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OPTION PRICING IN EMERGING MARKETS USING PURE JUMP PROCESSES: EXPLICIT CALIBRATION OF BIST30 EUROPEAN OPTION

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Bilgi Yimaz[1], A. Alper Hekimoglu[2]
[1]Kaisersluatern Technical University, Department of Mathematics, Kaiserslautern, Germany.
bilgiyilmaz07@gmail.com & yilmazb@rhrk.uni-kl.de z, ORCID: 0000-0002-9646-2757
[2]European Investment Bank, Model Validation Division, Luxembourg.
ahekimsfe@gmail.com & a.hekimoglu@eib.rog , ORCID: 0000-0003-3490-1985

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ABSTRACT

Purpose- This study aims to illustrate the efficiency of pure jump processes, more specifically Variance Gamma (VG) and Normal Inverse Gaussian models (NIG), in option pricing by comparing with the Black Scholes (BS) option pricing model for emerging markets.

Methodology- This study presents an alternative derivation of option pricing formulas for VG and NIG models. Then, it investigates the VG and NIG models' option pricing performance with the help of new derivation by comparing them with the BS option pricing model for emerging markets for an emerging country, Turkey. The data consists of the BIST30 index daily price and European options written on this index extend from 05 May 2018 to 05 May 2020 for given exercise prices with a maturity of 90 days. In this period, the European call options' strike prices range from 1200 to 1650, and the European put options' strike prices range from 1000 to 1400. To compare the models' efficiency, first, we calibrate the models by minimizing the sum of squared deviations between the observed and theoretical option prices. Second, we compute the option prices and compare the results with the observed option prices.

Findings- The significant contribution to the literature is the calibration of the pure jump processes (VG and NIG processes) using the characteristic functions, the continuous BS prices for an emerging market, and the computation of European options prices in BIST. We find that while the NIG process performs better than VG and BS models, the BS model is the worst in option pricing.

Conclusion- The pure jump processes (VG and NIG processes) can be calibrated using the characteristic functions, and option price estimations with them are better than the continuous BS prices for an emerging market. Thus, the pure jump processes are more efficient in market modeling than the BS model.

Keywords: Variance Gamma Process, Normal Inverse Gaussian Process, Black-Scholes Model, Options Pricing, Calibration
JEL Codes: C63, G12, D46

1. INTRODUCTION

The 2008 Global Financial Crisis (GFC) fundamental policy response was a sudden noble increase in the money supply, defined as quantitative easing by governments worldwide. Company and even country bailouts have occurred as an immediate consequence of the GFC. Since the GFC, the money supply has been escalating exponentially worldwide, breaking historical records of the money supply. On the other hand, nominal interest rates have been extraordinarily and persistently either negative or almost zero in most developed countries. Such a sustained response and humble policy rate cut are unlikely to feed bubbles rapidly in emerging economies. While financial markets and economic activities have picked up pace and display signs of health, we have anecdotal evidence that both depend crucially on the repeatedly reinforced perception through previous regulatory actions. Hence, policymakers assist endorsement gradually and thoroughly in their economies. However, skeptics, including those with notable influence and visibility, argue that risks are gathering on the horizon for yet another economic and financial crash, which may substantially impact the global financial system as the GFC.

The globalization of economies motivates a rapid increase in the expansion of derivatives and exchange markets. Such markets become extremely attractive to investors since they contain new investment vehicles used for hedging purposes. Furthermore, markets observe information on investors' views on option prices and share prices. In this respect, the options and shares are also used for speculative investments in the derivatives market in emerging markets (Alp, 2016). Investors are eager to invest in emergent investment vehicles for higher yields in emerging markets.

On the other hand, as previously Alp (2016) indicated, emerging markets are more volatile with superior risk premiums than developed markets. In this respect, the GFC primarily affected the developed economies and increased the investment speed in emerging markets. In particular, options tied to stock exchange indexes are regarded as one of the most charming investment tools for foreign investors since they provide appropriate exposure to local exchange markets.

On the other hand, such options in most emerging markets are comparatively in the early stage compared to those in the developed markets. As Alan et al. (2016) emphasized, previous studies on index options that analyze developed markets show that index options' pricing efficiency and hedging benefits are less efficient than throughout the early trading periods. Hence, derivatives markets and efficient pricing models of options and underlying assets are vital for hedging prospects in emerging markets.

Considering the rise in the flows of capital to the emerging markets following the GFC, this paper concerns the latter, namely the specification of the stochastic processes that underlying assets evolve and the estimation of European put and call option prices for an emerging market, more precisely Turkey's stock Exchange market. The study examines the fitting behaviors of the selected models to the BIST30 index price evolution. In an arbitrage-free market environment, the derivative price is nothing but the discounted value of expectation of the future payoffs under the specified risk-neutral measure. Therefore, such a pricing formula has three fundamental components: The risk-free rate (bank account), the contract specification (for instance, payoff function, etc.), and the stochastic process that the underlying asset evolves from (Eriksson et al., 2009). Hence, we start with identifying of stochastic processes that represent the dynamics of the BIST30 index. The most commonly used model in literature and real-life applications is the Black-Scholes model (BS) developed by Black and Scholes (1973). Hence, we compare the selected models with BS to identify their accuracy.

The implied volatility smile phenomenon reveals that the BS systematically leads to mispricing out-of-the-and-in-the-money options when the implied volatility of the at-the-money option is taken into account. Therefore, many stochastic volatility models, such as Hull&White (Hull & White, 1987), Stein&Stein (Stein & Stein, 1991), and Heston (Heston, 1993), have been designed to mirror the volatility smile phenomenon effect. However, such models still generate mispricing since they lack to capture jumps in the underlying asset price. Hence, Madan and Senata (1990) developed a continuous-time stochastic process called the Variance Gamma (VG) model to predict the uncertainty of the underlying asset return. They provide a practical and empirically relevant alternative to Brownian motion's role as the martingale component of the motion in log prices. Instead of just a distribution for log returns, the importance of introducing a stochastic process is crucial for applications to European option pricing that do not individually compute risk-neutral expectations but account for risk aversion via the identification of an exact change of measure (Harrison & Pliska, 1983). Also, Barndorff (1997) offered a normal-inverse Gaussian distribution (NIG) process, which is a continuous stochastic process. It is given as a normal variance-mean mixture where the mixing density is the inverse Gaussian distribution. Geman et al. (2001) showed that such improvement is motivated by a better fit to the data, improved option pricing and hedging strategies, and theoretical considerations.

