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A LITTLE MISTAKE IN A PAPER BY BOB GILMER ON RNGS

Alberto Facchini and Jennifer Parolin

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Abstract. We show that there is a little mistake in an implication in a paper of Bob Gilmer on rngs.

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By *rng* we mean a commutative ring not-necessarily with an identity. In [2], Bob Gilmer studied eleven conditions on commutative rngs. Two of these conditions are:

C : If A is a nonzero ideal of rng R such that $\sqrt{A} \neq R$, then R/A has an identity.

J : An ideal A of R such that $\sqrt{A}$ is maximal is primary.

Gilmer claims he proves that C $\Rightarrow$ J. This implication is also reported in D. D. Anderson's survey [1]. In this short note, we show that the implication C $\Rightarrow$ J does not hold, with the following counter-example.

Let $\mathbb{F}_2$ be the field of order 2, and let $\mathbb{Z}_2$ be the additive group of order 2 in which the product of any two elements is always zero (i.e., $\mathbb{Z}_2$ is the rng with two elements and trivial multiplication). Consider their direct product $R := \mathbb{F}_2 \times \mathbb{Z}_2$. Then R is a commutative ring without identity. The additive group of R is the Klein four-group, and has five subgroups: 0, R and the three cyclic subgroups M, P and B of order two generated by $(1,0)$, $(0,1)$ and $(1,1)$, respectively. The subgroup B is not an ideal, because $(1,1)(1,1) = (1,0) \notin B$. The ideal M is maximal but not prime

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because $R/M \cong \mathbb{Z}_2$. The ideal P is maximal and is the unique prime ideal of R (because $R/P \cong \mathbb{F}_2$).

The rng R satisfies Condition C, because the nonzero ideals of R are R, M and P, and their radicals are R, R and P, respectively. Thus R satisfies Condition C because $R/P \cong \mathbb{F}_2$ is a field with identity. Notice that all (nonzero) elements of R are zero-divisors.

Let us prove that R does not satisfy Condition J. Let A be the zero ideal. Its radical is P, which is maximal. But $A = 0$ is not a primary ideal, because: (1) $(0,1)(1,1) = (0,0) \in A$; (2) the element $(0,1)$ does not belong to the zero ideal A; and (3) all the elements $(1,1)^n = (1,0)$ also do not belong to the zero ideal A, for any positive integer n. This shows that implication C $\Rightarrow$ J does not hold.

The mistake in Gilmer's proof lies in the fact that in Condition C only *nonzero* ideals A are considered, and, in our example, J fails for the ideal $A = 0$.

We conclude with four remarks, related to our example above, about prime ideals, maximal ideals and multiplicatively closed subsets in arbitrary rngs. Recall that a subset S of a rng R is *multiplicatively closed* if $s, s' \in S$ implies $ss' \in S$. Clearly:

(1) *An ideal I of a rng R is prime if and only if its complement $R \setminus I$ is a non-empty multiplicatively closed subset.*

(2) *Let S be a non-empty multiplicatively closed subset of a rng R. Then every ideal I of R maximal with respect to the property $I \cap S = \emptyset$ is prime.* The proof is the same as for rings: If I is maximal with respect to the property $I \cap S = \emptyset$ and $x, x' \in R \setminus I$, then $(I + Rx + \mathbb{Z}x) \cap S \neq \emptyset$, so that there exist $i \in I$, $r \in R$, $z \in \mathbb{Z}$ and $s \in S$ such that $i + rx + zx = s$. Similarly, there exist $i' \in I$, $r' \in R$, $z' \in \mathbb{Z}$ and $s' \in S$ such that $i' + r'x' + z'x' = s'$. Then the product ss' is in $I + Rxx' + \mathbb{Z}xx'$, so that $xx' \notin I$.

(3) *In a rng R it is possible that there exist ideals that are maximal but not prime.* This is the case of the maximal ideal M in the example of rng $R = \mathbb{F}_2 \times \mathbb{Z}_2$ given above. In an arbitrary rng R, a maximal ideal M is prime if and only if $R^2 \nsubseteq M$.

(4) A subset S of a rng R is a *saturated multiplicatively closed subset of R* if, for every $r, r' \in R$, $rr' \in S \Leftrightarrow r \in S$ and $r' \in S$. We now prove that *a non-empty subset S of a rng R is a saturated multiplicatively closed subset of R if and only if its complement $R \setminus S$ is the union of a set of prime ideals of R.* Clearly, by (1), if $R \setminus S$ is the union of a set of prime ideals of R, then S is a saturated

multiplicatively closed subset. Conversely, assume $S \subseteq R$ non-empty, saturated and multiplicatively closed. Let us prove that $R \setminus S$ is the union of all prime ideals P of R such that $P \cap S = \emptyset$. One inclusion is trivial. Conversely, let x be an element of $R \setminus S$. We must prove that x belongs to a prime ideal of R disjoint from S. The ideal of R generated by x is $Rx + \mathbb{Z}x$. Let us prove that $(Rx+\mathbb{Z}x) \cap S = \emptyset$. If $rx + zx \in S$, then $(rx + zx)^2 \in S$ because S is multiplicatively closed. But $(rx + zx)^2 = (r^2x + 2zrx + z^2x)x$, and in this last product, both factors are in R. Since S is saturated in R, it follows that $x \in S$, which is a contradiction. This proves that $(Rx + \mathbb{Z}x) \cap S = \emptyset$. Applying Zorn's Lemma to the set of all the ideals of R that contain $Rx + \mathbb{Z}x$ and are disjoint from S (this set is non-empty, because it contains $Rx + \mathbb{Z}x$), we get from (2) that any maximal element P of this set is a prime ideal of R, which contains x and is disjoint from S. Hence the complement of S is a union of prime ideals, as we wanted to prove.

In (1), (2) and (4) the multiplicatively closed subset S is always non-empty. If $S = \emptyset$ is the empty set, then (1), (2) and (4) do not hold. For instance, in (4), the empty set $S = \emptyset$ is trivially a saturated multiplicatively closed subset, but the rng R can be either the union of a set of prime ideals, or not. For instance, in our first example above, the rng $R = \mathbb{F}_2 \times \mathbb{Z}_2$ *is not* a union of prime ideals of R. On the contrary, here is an example of a ring Q that *is* a union of prime ideals. Let T be a ring with a strictly ascending chain $P_0 \subset P_1 \subset P_2 \subset \dots$ of prime ideals, e.g. the ring $T := k[X]$ of polynomials with coefficients in a field k in an infinite set X of indeterminates. Then the union $Q := \bigcup_{i\geq 0} P_i$ of this infinite chain of primes is a rng, in which all the ideals P_i are prime as well. Thus this rng Q is an example of a rng that is the union of all its prime ideals.

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Alberto Facchini (Corresponding Author) and **Jennifer Parolin**
Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova
35131 Padova, Italy
e-mails: facchini@math.unipd.it (A. Facchini)
jenniferparolin@gmail.com (J. Parolin)