_j &= -\delta_x^2 v_j^n - 2v_j\psi_j, \\
\frac{d}{dt}v_j &= \delta_x^2 u_j^n + 2u_j\psi_j, \\
\frac{d}{dt}\psi_j &= \delta_x^2 f_j^n, \\
\frac{d}{dt}f_j &= \psi_j - (u_j^2 + v_j^2).
\end{aligned} \tag{2.1}$$

The system (2.1) can be written as a finite-dimensional Hamiltonian system

$$\frac{d}{dt}Z_j = J\nabla H(z_j),$$

with $Z_j = (u_j, v_j, \psi_j, f_j)^T$ and

$$J = \begin{bmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{bmatrix}.$$

Here, I is the $N \times N$ identity matrix

$$H(Z) = \sum_{j=1}^{N} \frac{1}{2}[(\delta_x u_j)^2 + (\delta_x v_j)^2 + (\delta_x f_j)^2 + \psi^2 - 2\psi_j(u_j^2 + v_j^2).$$

The second-order AVF method for (2.1) is written as

$$\begin{aligned}
\delta_t^+ u_j^n &= -\delta_x^2 v_j^{n+1/2} - \frac{2}{3}\left[v_j^{n+1}\psi_j^{n+1} + \frac{1}{2}v_j^{n+1}\psi_j^n + \frac{1}{2}v_j^n\psi^{n+1} + v_j^n\psi_j^n\right], \\
\delta_t^+ v_j^n &= \delta_x^2 u_j^{n+1/2} - \frac{2}{3}\left[u_j^{n+1}\psi_j^{n+1} + \frac{1}{2}u_j^{n+1}\psi_j^n + \frac{1}{2}u_j^n\psi_j^{n+1} + u_j^n\psi_j^n\right], \\
\delta_t^+ \psi_j^n &= \delta_x^2 f_j^{n+1/2}, \\
\delta_t^+ f_j^n &= \psi_j^{n+1/2} - \frac{1}{3}\left[(u_j^{n+1})^2 + u_j^{n+1}u_j^n + (u_j^n)^2\right] - \frac{1}{3}\left[(v_j^{n+1})^2 + v_j^{n+1}v_j^n + (v_j^n)^2\right].
\end{aligned} \tag{2.2}$$

It is clear that the AVF method (2.2) is fully implicit and requires an iterative solver at each time step. In this work, we used Newton's method. Because a system of nonlinear equations is solved at each time step, the AVF method (2.2) has a very high cost per time step.

The PAVF method for (2.1) is

$$\begin{aligned}
\delta_t^+ u_j^n &= -\delta_x^2 v_j^{n+1/2} - (v_j^{n+1} + v_j^n)\psi_j^n, \\
\delta_t^+ v_j^n &= \delta_x^2 u_j^{n+1/2} + (u_j^{n+1} + u_j^n)\psi_j^n, \\
\delta_t^+ \psi_j^n &= \delta_x^2 f_j^{n+1/2}, \\
\delta_t^+ f_j^n &= \psi_j^{n+1/2} - [(u_j^{n+1})^2 + (v_j^{n+1})^2].
\end{aligned} \tag{2.3}$$

It can be seen that the PAVF scheme (2.3) is linearly implicit and does not require any iterative method such as Newton's method. The values u^{n+1} and v^{n+1} can be obtained from the first two equations of (2.3) explicitly and substituted into the last two equations. Thus, the values ψ^{n+1} and f^{n+1} can be solved explicitly from the last two equations of (2.3). Because of explicit structure, unlike the AVF method (2.2), the PAVF method (2.3) is extremely easy to implement and has a very low cost per time step.

Theorem 2.1. *The PAVF (2.3) for the Zakharov equation conserves the mass (1.6) in the sense that*

$$M(u_j^n, v_j^n) = M(u_j^0, v_j^0), \quad n = 1, 2, ..., M, \ j = 1, 2, \cdots, N+1.$$

Proof. Multiplying the first two equations in (2.3) by $(u_j^{n+1}+u_j^n)^T$ and $(v_j^{n+1}+v_j^n)^T$, respectively, we get

$$\frac{1}{\tau}(u_j^{n+1}+u_j^n)^T(u_j^{n+1}-u_j^n)=(u_j^{n+1}+u_j^n)^T\left(-\delta_x^2 v_j^{n+1/2}-(v_j^{n+1}+v_j^n)\psi_j^n\right), \tag{2.4}$$

and

$$\frac{1}{\tau}(v_j^{n+1}+v_j^n)^T(v_j^{n+1}-v_j^n)=(v_j^{n+1}+v_j^n)^T\left(\delta_x^2 u_j^{n+1/2}+(u_j^{n+1}+u_j^n)\psi_j^n\right). \tag{2.5}$$

Summing (2.4) and (2.5), we obtain the mass conservation

$$\sum_{j=1}^{N+1}\left[\left(u_j^{n+1}\right)^2+\left(v_j^{n+1}\right)^2\right]=\sum_{j=1}^{N+1}\left[\left(u_j^{n}\right)^2+\left(v_j^{n}\right)^2\right].$$

This completes the proof. □

The adjoint PAVF method for (2.1) is

$$\begin{aligned}
\delta_t^+ u^n &= -\delta_x^2 v^{n+1/2}-(v^{n+1}+v^n)\psi^{n+1},\\
\delta_t^+ v^n &= \delta_x^2 u^{n+1/2}+(u^{n+1}+u^n)\psi^{n+1},\\
\delta_t^+ \psi^n &= \delta_x^2 f^{n+1/2},\\
\delta_t^+ f^n &= \psi^{n+1/2}-[(u^n)^2+(v^n)^2].
\end{aligned} \tag{2.6}$$

We notice that the adjoint method (2.6) is also explicit. The values ψ^{n+1} and f^{n+1} can be solved from the last two equations of (2.6), and then substituted into the first two equations. Thus, u^{n+1} and v^{n+1} can be solved explicitly, too.

Using the PAVF method (2.3) with its adjoint method (2.6), we can write the second-order energy- preserving PAVF-C method

$$\begin{aligned}
\frac{2}{\tau}(u^*-u^n) &= -\frac{1}{2}(v^*+v^n)-(v^*+v^n)\psi^n,\\
\frac{2}{\tau}(v^*-v^n) &= \frac{1}{2}\delta_x^2(u^*+u^n)+(u^*+u^n)\psi^n,\\
\frac{2}{\tau}(\psi^*-\psi^n) &= \frac{1}{2}\delta_x^2(f^*+f^n),\\
\frac{2}{\tau}(f^*-f^n) &= \frac{1}{2}(\psi^*+\psi^n)-[(u^*)^2+(v^*)^2],\\
\frac{2}{\tau}(u^{n+1}-u^*) &= -\frac{1}{2}\delta_x^2(v^{n+1}+v^*)-(v^{n+1}+v^*)\psi^{n+1},\\
\frac{2}{\tau}(v^{n+1}-v^*) &= \frac{1}{2}\delta_x^2(u^{n+1}+u^*)+(u^{n+1}+u^*)\psi^{n+1},\\
\frac{2}{\tau}(\psi^{n+1}-\psi^*) &= \frac{1}{2}\delta_x^2(f^{n+1}+f^*),\\
\frac{2}{\tau}(f^{n+1}-f^*) &= \frac{1}{2}(\psi^{n+1}+\psi^*)-[(u^*)^2+(v^*)^2],
\end{aligned} \tag{2.7}$$

for the for Zakharov system (2.1). It can be seen that the method (2.7) is also explicit since it is the composition of two explicit schemes.