The research on some of the fundamental problems of emerging markets needs to be improved, particularly Borsa Istanbul Stock Exchange (BIST). We have recognized a literature gap about the performance of various models for the BIST30 index returns and pricing options tied to this index. While a substantial and expanding body of literature has examined CAPM and FAMA FRENCH models (i.e., Coskun et al. (2017) and references therein), the models based on the stochastic processes are limited for BIST to our best knowledge. In contrast to those studies, this study contains three critical contributions to the literature: First, we fit two Lévy (pure jump) models (VG and NIG) to an emerging market, namely the BIST30 index, and estimate the option prices appealing interpretation and tractability. Hence, throughout our analysis, we calibrate (determine the models' parameters) the pure jump processes and compare their results with the classical BS. Second, we compute the option prices using the calibrated processes and compare their results with the observed options' prices. This analysis shows that the pure jump processes are better than the classical BS in option pricing for the BIST30 index. Therefore, using the VG and NIG models for emerging markets is better. Third, we find delta hedging coefficients corresponding to the models we consider and show that the delta hedging for pure jump processes is superior to the classical BS.

The remainder of the study is outlined as follows. Section 2 summarizes the literature on stochastic models and their usage. Section 3 is dedicated to giving a brief on BIST and related derivatives markets. Section 4 briefly describes the structure of the Lévy processes to be employed and their primary properties and relevance to option pricing theory. In Section 5, we give the details of the data. Section 6 presents the calibration results and estimated option prices, contains the models pricing efficiency, and Section 7 concludes the study.

2. LITERATURE REVIEW

Stochastic processes are at the center of option pricing theory. These processes are classified into two broad categories based on their sample paths: i) continuous processes and ii) discontinuous processes. In this study, we are dealing with the pure jump processes, VG, and NIG processes. Hence, this section concentrates on only the literature on the two critical and popular exponential Lévy models.

On a typical underlying asset, for instance, the S&P 500 index, approximately 200 option prices range across twenty strikes and ten maturities at any instant, which is defined by the considered model consistently through the calendar time. From this viewpoint, the BS model's simplicity is unrealistically glaring (Konikov & Madan, 2002) since, generally, log returns show deviations from the normality assumption. Therefore, most research on the jump-diffusion models has emphasized using a diffusion component to describe the relatively large number of small price movements. In contrast, an orthogonal Poisson process with a finite number of sizable moves per unit of time is used to model the large and relatively rarer log return movements (Konikov & Madan, 2002).

More attention has focused on providing the search for special Lévy models to outperform the BS model was initiated since Mandelbrot (1963). Lévy processes and the jump-diffusion models are generally successful at fitting the volatility smile phenomena for a single maturity since these models can incorporate both the skewness and kurtosis properties into the marginal distribution of underlying assets. However, these processes fail to fit in the calibration of multiple maturities. Cont and Fonseca (2002) contributed a considerable model to the literature; contrary to the Lévy processes, Brownian motion has zero skewness and excess kurtosis, which possibly causes the volatility smile phenomena. After Madan and Senata (1987) considered the key findings of Praetz (1972) and issued the prior symmetric edition of the VG process with zero mean, a broader nonnegligible improvement has been adopted to the VG process, a Lévy process (e.g., the hyperbolic/ NIG process in Barndorff-Nielsen (1977) or Eberlein et al. (1995)), as alternatives to the standard BS model.

The literature on the VG process has highlighted two significant parts; theory and application. The univariate case where Madan and Senata (1990) extended the BS model by applying the VG process in the option pricing framework. Madan et al. (1998) concluded that the VG option pricing decreases the bias of option pricing contrary to the BS model, as the VG process controls the excess kurtosis caused by the jumps in the log returns. Daal and Madan (2005) used such a novel assessment for a numerical illustration of the VG option pricing model, the classical BS model, and Merton (1976)'s jump-diffusion model for the options written on foreign currencies. The authors justified Madan and Senata (1990)' conclusions that the VG option pricing model is superior to the other models. For further numerical illustrations of the VG process, interested readers may find more details in Leicht and Rathgeber (2014).

Several VG process variations, like the Carr Geman Madan Yor (CGMY) process introduced by Carr et al. (2002), are introduced to the literature. The multivariate case concerns the integrating correlations and the relation among the Lévy processes. For a general view, see, for instance (Luciano & Semeraro, 2013; Luciano & Schoutens, 2006; Luciano & Semeraro, 2010; Semeraro, 2008). To summarize, the VG process, like the other Lévy processes, suggests many possibilities for asset pricing and modeling risk by decreasing the pricing error or miscalibrations of the models that the underlying assets evolve. Such models help involve jumps, map a realist market behavior compared to the traditional models, and are essential instruments from financial mathematics.

The NIG model introduced in Barndorff (1977) is also one of the most popular Lévy models due to its flexibility. At the same time as its relevance to practice, the NIG process is challenging for mathematical illustrations. However, this modeling family has been widely used in mathematical finance (Eberlein, 2001).

Many computational and statistical procedures are developed for European option pricing in this context. We have three major types of numerical valuation methods: (i) The method of Monte Carlo simulation, (ii) the numerical solution of the partial integration-differential equations related to the model, and (iii) Fourier transformation methods (Eberlein, 2014). Additionally, Ivanov (2013) has given analytical solutions for European Call and digital options under the assumption that the underlying asset price dynamics evolve from the exponential NIG model. The application of the NIG distributions is defined by considering the moments; mean, variance, skewness, and kurtosis. Such moments are essential to many risk management applications. One strength of this class is that authors associate individual derivatives pricing to these risk-neutral distribution moments, which intuitively reviews how the moments can interpret the derivative price behaviors (Eriksson et al., 2009).

For the '*symmetric case'*, a reasonable hypothesis for the price of equities, these models need only one extra parameter given by κ, compared to the two-parameter the BS model. Such an extra parameter corresponds to the percentage of excess kurtosis connected to the normal distribution. Therefore, κ primarily controls the tail thickness of the underlying asset log return distribution. Therefore, it determines the '*excessively*' large positive or negative log return frequency of the underlying asset. Both VG and NIG models are in the family of pure-jump stochastic models with infinite jump activity (i.e., models having infinitely many jumps during survival time $[0,T], T < \infty$). Even so, σ controls the log returns variability of the underlying asset. Consequently, σ is considered the price process volatility (Viens et al., 2011). Various papers focusing on empirical analysis have confirmed that some parametric exponential Lévy models (ELM), such as VG and NIG models, can fit daily log returns of underlying assets unbelievably well with the classical calibration methods, e.g., maximum likelihood estimators (MLE) or method of moment estimators (MME) (Barndorff, 1997; Behr and Pötter, 2009; Eberlein, 1995; Madan et al., 1998).