Finally, the second-order energy-preserving PAVF-P method can be written as

$$\begin{aligned}
\delta_t^+ u^n &= -\delta_x^2(v^{n+1/2}-\frac{1}{2}(v^{n+1}+v^n)(\psi^n+\psi^{n+1}),\\
\delta_t^+ v^n &= \delta_x^2(u^{n+1/2}+\frac{1}{2}(u^{n+1}+u^n)(\psi^n+\psi^{n+1}),\\
\delta_t^+ \psi^n &= \delta_x^2(f^{n+1/2},\\
\delta_t^+ f^n &= \frac{1}{2}(\psi^{n+1}+\psi^n)-\frac{1}{2}[(u^{n+1})^2+(v^{n+1})^2+(u^n)^2+(v^n)^2].
\end{aligned} \tag{2.8}$$

It is clear that, like the AVF method (2.2), the PAVF-P method (2.8) is fully implicit.

TABLE 1. Temporal convergence rates of the energy preserving methods at $T = 1$ with $\Delta x = 0.01$

	τ	L_∞	order	L_2	order
AVF	1/4	$4.966e-3$	....	$2.407e-2$	....
	1/8	$1.276e-3$	1.95	$6.144e-3$	1.97
	1/10	$8.203e-4$	1.98	$3.932e-3$	2.00
	1/12	$5.697e-4$	2.00	$2.730e-3$	2.00
PAVF	1/4	$8.514e-3$	....	$5.637e-2$	....
	1/8	$4.594e-3$	0.89	$2.780e-2$	1.02
	1/10	$3.699e-3$	0.97	$2.219e-2$	1.01
	1/12	$3.088e-3$	0.99	$1.849e-2$	1.00
PAVF-C	1/4	$4.710e-4$	....	$3.444e-3$	....
	1/8	$1.132e-4$	2.08	$8.239e-5$	2.09
	1/10	$4.312e-5$	2.10	$3.138e-5$	2.10
	1/12	$1.703e-5$	2.11	$1.233e-5$	2.12
PAVF-P	1/4	$4.690e-4$	....	$3.324e-3$	....
	1/8	$1.205e-4$	0.95	$8.195e-4$	2.02
	1/10	$7.729e-5$	1.99	$5.233e-4$	2.01
	1/12	$5.367e-5$	2.00	$3.624e-4$	2.00

3. NUMERICAL RESULTS

In order to show the temporal accuracy, we use the errors [16]

$$\begin{aligned} L_\infty(\tau) &= \max_{1\le n\le M}\left\{|\phi(x_j,t_n)-\phi_j^n|+|\psi(x_j,t_n)-\psi_j^n|\right\}, \\ L_2(\tau) &= \left(\tau\sum_{j=1}^{N}|\phi(x_j,t_n)-\phi_j^n|^2\right)^{1/2}+\left(\tau\sum_{j=1}^{N}|\psi(x_j,t_n)-\psi_j^n|^2\right)^{1/2}, \end{aligned}$$

where $\phi(x_j,t_n)$, $\psi(x_j,t_n)$ are exact solutions and ϕ_j^n, ψ_j^n are approximate solution at the point (x_j,t_n). L_∞ and L_2 errors for spatial accuracy can be defined analogously. The rate of convergence in time discretization is obtained by using

$$\text{order} \approx \ln\left(\frac{L(\tau_1)}{L(\tau_2)}\right)/\ln\left(\frac{\tau_1}{\tau_2}\right),$$

where L is the L_∞ error or L_2 error. The rate of convergence in space discretization can be defined analogously. The relative discrete Hamiltonian energy error and the discrete mass error of the Zakharov equation is defined as

$$R_H(t)=\left|\frac{H(Z^n)-H(Z^0)}{H(Z^0)}\right|, \qquad R_M(t)=\left|\frac{M(Z^n)-M(Z^0)}{M(Z^0)}\right|,$$

where $H(Z^n)$ and $M(Z^n)$ are the discrete Hamiltonian and mass at time $t_n = n\Delta t$ and $H(Z^0)$ and $M(Z^0)$ are the initial discrete Hamiltonian and the mass, respectively. The initial conditions are taken from the Langmuir soliton solution (1.4). To demonstrate the long time behavior of the energy-preserving schemes, we take periodic boundary conditions

$$\phi(x_L,t)=\phi(x_R,t), \qquad \psi(x_L,t)=\psi(x_L,t),$$

in all experiments. For simplicity, we present the wave profiles of the PAVF (2.3) scheme in all computations, since the other schemes AVF (2.2), PAVF-C (2.7) and PAVF-P (2.8) produce the same profile.

3.1. **Travelling of Single Solitary Wave.** First, we present some numerical results to illustrate the accuracy and correctness of the proposed methods. We solve the Zakharov system (1.1) with the initial conditions

$$\begin{aligned} \phi(x,0) &= Bi\sqrt{1-c^2}\,sech(2B(x-x_0))\times exp\left(i\left(\frac{c(x-x_0)}{2}\right)\right), \\ \psi(x,0) &= 4B^2 sech^2(2B(x-x_0)), \end{aligned}$$

on the region $-10 \le x \le 10$ by using the set of parameters $c = 0.1$, $x_0 = 0$ and $B = 0.5$. Table 1 and 2 represent the errors. We can see that the methods provide a good numerical solution. The errors were reduced by reducing the

TABLE 2. Spatial convergence rates of the energy preserving methods at $T = 1$ with $\Delta t = 0.002$

	τ	L_∞	order	L_2	order
AVF	0.5	$8.488e-2$	....	$8.189e-2$	....
	0.25	$1.839e-2$	2.20	$1.913e-2$	2.10
	0.125	$4.465e-3$	2.04	$4.722e-3$	2.02
	0.0625	$1.109e-3$	2.01	$1.171e-3$	2.01
PAVF	0.5	$8.496e-2$	....	$8.197e-2$	....
	0.25	$1.841e-2$	2.21	$1.914e-2$	2.10
	0.125	$4.464e-3$	2.04	$4.733e-3$	2.02
	0.0625	$1.111e-3$	2.01	$1.174e-3$	2.01
PAVF-C	0.5	$8.488e-2$	....	$8.189e-2$	....
	0.25	$1.839e-2$	2.20	$1.913e-2$	2.10
	0.125	$4.462e-3$	2.04	$4.722e-3$	2.02
	0.0625	$1.109e-3$	2.01	$1.171e-3$	2.01
PAVF-P	0.5	$8.488e-2$	....	$8.189e-2$	....
	0.25	$1.839e-2$	2.20	$1.913e-2$	2.10
	0.125	$4.462e-3$	2.04	$4.722e-3$	2.02
	0.0625	$1.109e-3$	2.01	$1.171e-3$	2.01

TABLE 3. Computational costs of one solitary wave solution by the energy preserving schemes with $\tau = 0.01$ at $T = 1$

h	AVF	PAVF	PAVF-C	PAVF-P
1/2	3.26	0.50	0.75	3.56
1/5	25.07	1.55	2.74	26.03
1/10	159.33	7.03	13.68	156.06
1/15	426.64	16.23	35.08	367.29

step sizes. We can see that all methods have second-order accuracy in time except the PAVF method which has only first-order accuracy. The accuracy test validates the correctness of our methods. From the tables, we can see that the PAVF-C scheme is the most accurate of all the other schemes.

Furthermore, the CPU times are reported in Table 3. Since the AVF method and PAVF-P methods are fully implicit, their CPU times are more than the linearly implicit methods PAVF and PAVF-C methods. Although the PAVF-C method is composition of the PAVF method and its adjoint method, it is surprising that its CPU time is much smaller than twice that of the PAVF method.

Next, we investigate the stability of the numerical solution with respect to the initial condition [30, 42] . We slightly perturbed the initial condition as

$$\phi_{noise}(x,0) = \phi(x,0)(1+\lambda\theta), \qquad \psi_{noise}(x,0) = \psi(x,0)(1+\lambda\theta), \tag{3.1}$$

where λ is the percentage of the noise and θ is the random variable generated from a uniform distribution in the interval $[-0.5, 0.5]$. Figure 1 represents the solitary wave of the Zakharov equation (1.1) with and without noise using the perturbed initial condition (3.1) with $\lambda = 0.02$. Figure 2 represents the solitary wave obtained by the PAVF scheme (2.3) using the perturbed initial condition (3.1) for $0 \le t \le 10$. From the figure, we see that the traveling of the solitary wave is not affect by perturbation. Figure 3 represents the residuals of the methods with perturbed initial conditions (3.1). We see that this small perturbation in the initial data does not have any significant effect on the preservation of the energy that is the energy is preserved exactly by the schemes. This investigates the stability of our schemes.

The aliasing instability introduces significant numerical errors over a long time. Thus, to address this issue, we utilize the long-time integration of the solitary wave and evaluate the conservation properties of the proposed methods. To this end, we choose the parameters $h = 0.1$, $\tau = 0.02$, $x_0 = -10$, $c = 0.5$ and $B = 0.5$ on the interval $-32 \le x \le 32$. Figure 4 represents the numerical solution of the PAVF scheme (2.3) for $0 \le t \le 150$. The effects of the periodic boundary conditions can be seen in the figure. We see that single wave moves in the right direction, hits the right

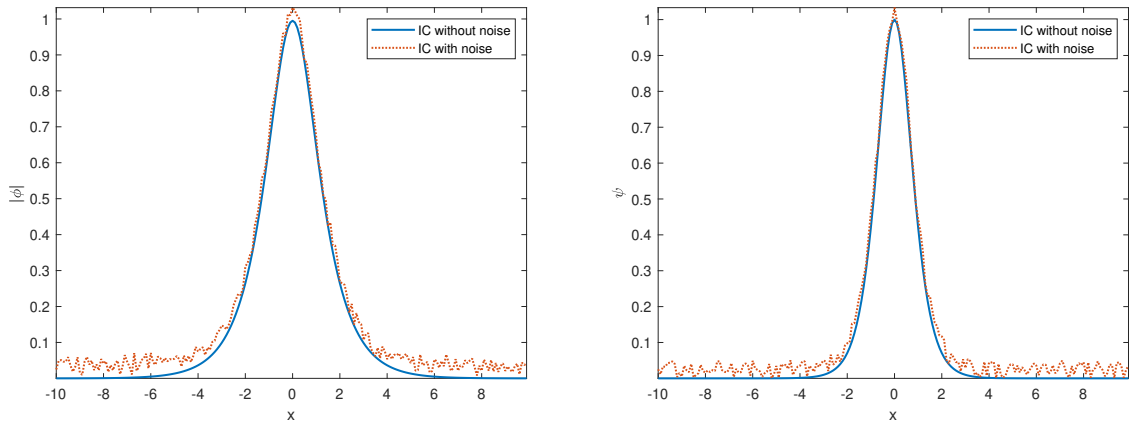


FIGURE 1. Initial condition with and without noise.

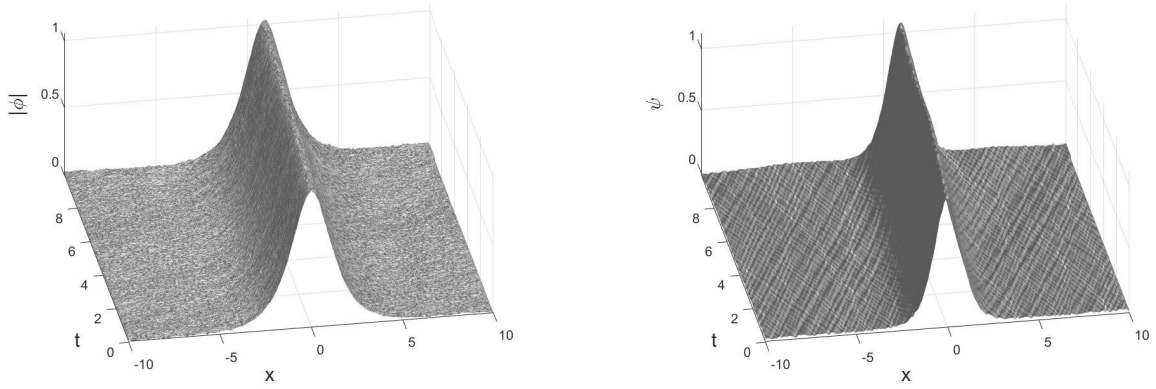


FIGURE 2. Stability test of the PAVF method.

boundary, and reappears from the left due to the periodic boundary condition. Then the wave continues to move in the right direction without any changes in its shape. This shows that the proposed schemes are reliable and well simulate the solitary wave. Figure 5 shows the corresponding relative energy errors. From the figure, we see that all methods preserve the energy exactly. Figure 6 represents corresponding relative mass errors. From the figure we see that all new energy-preserving schemes preserve the mass exactly up to the machine round-off error except the AVF method which is not mass conserving. We see that the relative mass error of the AVF scheme is bounded during the time evolution.