3.TURKEY'S STOCK EXCHANGE AND DERIVATIVE MARKETS

The Borsa İstanbul Stock Exchange (BIST) is the single exchange market in Turkey. It is organized to supply trading in bills, equities, revenue-sharing certificates, bonds, and international securities. Turkey's securities exchange legal framework was completed in 1982, and started its operation with 40 listed corporations in 1986. Until a manual system was authorized in late 1987, the trade floor activities were limited to licensed brokers, but unlicensed investors could directly execute their orders. In 1989, Turkey's financial system changed to a liberalization system, and then, foreign investors became allowed to invest in portfolios that consisted of stocks traded in BIST. Since November 1994, the number of assets in the market increased drastically by 2003. The daily trading volume of the market has reached an amount of 2.972 billion US dollars. As Bastı et al. (2015) emphasized, BIST has got into the first thirty largest exchange markets among the stock exchanges worldwide and has new memberships in numerous international federations and associations (e.g., the World Federation of Exchanges, Federation of Euro-Asian Stock Exchanges, Federation of European Securities Exchanges, and International Capital Market Association).

There are five sub-markets in BIST that investors can operate. Namely, the equity, futures, and options written on stocks and indexes, the debt securities, the emerging companies, and the precious metals and diamond markets. Additionally, there are eight sub-markets under the equity market: the national, the collective products, the secondary national, the watch-list companies, the primary, the wholesale, the rights coupon markets, and the free trade platform (Bastı et al., (2015).

BIST may be described as regulated by restrictive monetary policy and is led by high-interest rates and large budget deficits. There are limited numbers of studies explaining stock returns and option prices using stochastic processes during the BIST's short history. Even though many studies investigate option pricing for developed and many emerging economies, the studies focusing on option pricing in Turkey's derivatives market are extremely limited. Demir and Tutek (2004) analyzed the applicability of the numerical martingale simulation method for pricing the options tied to the options that are traded in BIST. However, instead of studying real options, the authors hypothetically generate a set of options tied to the BIST Composite Index. In the end, the authors highlighted the method that outputs option prices closer to those driven by the BS model. Later, Akyapı (2014) investigated the differences between real and hypothetical option prices that are again estimated from the BS option pricing formula in the BIST30 index. Akyapı (2014) showed that Turkey's options market permits arbitrage opportunities. The author also observed that the observed option prices are unequal to prices computed by the BS option pricing method.

Tokat (2009) searched the volatility of the BIST30 index for January 1990-April 2007. The author perceived unexpected changes in the volatility of the log return and leptokurtic distribution with additional kurtosis. On the other hand, Kayalidere et al. (2012) investigate the effect of GARCH, the tradeoff of risk and return, and the effect of day-of-the-week on the BIST30 future contracts for the 2006-2011 period. They found that the BIST30 index has fat-tailed distribution with negative skewness. Also, Gokgoz and Sezgin-Alp (2014) modeled Turkey's BIST100 market index under the Arbitrage Pricing Theory assumption using the Artificial Neural Networks method. They emphasized that the BIST30 index has a leptokurtic feature. Kayalidere et al. (2012) reflected that the BIST30 futures volatility is affected more by unfavorable news than favorable news. The tradeoff of risk-return is irrational, and the market is not weak-form efficient. Ersoy and Bayraktaroglu (2013) examined the lead-lag link between the spot and future markets utilizing the daily closing prices of the BIST30 index and futures contracts tied to this index. They conclude that there is not a lead-lag association among these markets. Akyapı (2014) investigated the deviations within the real and hypothetical prices derived from the BS option pricing formula in the BIST30 index. He revealed that options markets might be exposed to arbitrage opportunities in BIST. Also, he showed that, generally, the real option prices are unequal to the numerical option prices.

4.THE STRUCTURE OF VG AND NIG PROCESSES

4.1. Variance Gamma (VG) Process

The VG process considers both the symmetric increase in the left and right tail probabilities of the log return distribution (kurtosis) and the asymmetry of the left and right tails of the log return density (skewness). These properties allow a more accurate representation of stock returns (Rathgeber et al., 2016). Based on Madan et al. (1998), we can write the call option price for a VG process in the BS manner as in the following proposition.

Proposition 1: Let the stock price process $S(t)$ be the VG process from which the underlying asset evolves. Then, the European call option price equals to

$$C(S(0),K,r,T,\theta,\nu,\sigma) = S(0)F^S(X,\theta_S,\sigma,T,\nu_S) - Ke^{-rT}F(X,\theta,\sigma,T,\nu),$$

where $X = log\,(S(t)/K\,) + (r - \phi_{VG}(-i))T$ and $\nu_S, \theta_S,\ \sigma$ represent parameters of the VG process, and ϕ is the log-characteristic function, respectively.

Proof: The proof is given in Appendix B.

Here, we present an alternative derivation given by Madan et al. (1998) for the VG option pricing formula. However, our new derivation procedure helps write a similar structure for the NIG process in a BS option price formula.

We introduce the CDFs that are used to calculate the VG model option pricing by using the densities that we derive in Appendix B for the VG process,

$$F^S(X,\theta_S,\sigma,T,\nu_S,\nu) = \int_{-\infty}^{X} \frac{2\exp\left(\frac{\theta_S\,x}{\sigma^2}\right)}{\Gamma\left(\frac{T}{\nu}\right)\sqrt{2\pi\nu^{\frac{T}{\nu}-1}}} \left(\frac{x^2}{\frac{2\sigma^2}{\nu_S}+\theta_S^2}\right)^{\frac{T}{2\nu_S}-0.25} \times \frac{K_{\frac{T}{\nu}-0.5}\left(\frac{x^2}{\sigma^2}\left(\frac{2\sigma^2}{\nu_S}+\theta_S^2\right)\right)}{\sqrt{x^2+\frac{T^2}{\nu}}}dx,$$

$$F(X,\theta,\sigma,T,\nu_S,\nu) = \int_{-\infty}^{X} \frac{2\exp\left(\frac{\theta\,x}{\sigma^2}\right)}{\Gamma\left(\frac{T}{\nu}\right)\sqrt{2\pi\nu^{\frac{T}{\nu}-1}}} \left(\frac{x^2}{\frac{2\sigma^2}{\nu}+\theta}\right)^{\frac{T}{2\nu_S}-0.25} \times \frac{K_{\frac{T}{\nu}-0.5}\left(\frac{x^2}{\sigma^2}\left(\frac{2\sigma^2}{\nu}+\theta\right)\right)}{\sqrt{x^2+\frac{T^2}{\nu}}}dx,$$

where θ_S, ν_S, and σ are VG parameters under $\mathbb{Q}^S$ measure, and θ, ν, and σ are parameters under $\mathbb{Q}^S$ measure and $K_{\frac{T}{\nu}-0.5}$ represents the Modified Bessel function of the second kind (MacDonald's function).