3.2. **Collision of Two Solitary Waves.** In this section, we will study the collision of two solitary waves. The initial conditions are taken as

$$\begin{aligned}\phi(x,0) &= i\sum_{j=1}^{2} B_j\sqrt{1-c_j^2}\,sech(2B_j(x-x_j))\times exp\left(i\left(\frac{c_j(x-x_j)}{2}\right)\right),\\ \psi(x,0) &= \sum_{j=1}^{2} 4B_j^2\,sech^2(2B_j(x-x_j)).\end{aligned}$$

These equations represent two solitary waves, one initially located to the position x_1 and moving with the velocity c_1, and the other initially located to the position x_2 and moving with the velocity c_2. We solve the problem using the

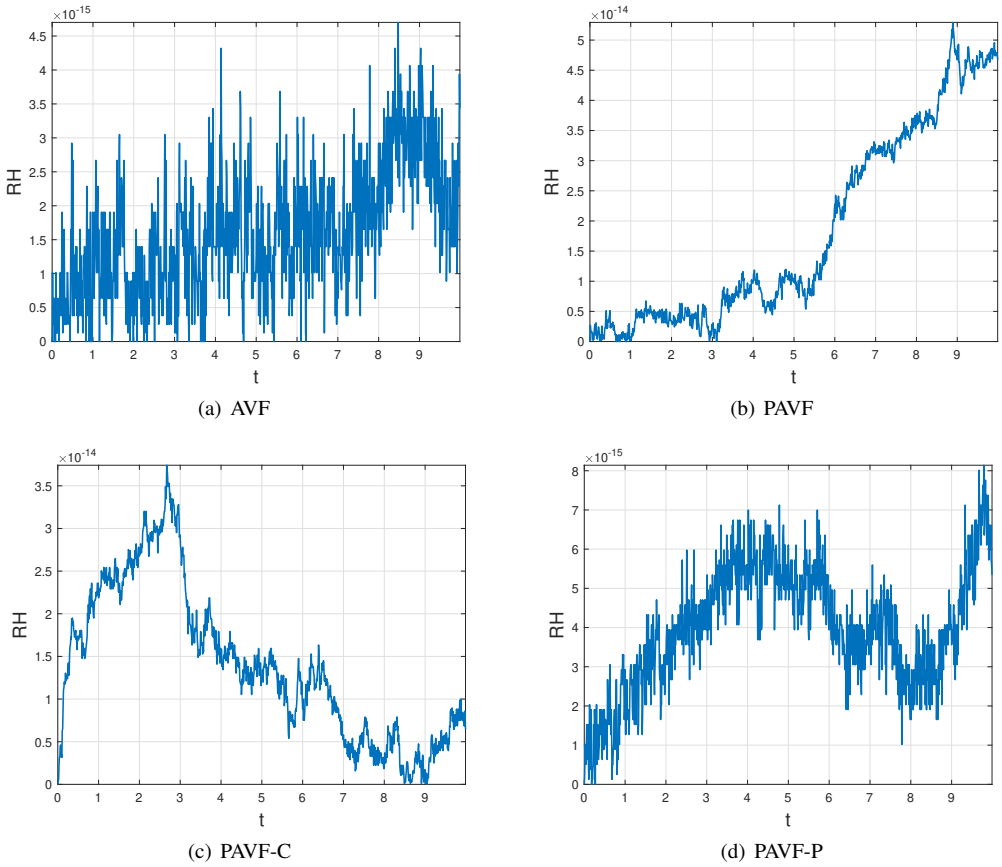


FIGURE 3. Stability test of the Hamiltonian error for the propagation of one soliton.

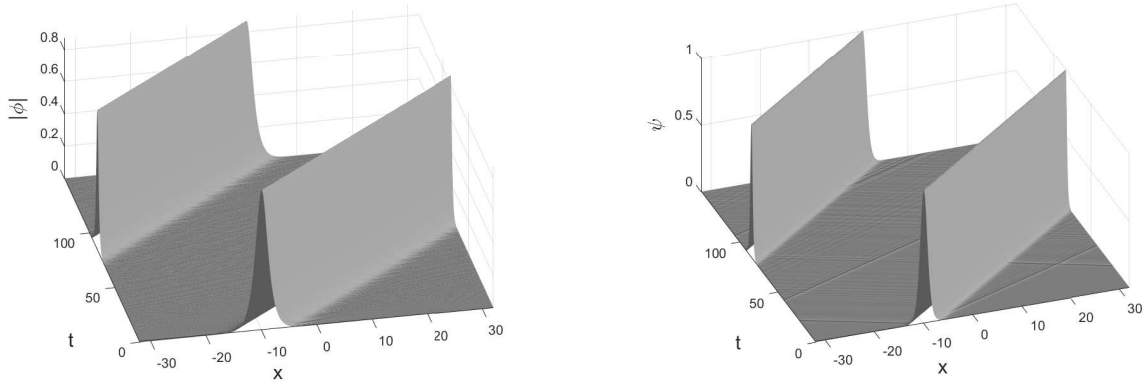
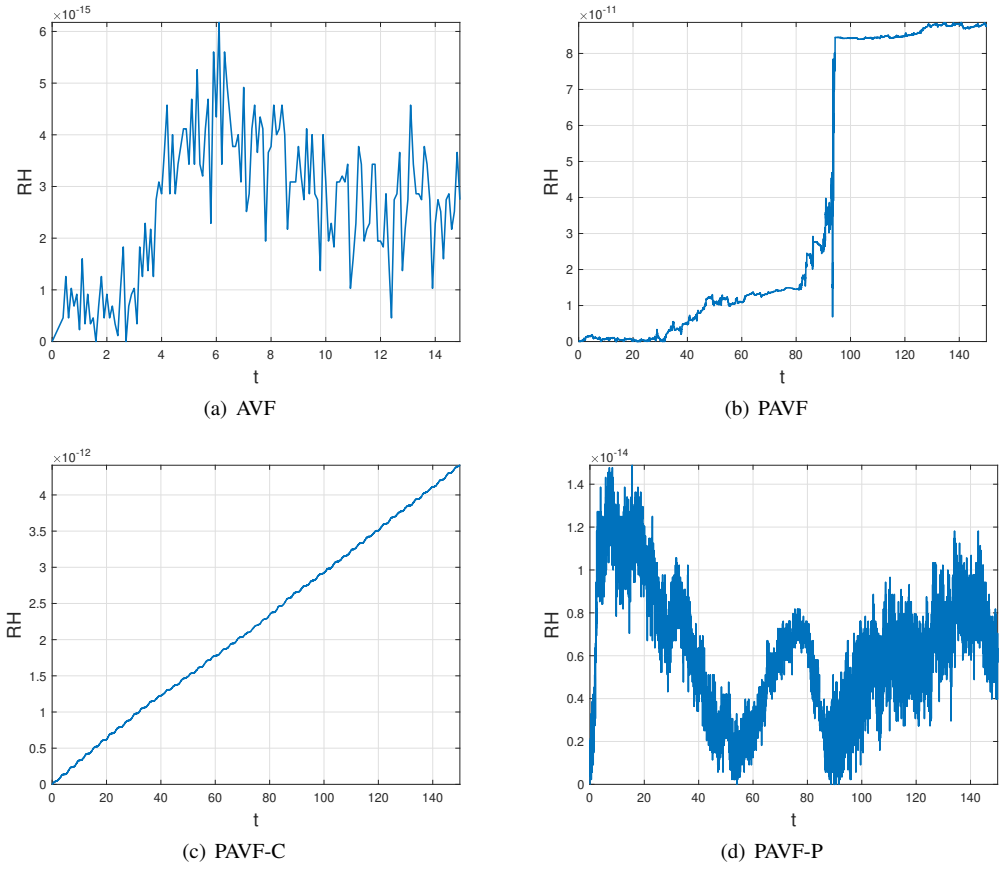


FIGURE 4. Propagation of one soliton by PAVF method.

proposed schemes on the region $-32 \leq x \leq 32$ and $0 \leq t \leq 80$. All computations are performed with the step sizes $h = 0.1$, and $\tau = 0.02$. Two special cases will be considered:



(a) AVF (b) PAVF

(c) PAVF-C (d) PAVF-P

FIGURE 5. Relative Hamiltonian error for the propagation of one soliton.

I. Collision of two solitary waves traveling in opposite directions with equal amplitudes and same velocities. We choose

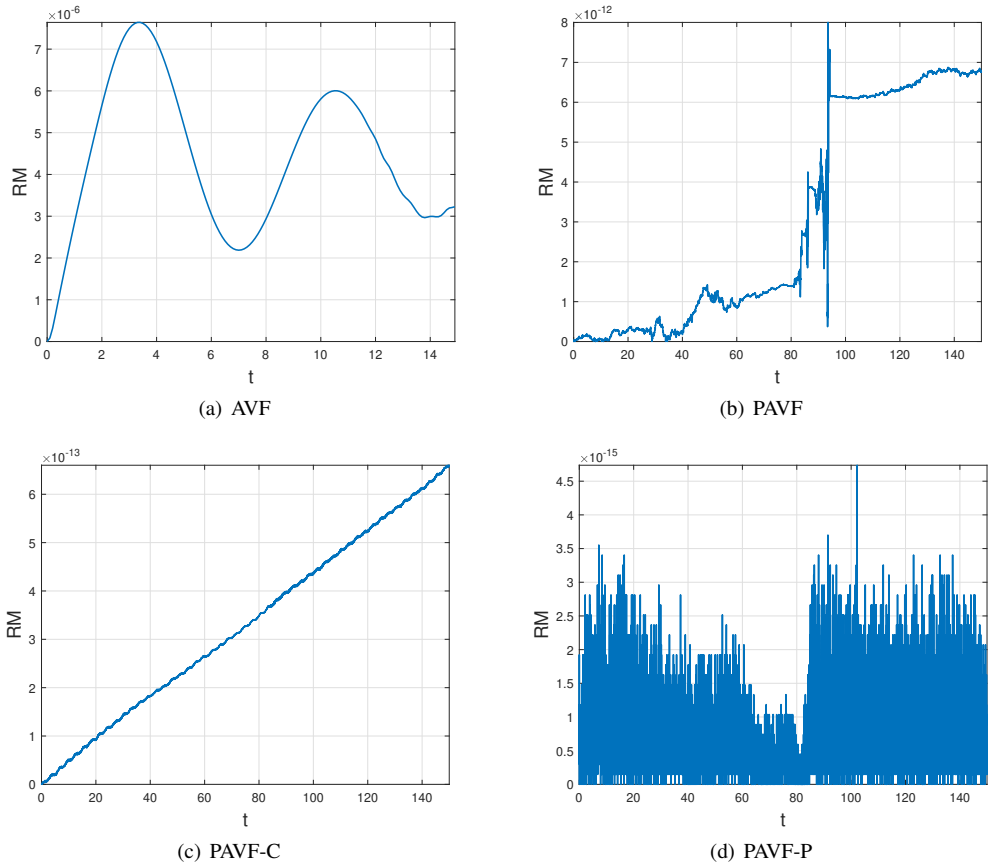
$$x_1 = -x_2 = 20, c_1 = -c_2 = 0.5, B_1 = B_2 = 0.5.$$

II. Collision of two solitary waves traveling in the same directions with different amplitudes and different velocities. We choose

$$x_1 = -x_2 = 20, c_1 = 0.5, c_2 = -0.3, B_1 = 0.5, B_2 = 0.3.$$

Figure 7 represents the collision scenario for case I. From the figure, we see that the solitons move to each other without changing their shapes and then collision takes place at the time around $t \approx 36$. We see that after collision two waves merge and move together by leaving some oscillations behind them. This is a head-on collision or symmetric collision. The collision results in the fusion of two waves. After the collision, we also see that the amplitudes of the waves increase. Figure 8 and 9 display the corresponding errors of energy and mass, respectively. It can be observed from the figures that the relative errors in energy are all on the scale of between 10^{-12} and 10^{-14}. These numerical results verify that the present four schemes are energy-preserving. The errors in mass for PAVF, PAVF-C, and PAVF-P methods all oscillate near zeros in the scale of at most 10^{-13}. This figure verifies that contrary to the AVF method, the methods PAVF, PAVF-C, and PAVF-P are mass-conserving methods. Obviously, the three methods PAVF, PAVF, and PAVF-P all preserve the energy and mass conservation law exactly.

Figure 10 represents the collision scenario for case II. From the figure, we can see that the solitary wave with a larger amplitude absorbs the part of the wave with a smaller amplitude after the collision. Figure 11 and 12 display the corresponding errors of energy and mass, respectively.



(a) AVF (b) PAVF

(c) PAVF-C (d) PAVF-P

FIGURE 6. Relative mass error for the propagation of one soliton.