2.2. Normal Inverse Gaussian (NIG) Process

As a model of underlying asset return evolution, the NIG process is a particular case of the generalized hyperbolic distributions, primarily introduced by Barndorff (1997). Barndorff (1997) analyzes the NIG process, including the derivation of the Lévy measure of this process, obtaining its properties, and proposing an Ornstein-Uhlenbeck process of the NIG type and an NIG type stochastic volatility model. The distribution of the NIG characterization is done by a normal inverse Gaussian mixing distribution.

Definition 1: Let Y be a random variable that follows an inverse Gaussian probability law (IG) given as in Eriksson et al. (2009)

$$\mathcal{L}(Y) = IG(\delta, \sqrt{\alpha^2-\beta^2}).$$

Now, suppose that X is a conditional process on Y, and it is normally distributed with mean $\mu + \beta\,Y$ and variance Y ($L(X|Y) = N(\mu + \beta\,Y, Y)$). Then, the conditional density X is an NIG

$$\mathcal{L}(X) = NIG(\alpha,\beta,\mu,\delta).$$

The NIG X has a density function given as in the following theorem.

Theorem 1: The $NIG(\alpha,\beta,\mu,\delta)$ distribution, given for the parameters $\alpha,\delta \geq\ 0$, $|\beta| \leq \alpha, \mu \in \mathbb{R}$ has a density

$$f(x) = \frac{\alpha\delta}{\pi}\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}e^{\delta\gamma+\beta(x-\mu)},$$

where $\gamma = \sqrt{\alpha^2-\beta^2}$ and K_1 is the modified Bessel function of the third kind. Then, any process X_t that has a $NIG(\alpha,\beta,\mu\,t,\delta t)$ distribution is called a NIG Lévy process. Such processes are pure jump processes, and their Lévy measure has the following density

$$\nu(x;\alpha,\beta,\delta) = \frac{\delta\alpha}{\pi|x|}e^{\beta\,x}K_1(\alpha|x|).$$

Proof: The proofs may be found in Barndorff (1997). □

Remark 1: An $NIG(\alpha,\beta,\delta,\mu)$ distributed random variable has the moments:

- Mean: $\frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}} + \mu,$
- Variance: $\frac{\alpha^2\gamma}{(\alpha^2-\beta^2)^{\frac{3}{2}}},$

- Skewness: $\frac{3\beta}{\alpha\sqrt{\delta}(\alpha^2-\beta^2)^{\frac{1}{4}}}$,
- Kurtosis: $3(1+\frac{\alpha^{+}4\beta^2}{\delta\alpha^2\sqrt{\delta}(\alpha^2-\beta^2)})$.

Proof: See the detailed proof in Barndorff (1997). □

Remark 2: The characteristic exponent of X is given as

$$\kappa(\xi)=\mu\xi+\delta[\sqrt{(\alpha^2\beta^2)}-\sqrt{\alpha^2-(\beta+\xi)^2}].$$

The NIG class of densities has two key properties:

- Scaling property,

$$L_{NIG}(X)=NIG(\alpha,\beta,\mu,\delta)\Leftrightarrow \mathcal{L}_{NIG}(cX)=NIG(\frac{\alpha}{c},\frac{\beta}{c},\frac{\mu}{c},c\delta).$$

- A closure under convolution property

$$NIG(\alpha,\beta,\mu_1,\delta_1)*NIG(\alpha,\beta,\mu_2,\delta_2)=NIG(\alpha,\beta,\mu_1+\mu_2,\delta_1+\delta_2).$$

The NIG distribution is infinitely divisible and hence generates a Lévy process $(Z_t), t\geq 0$ (i.e., a stochastic process with stationary and independent increments, $Z_0=0$ a.s. and Z_1 is NIG-distributed). Now, let S_t, for $t\geq 0$, denotes the price of a non-dividend-paying stock at time t, and it postulates the following dynamics for the stock price

$$dS_t=S_{t^-}(dZ_t+e^{\Delta Z_t}-1-\Delta Z_t),$$

where $(Z_t), t\geq 0$ denotes the NIG Lévy motion, Z_{t^-} the left-hand limit of the path at time t, and $\Delta Z_t=Z_t-Z_{t^-}$ the jump at time t. Then, the solution of this stochastic differential equation is $S_t=S_0e^{Z_t}$, and it follows that the log returns, $ln(S_t/S_{t-1})$ are indeed NIG-distributed.

Our objective is the risk-neutral valuation of derivative securities in this model environment. Hence, we must adopt an equivalent martingale measure since the NIG model is incomplete. In this study, we select the method of characteristic functions to determine an equivalent martingale measure. The characteristic function approach is appropriate whenever the stochastic process (Z_t), $t\geq 0$, has stationary and independent increments (Eberlein et al., 1995).

Proposition 2: Let $S(t)$ be the NIG process that the underlying asset evolves. Then, the European call option price equals to

$$C(S(0),K,r,T,\theta,\theta_S,\kappa,\sigma)=S(0)F^S(X,\theta_S,\sigma,T,\kappa)-Ke^{-rT}F(X,\theta,\sigma,T,\kappa),$$

where $X=log(\frac{S(t)}{K})+(r-\phi_{NIG}(-i))T$.

Proof: We illustrate a detailed proof in Appendix B. □

The CDFs used to calculate the NIG model option price, using the densities derived in Appendix B for NIG process,

$$F^S(X,\theta_S,\sigma,T,\kappa)=\int_{-\infty}^{X}\frac{T}{\pi}exp\left(\frac{T}{\sqrt{k\omega}}+\frac{\theta_S x}{\sigma^2}\right)\times\frac{K_1(\sqrt{\theta_S^2+\frac{\sigma^2}{\omega}},\frac{1}{\sigma^2}\sqrt{x^2+\frac{T^2}{\sigma^2\kappa_S}})}{\sqrt{x^2+\frac{\sigma^2T^2}{\kappa}}}\,dx,$$

$$F(X,\theta,\sigma,T,\kappa)=\int_{-\infty}^{X}\frac{T}{\pi}exp\left(\frac{T}{\sqrt{k\omega}}+\frac{\theta x}{\sigma^2}\right)\times\frac{K_1(\sqrt{\theta^2+\frac{\sigma^2}{\omega}},\frac{1}{\sigma^2}\sqrt{x^2+\frac{T^2}{\sigma^2\kappa}})}{\sqrt{x^2+\frac{\sigma^2T^2}{\kappa}}}\,dx,$$

where θ_S,κ_S,σ are NIG parameters under the $\mathbb{Q}^S$ measure and θ, κ, σ are parameters under the $\mathbb{Q}$ measure. Also, $\omega=\frac{1}{1-k(\sigma^2-2\theta}$ as noted in Appendix B, and again, K_1 represents the Modified Bessel function of the second kind (MacDonald's function).