4. CONCLUSION

In this study, the Average Vector Field (AVF) method and Partitioned Average Vector Field (PAVF) method are applied to the Zakharov system. The classic second-order AVF method is fully implicit and hence leads to a nonlinear system of algebraic equation. Using the adjoint method of the PAVF method, PAVF composition (PAVF-C) and PAVF plus (PAVF-P) methods are presented. Some numerical results are presented for the single and colliding solitary wave solution of the Zakharov system. Numerical results show that four schemes are energy preserving. In addition, we further show that the PAVF method preserves the mass of the Zakharov system while the classical second-order AVF method cannot. Errors of the methods are compared. We find that errors of the PAVF-C scheme are less than the that of PAVF scheme. The computational costs of the methods are also compared. We find that CPU times of the fully implicit AVF and PAVF-P methods are much larger than linearly implicit methods PAVF and PAVF-C. Therefore, we have concluded that the linearly implicit PAVF and PAVF-C methods are convenient energy-preserving and mass-preserving methods for the numerical solution of the Zakharov system.

AUTHORS CONTRIBUTION STATEMENT

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

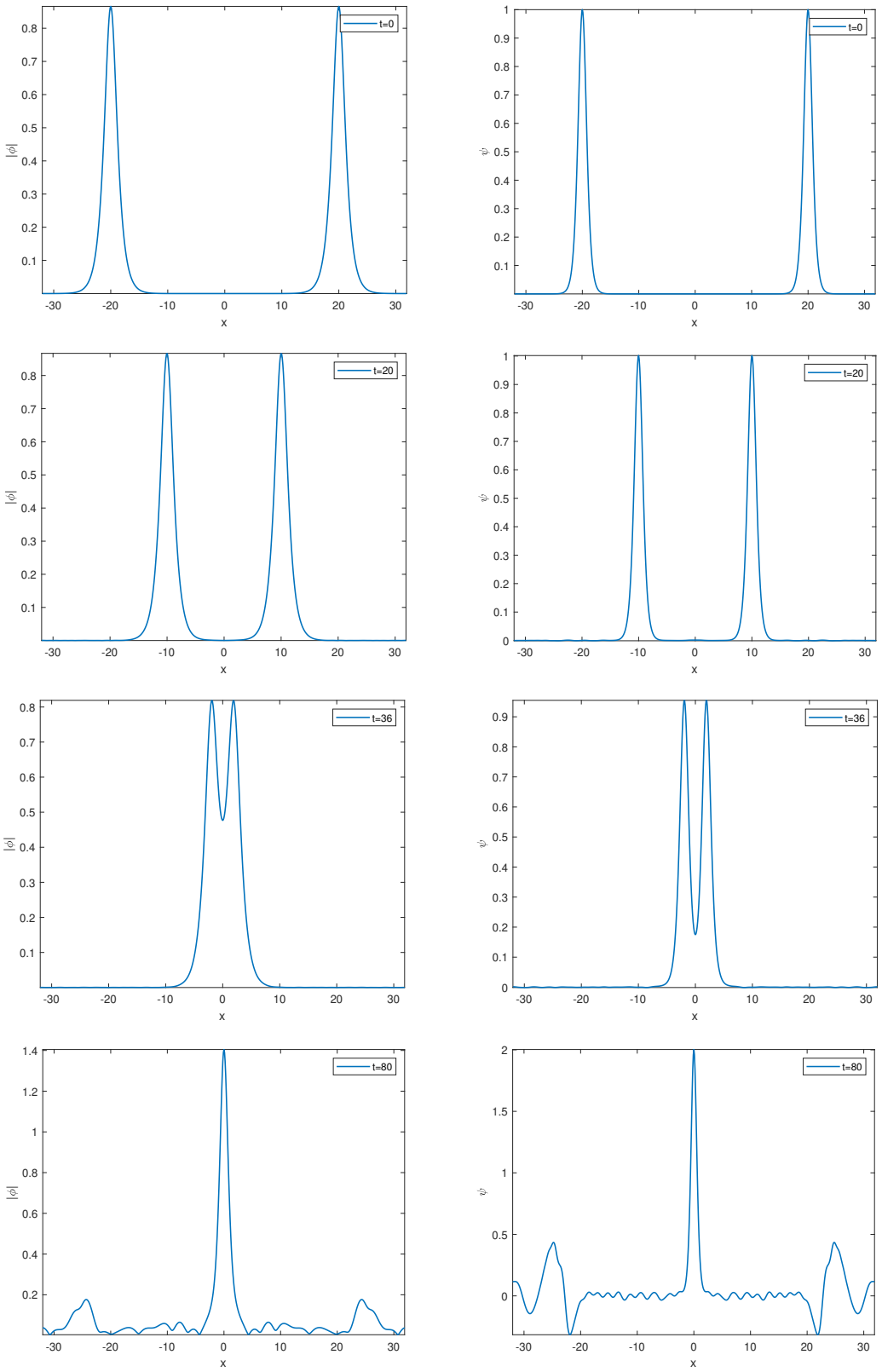


FIGURE 7. Propagation of two solitons by PAVF method: Left $|\phi(t,x)|$, Right $\psi(x,t)$.

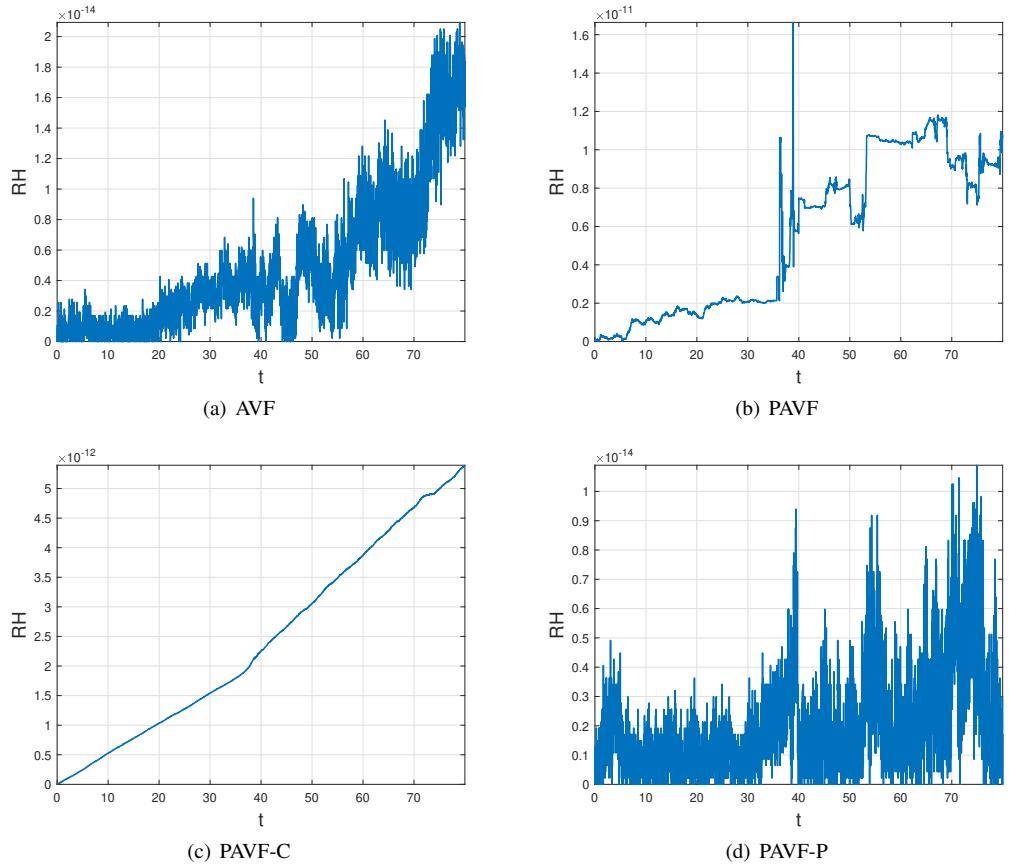


FIGURE 8. Relative Hamiltonian error for the propagation of two solitons.