3. DATA

In this study, we analyze the BIST30 index daily prices and its log returns and European call and put options written on it for 05 May 2018 - 05 May 2020. We consider various European put (9 put options) and call (10 call options) options with various strike prices (1200 to 1650 for call options and 1000 to 1400 for put options) with a 90-day maturity and gather the relevant data from the Bloomberg data stream for the empirical analysis given in this section.

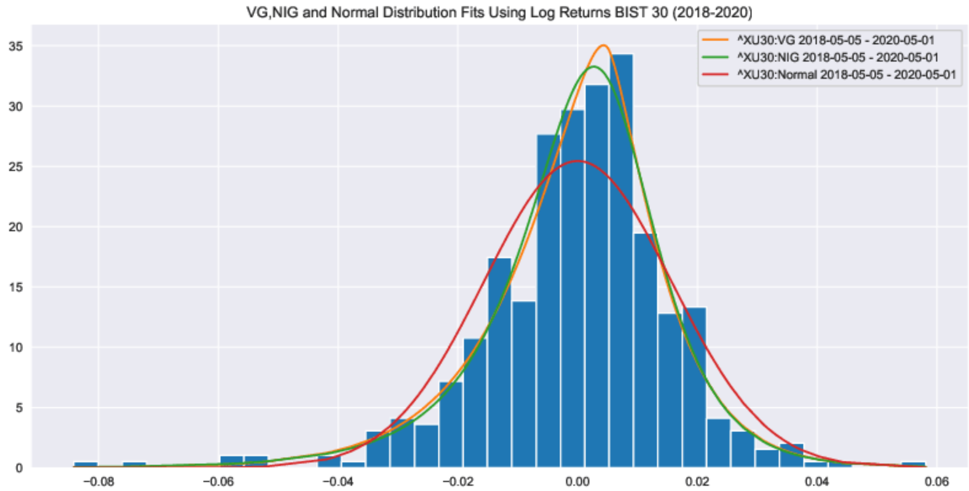
The descriptive statistics of the daily BIST30 index prices and its log return series are depicted in Table 1. The table shows that the average index price and its standard deviation are 99274.371 and 8132.956, respectively. Also, the index price series is highly right-skewed (skewness=1.0013). As a result, the BIST30 index price has a non-normal distribution property. On the other hand, its' log return series' mean and standard deviation are 0.00119 and 0.015090, respectively. Further, the log return series is moderately left-skewed (skewness=-0.7436).

Additionally, the log return series is too peaked and has a heavy-tailed distribution (kurtosis= 3.674>1) compared to the index price series. More importantly, the table shows that the BIST30 index price has negative log returns that may cause swear losses to investors if they do not hedge their positions. The descriptive statistics of the BIST30 price and its' log return series show that both series are not normally distributed and have tail properties. At this stage, we also present the histogram of the log return series in Figure 1. The figure also reveals that the log return series comes from a non-normal distribution family.

Table 1: Descriptive Statistics of BIST30 and its Log Return Series

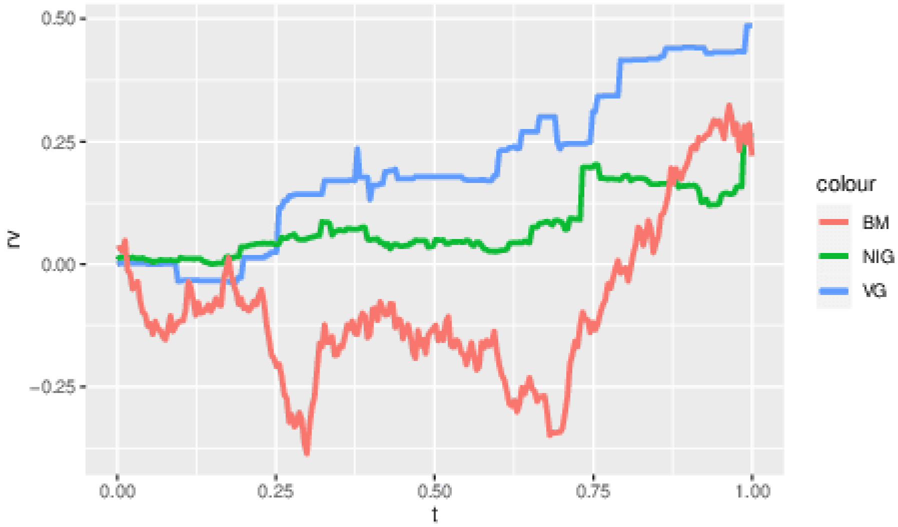
	BIST30	Log return
Std	8132.956	0.015090
skewness	1.0013	-0.7436
kurtosis	0.89194	3.674
Mean	99274.371	0.00119
Max	123556.102	0.05983
Min	83675.296875	-0.08072

Figure 1: BIST30 Log Return Distribution Characteristic of VG, NIG, and Normal Distributions



To illustrate all three models' behaviors, we graph their paths that are simulated using the calibrated parameters in Figure 2. Here, it is worth emphasizing that the figure shows that the VG and NIG models show jump behaviors better than the BS and, hence, both are good candidates to capture the jumps, skew, and fat tails properties of the observed data.

Figure 2: A Simulated Illustration of VG, NIG, and BM Processes with the Fitted Parameters



4. EMPIRICAL ANALYSIS OF OPTION PRICING

As it is well known, instead of working on prices, working on returns is more convenient in financial data analysis due to the stationary problem. Hence, in our empirical analysis, we also used the log return series of the BIST30 index price.

We illustrate the histogram of the BIST30 log return series along with its distribution fitting results of the VG, NIG models, and normal distribution in Figure 1. The figure combines a normalized histogram and density plots of the models to highlight the log return series statistical properties. In this figure, while the orange and green curve illustrates the fitting performance of the pure jump models VG and NIG, the red curve represents the normal distribution (GBM) fitting performance. Due to its Gaussian property, the BS model has a normal distribution. However, as evident from the plot, the BIST30 log return series is slightly left-skewed, with a peak at %0.01. There were also a few tiny peaks seen close to zero. Hence, it is not normally distributed. Therefore, we can conclude that VG and NIG fit better into the log return series than the Normal distribution. This is an expected result since the BIST30 log return series is not normally distributed. The BS model is based on GBM as an asset price process where the returns have Normal distribution.

The parameters and their p-values that we find in the fitting process of the distributions are given in Table 2. The values in parenthesis are the p-values of the parameters. The standard deviation of the NIG and BS models are close to each other ($\sigma = 0.015$ and $\sigma = 0.01568$, respectively). In contrast, the VG model's standard deviation ($\sigma = 0.011856$) is lower than both NIG and BS models. On the other hand, μ values of the models vary. The corresponding p-values show that all parameters are statistically significant except the BS mean value μ. Note also that the (-) sign means that the model does not include the corresponding parameter in this and the following tables. For instance, while all models include σ, only the VG includes μ. The values of these parameters correspond to the distribution parameters we graph in Figure 1. Hence, it is worth emphasizing, to not confuse the readers, that these parameters are not the calibrated parameters. The calibration results are introduced in the following section.

Table 2: The Distribution Fitted Parameters of the Models (Fitted to Log Returns)

	VG	NIG	BS
σ	0.011856 (6.084-13	0.015 (0)	0.01568 (0)
ν	1.584 (0)	-	-
κ	-	0.868 (0.0128)	-
θ	-0.0032 (0.00189)	-0.0045 (0)	-
μ	0.0051 (4.951-05)	0.0045 (0)	0.000005032 (0.10)
$L(\mu,\sigma,\theta,\nu)$	-1364.821	-1366.754	-

Here, it is worth mentioning that our fitting procedure involves the maximum likelihood estimation (MLE) through the optimization of the following log-likelihood function introduced by Loregian and Rroji (V2012),

$$L(\mu,\theta,\sigma,\nu)=\frac{T}{2}log\left(\frac{2}{\pi}\right)+\sum_{t=1}^{T}\frac{(x_t-\mu)\theta}{\sigma^2}-\sum_{t=1}^{T}\log(\Gamma(\nu)\sigma)+\sum_{t=1}^{T}\log\left(K_{\nu-0.5}\left(\frac{\sqrt{2\sigma^2+\theta^2}|x_t-\mu|}{\sigma^2}\right)\right)$$
$$+\sum_{t=1}^{T}(\nu-0.5)\left[log(\,|x_t-\mu|-0.5log(2\sigma^2+\theta^2))\right].$$

As it is well-known, the optimization cost is affected significantly (positively) if the initial values of the models' parameters are chosen close to the local/global maximum points. Therefore, a relatively well-specified initial value is crucial in such an optimization procedure. Consequently, we introduce the following analytical formulas for determining the initial values with the help of the method of moments (MM) for both VG and NIG distributions, respectively.

$$\nu=\frac{3}{\mathbb{K}(x)-3},\qquad \sigma_{VG}=\sqrt{\frac{\mathbb{V}(x)(\mathbb{K}(x)-3)}{3}},\ \ \theta_{VG}=\frac{\mathbb{S}(x)\sqrt{\mathbb{V}(x)}}{3},\mu_{VG}=\mathbb{E}(x)-\frac{\mathbb{S}(x)\sqrt{\mathbb{V}(x)}}{\mathbb{K}(x)-3},$$

$$k=\frac{\mathbb{K}(x)}{3}-1,\quad \sigma_{NIG}=\frac{\sqrt{\mathbb{V}(x)}}{\left(1+\left(\frac{\mathbb{S}(x)}{3\mathbb{K}(x)}\right)^2k\right)},\theta_{NIG}=\frac{\sigma_{NIG}\mathbb{S}(x)}{3\mathbb{K}(x)},\qquad \mu_{NIG}=\mathbb{E}(x)-\theta_{NIG},$$

where $\mathbb{E}(x)$, $\mathbb{V}(x)$, $\mathbb{S}(x)$, and $\mathbb{K}(x)$ correspond to the moments: mean, variance, skewness, and kurtosis, respectively. Here, note that we observe significant improvements in log-likelihood convergence after introducing these initial parameters to our optimization procedure. The process takes only three iterations to converge both VG and NIG models.

Now, we may apply an optimization procedure to calibrate the models. The VG, NIG, and BS model parameters illustrate numerical illustrations of the VG and NIG processes' accuracy in the European put and call options pricing and underlying asset price prediction by comparing their results with the BS and observed option prices.

Using the NIG and VG option price formulas, we calibrate the models from the daily BIST30 index price series log returns, compute European call, and put option prices using data for 05 May 2018 - 05 May 2020. The calibration result of the models is given in Table 3. Afterward, we compute the European Call and Put options for all three models using the parameters in this table. The root means square error (RMSE) shows that the performance of the NIG model is superior to both VG and BS. Also, the RMSE shows that the BS has the worst performance in log return fitting of the BIST30.

Table 3: The Models' Parameters Obtained from the Calibration

Parameter	VG	NIG	BS
σ	0.215342	0.460616	0.391743
ν	0.105535	-	-
κ	-	0.348656	-
θ	0.454742	0.084060	-
RMSE	0.00029766	0.00003343715	0.031066734

Figure 3 illustrates European Call (right) and Put (left) options prices with various strikes. The figure shows that the pure jump models, VG and NIG, are superior to the continuous BS model for European Call and Put options. It is clear that both VG and NIG models capture the real option values with the calibrated parameters, summarized in Table 3. It is also important to highlight here that even though the VG model's option price estimation is almost identical to the NIG model option price estimation, the VG is computationally more expensive than the NIG model's (The calibration of the VG model took 0.431 seconds with the error of 0.000298 whereas the former took 0.349 seconds and the latter was 0.0000334 for NIG). Therefore, from the computation cost perspective, we can claim that the NIG process is superior to the VG process.