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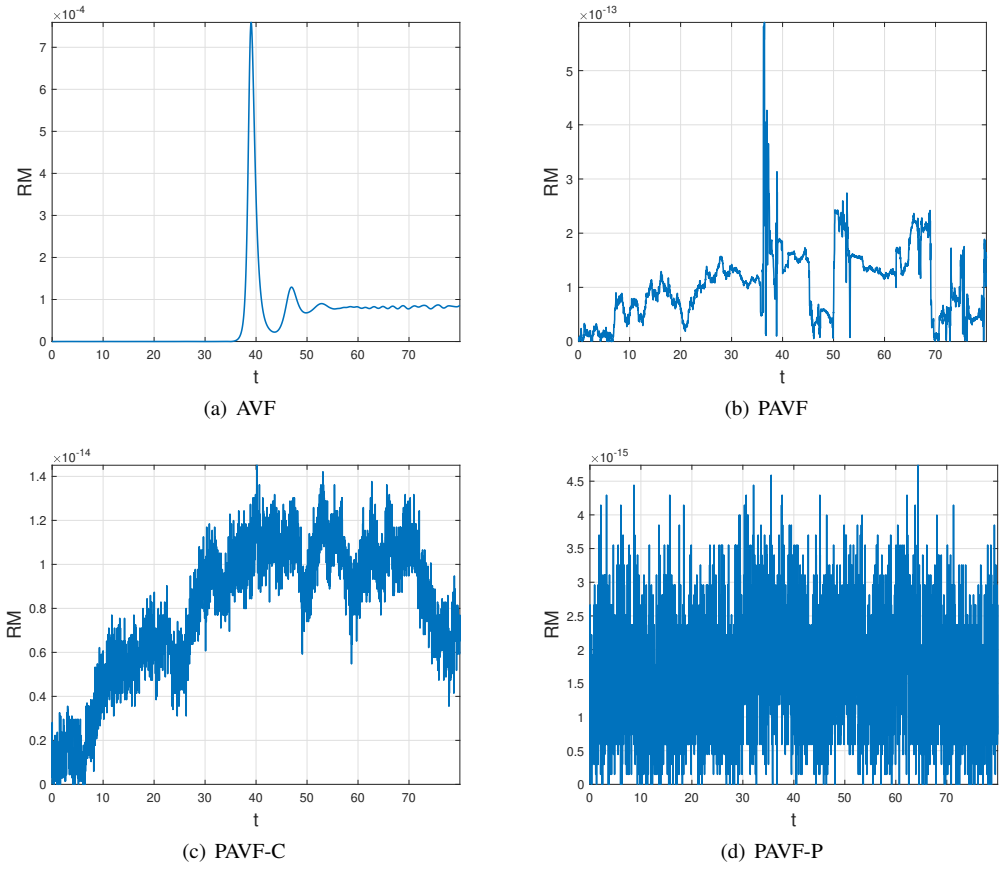

FIGURE 9. Relative mass error for the propagation of two solitons.

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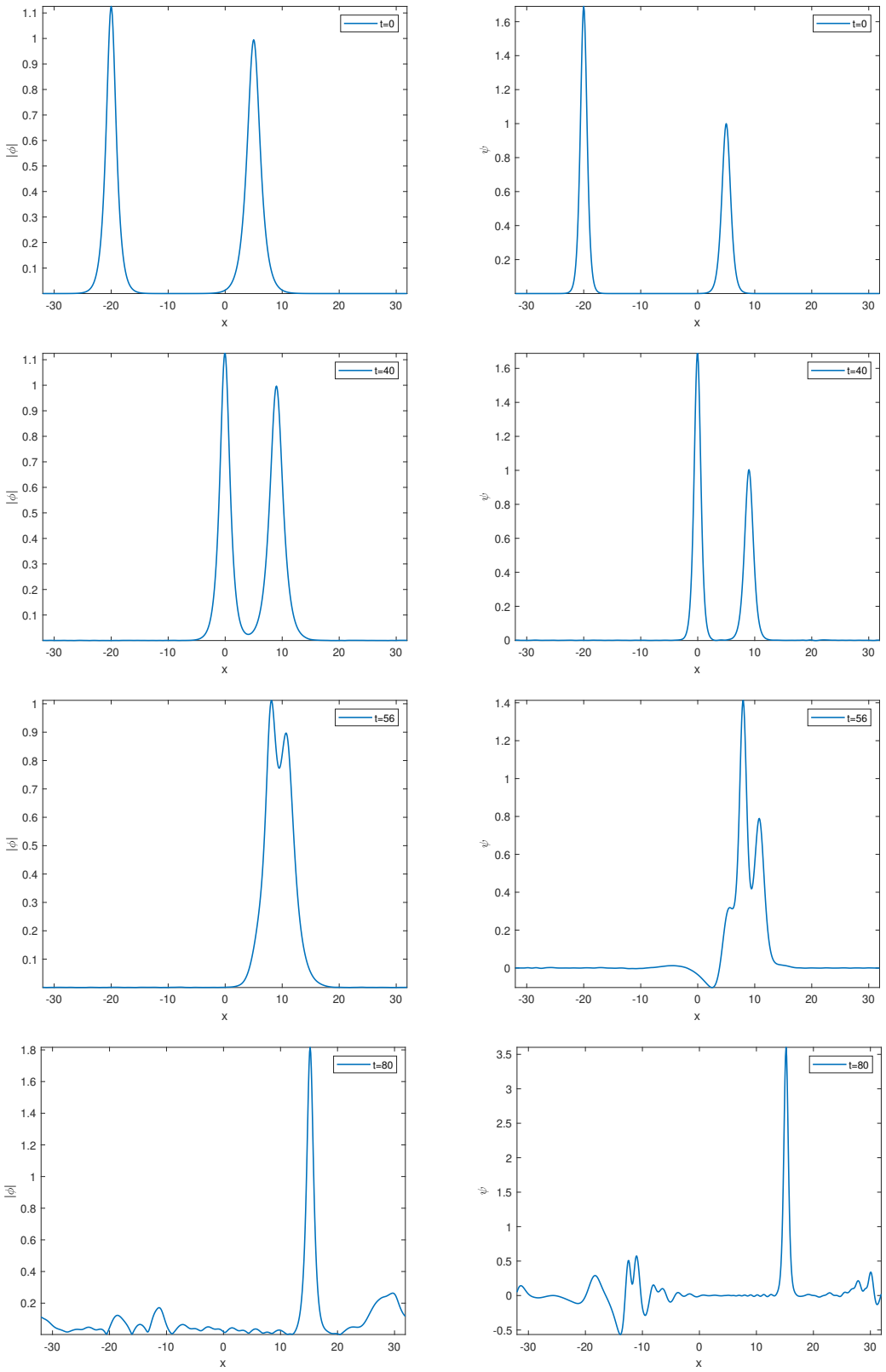

FIGURE 10. Propagation of two solitons by PAVF method: Left $|\phi(t,x)|$, Right $\psi(x,t)$.