Figure 3: BIST30 European Call and Put Option Price Estimations under VG, NIG, and BS Model Assumption

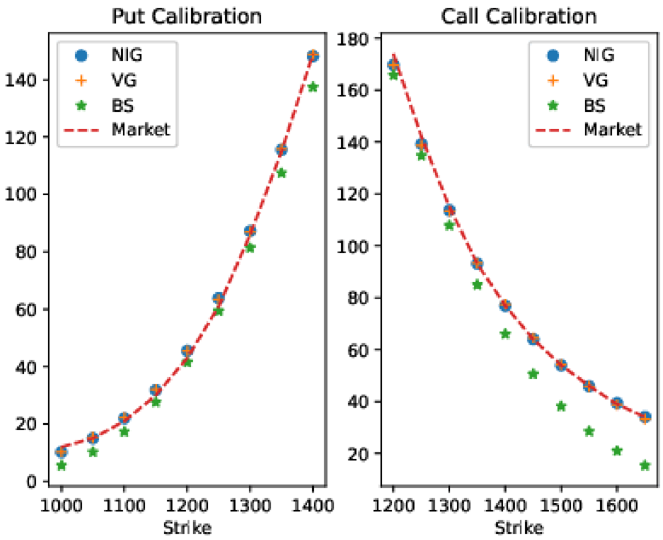


Figure 4 illustrates that both the VG and NIG are successfully capture the volatility smile phenomena of the option pricing since plotting the implied volatility surface as considering the option parameters, strike price, and time to maturity is helpful. Here, we end up with a two-dimensional curved surface graphed in three dimensions. The implied volatility surface of the market (z-axis) of European Call options on the underlying asset is plotted against option prices (y-axis) and time to maturity (x-axis). Such a representation describes the absolute value of the implied volatility surface. By adjusting the coordinates, the option price is returned by delta yields the relative implied volatility surface. More importantly, the implied volatility surface displays both the volatility smile and the volatility term structure simultaneously. Option investors use an implied volatility plot to rapidly select the implied volatility surface's shape and specify any region where the plot's slope (and consequently relative implied volatilities) was out of line. The figure illustrates the implied volatility surface for all European call options on a certain underlying asset price. In the figure, the z-axis corresponds to percent values of implied volatilities while the x-axes and y-axes correspond to the delta of the option and time to maturity. To satisfy the put-call parity, a 20-delta put should be equal to the same implied volatility as an 80-delta call. Given this implied volatility surface, we may conclude that the underlying asset has both volatility skew (a tilt along the delta axis) and a volatility term structure that indicates an anticipated event soon.

Figure 4: Volatility Surface of VG and NIG Model

(a) VG process **(b) NIG process**

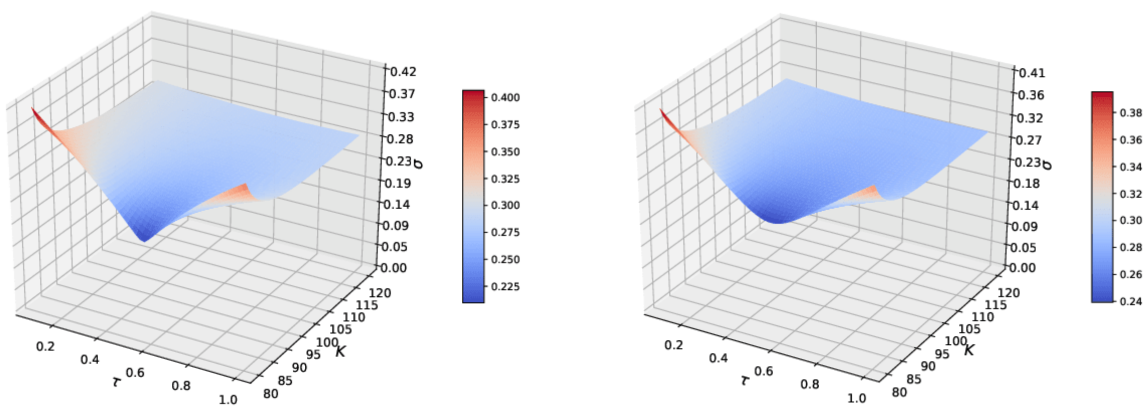
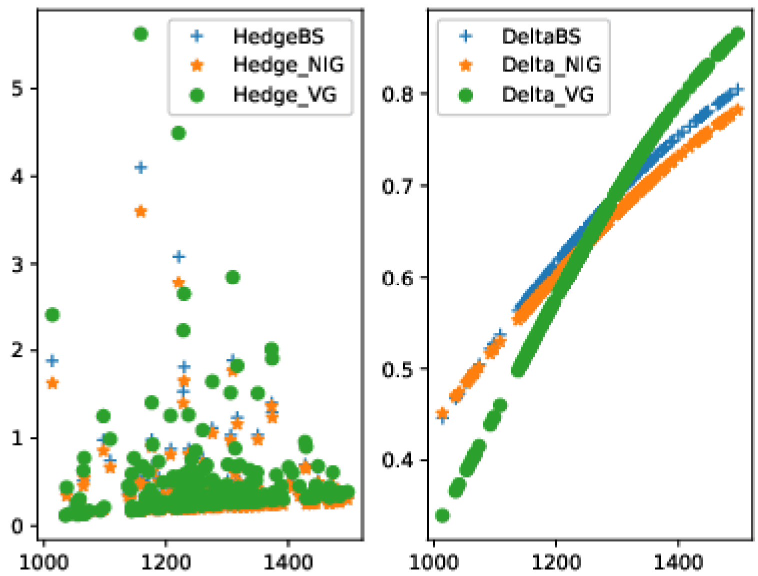


Figure 5 graphs the hedging performance of the models for the options. The first figure (on the left) shows the delta hedging coefficients corresponding to the stochastic models and the European call options. In this figure, the values on the x-axis show the strike price of a European call option, and values on the y-axis indicate the delta hedging coefficients. Here, the critical interpretation is that the NIG and BS models have almost the same hedging coefficient, while the hedging coefficient corresponding to the VG varies from the BS. The second figure (on the right) shows the stochastic models' dynamic hedging performance and European call options. The figure reveals that the hedging benefit of the VG process increases as the option strike price increases. On the other hand, the NIG model dynamic hedging is almost identical to the BS model.

Figure 5: Hedging Performance of VG, NIG, and BS Model



5. CONCLUSION AND IMPLICATIONS

The potential hedging benefits of options tied to the BIST30 index provide to investors become even more crucial in emerging markets where the cash markets are more vulnerable and prone to extreme volatility levels. Therefore, an efficient derivative market becomes even more relevant to domestic and foreign investors. There is sufficient literature on the various dynamics of derivatives and their hedging benefits. However, this study's bulk is conducted on emerging markets, particularly the BIST30 index and European options tied to this index.

Our study's significant contribution to the literature is the calibration of the pure jump processes (VG and NIG processes) using the characteristic functions, the continuous BS prices for an emerging market, and the computation of European options prices in this market. Such a study is the first study that investigates the pure jump processes accuracy for BIST. The calibration method can be repeated with other emerging markets by using the derivation we made for the characteristic functions. The essential point is the demonstration that the VG and NIG models can be fitted for emerging markets, and option prices in these markets can be estimated more adequately compared to the classical BS. We conclude that the NIG and VG processes are both attractive and tractable ways to incorporate the phenomenon of option pricing to derive actionable insights and investment decisions from the data. Hence, these pure jump models are becoming imperative for investors and portfolio managers to tackle predicting option prices and hedging in emerging markets.