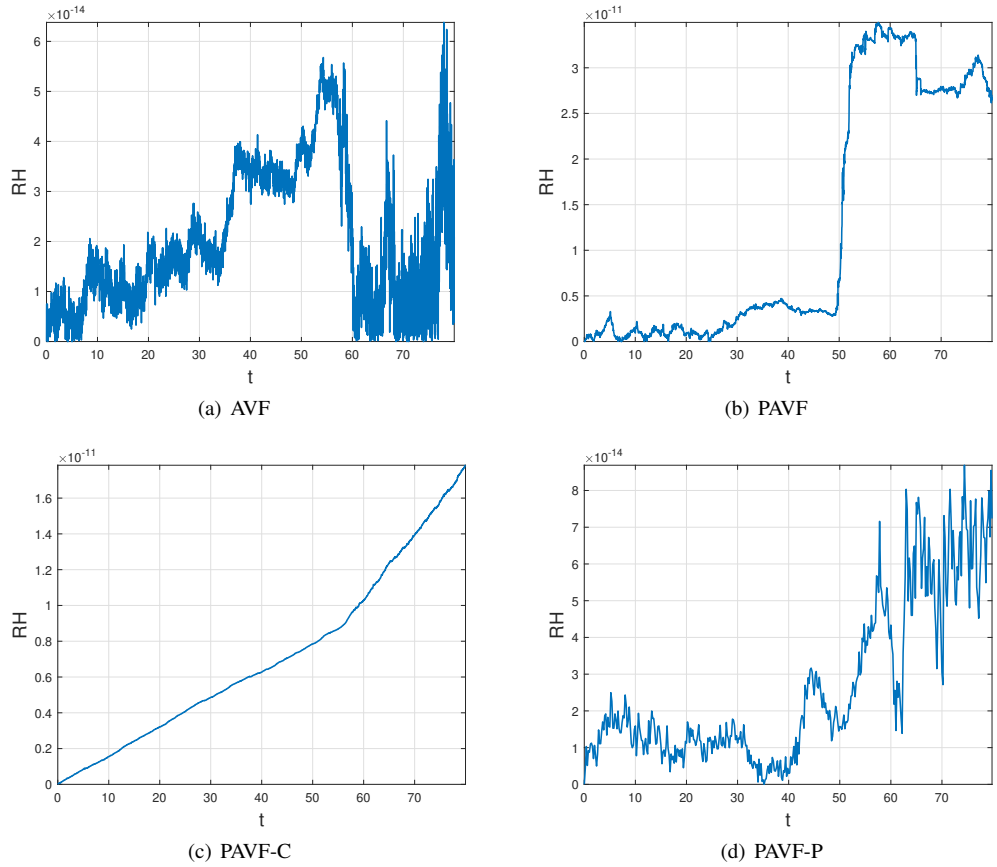

FIGURE 11. Relative Hamiltonian error for the propagation of two solitons.

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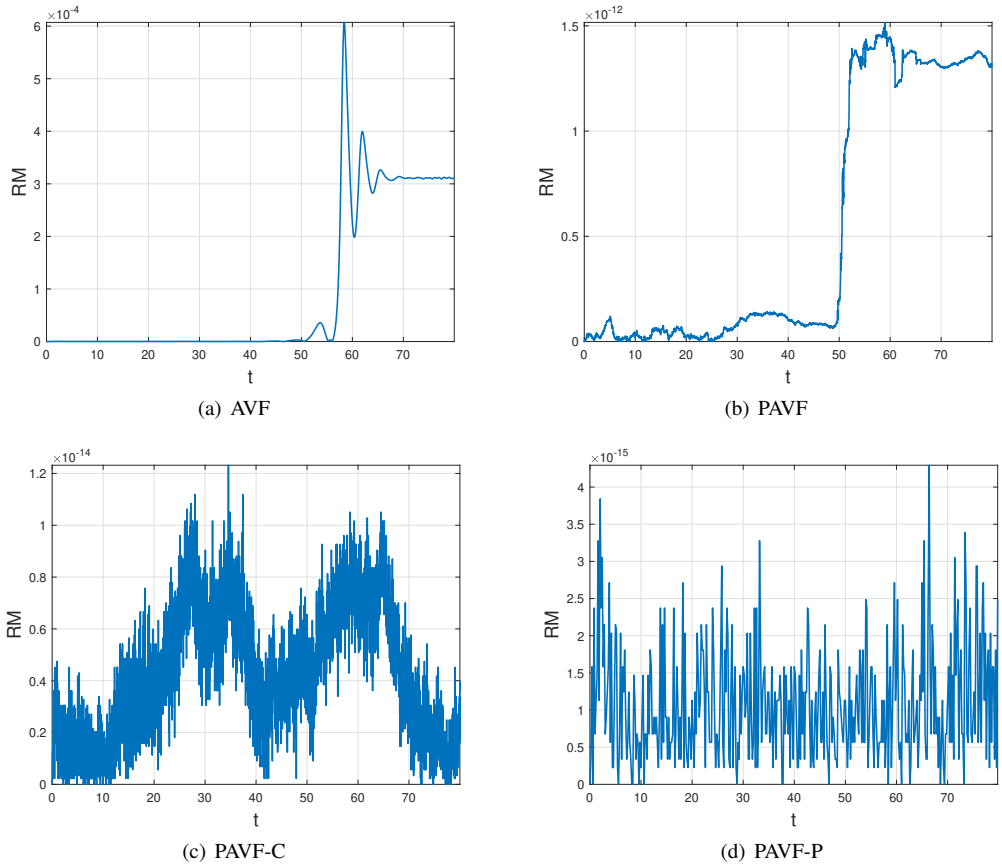

(a) AVF (b) PAVF

(c) PAVF-C (d) PAVF-P

FIGURE 12. Relative mass error for the propagation of two solitons.

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