REFERENCES

Akyapı, B. (2014). An Analysis of BIST30 index options market / [M.S. - Master of Science]. Middle East Technical University. http://etd.lib.metu.edu.tr/upload/12617381/index.pdf

Alan, N. S., Karagozoglu, A. K., & Korkmaz, S. (2016). Growing pains: The evolution of new stock index futures in emerging markets. Research in International Business and Finance, 37, 1-16.

Alp, Ö. S. (2016). The Performance of Skewness and Kurtosis Adjusted Option Pricing Model in Emerging Markets: A case of Turkish Derivatives Market. International Journal of Finance & Banking Studies, 5(3), 70-79.

Barndorff-Nielsen, O. (1977). Exponentially decreasing distributions for the logarithm of particle size. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 353(1674), 401-419.

Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. Scandinavian Journal of statistics, 24(1), 1-13.

Barndorff-Nielsen, O. E. (1997). Processes of normal inverse Gaussian type. Finance and stochastics, 2(1), 41-68.

Bastı, E., Kuzey, C., & Delen, D. (2015). Analyzing initial public offerings' short-term performance using decision trees and SVMs. Decision Support Systems, 73, 15-27.

Behr, A., & Pötter, U. (2009). Alternatives to the normal model of stock returns: Gaussian mixture, generalised logF and generalised hyperbolic models. Annals of Finance, 5(1), 49-68.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. Journal of political economy, 81(3), 637-654.

Carr, P., Geman, H., Madan, D. B., & Yor, M. (2002). The fine structure of asset returns: An empirical investigation. The Journal of Business, 75(2), 305-332.

Cont, R., & Da Fonseca, J. (2002). Dynamics of implied volatility surfaces. Quantitative finance, 2(1), 45.

Coşkun, Y., Selcuk-Kestel, A. S., & Yilmaz, B. (2017). Diversification benefit and return performance of REITs using CAPM and Fama-French: Evidence from Turkey. Borsa Istanbul Review, 17(4), 199-215.

Daal, E. A., & Madan, D. B. (2005). An empirical examination of the variance-gamma model for foreign currency options. The Journal of Business, 78(6), 2121-2152.

Demir, S., & Tutek, H. (2004). Pricing of options in emerging financial markets using martingale simulation: An example from Turkey. WIT Transactions on Modelling and Simulation, 38.

Eberlein, E. (2001). Application of generalized hyperbolic Lévy motions to finance. In Lévy processes (pp. 319-336). Birkhäuser, Boston, MA.

Eberlein, E. (2014). Fourier-based valuation methods in mathematical finance. In Quantitative energy finance (pp. 85-114). Springer, New York, NY.

Eberlein, E., & Keller, U. (1995). Hyperbolic distributions in finance. Bernoulli, 281-299.

Eriksson, A., Ghysels, E., & Wang, F. (2009). The normal inverse Gaussian distribution and the pricing of derivatives. The Journal of Derivatives, 16(3), 23-37.

Ersoy, E., & Bayrakdaroğlu, A. (2013). The lead-lag relationship between ISE 30 index and the TURKDEX-ISE 30 index futures contracts. İstanbul Üniversitesi İşletme Fakültesi Dergisi, 42(1), 26-40.

Geman, H., Madan, D. B., & Yor, M. (2001). Asset prices are Brownian motion: only in business time. In Quantitative Analysis In Financial Markets: Collected Papers of the New York University Mathematical Finance Seminar (Volume II) (pp. 103-146).

Gokgoz, F., & Sezgin-Alp, O. (2014). Estimating the Turkish sectoral market returns via arbitrage pricing model under neural network approach. International Journal of Economics and Finance, 7(1), 154.

Harrison, J. M., & Pliska, S. R. (1983). A stochastic calculus model of continuous trading: complete markets. Stochastic processes and their applications, 15(3), 313-316.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. The review of financial studies, 6(2), 327-343.

Hull, J., & White, A. (1987). The pricing of options on assets with stochastic volatilities. The journal of finance, 42(2), 281-300.

Ivanov, R. V. (2013). Closed form pricing of European options for a family of normal-inverse Gaussian processes. Stochastic Models, 29(4), 435-450.

Kayalidere, K., Araci, H., & Aktaş, H. (2012). Türev ve spot piyasalar arasındaki etkileşim: VOB üzerine bir inceleme. Muhasebe ve Finansman Dergisi, (56), 137-154.

Konikov, M., & Madan, D. B. (2002). Option pricing using variance gamma Markov chains. Review of Derivatives Research, 5(1), 81-115.

Leicht, J. J., & Rathgeber, A. W. (2014). Guaranteed stop orders as portfolio insurance–An analysis for the German stock market. Journal of Derivatives & Hedge Funds, 20(4), 257-278.

Loregian, A., Mercuri, L., & Rroji, E. (2012). Approximation of the variance gamma model with a finite mixture of normals. Statistics & Probability Letters, 82(2), 217-224.

Luciano, E., Marena, M., & Semeraro, P. (2016). Dependence calibration and portfolio fit with factor-based subordinators. Quantitative Finance, 16(7), 1037-1052.

Luciano, E., & Schoutens, W. (2006). A multivariate jump-driven financial asset model. Quantitative finance, 6(5), 385-402.

Luciano, E., & Semeraro, P. (2010). Multivariate Variance Gamma and Gaussian dependence: a study with copulas. In Mathematical and Statistical Methods for Actuarial Sciences and Finance (pp. 193-203). Springer, Milano.

Luciano, E., Marena, M., & Semeraro, P. (2013). Dependence calibration and portfolio fit with factor-based time changes. Carlo Alberto Notebooks, (307).

Madan, D. B., & Seneta, E. (1987). Simulation of estimates using the empirical characteristic function. International Statistical Review/Revue Internationale de Statistique, 153-161.

Madan, D. B., & Seneta, E. (1990). The variance gamma (VG) model for share market returns. Journal of business, 511-524.

Madan, D. B., Carr, P. P., & Chang, E. C. (1998). The variance gamma process and option pricing. Review of Finance, 2(1), 79-105.

Mandelbrot, B. (1963). New methods in statistical economics. Journal of political economy, 71(5), 421-440.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. Journal of financial economics, 3(1-2), 125-144.

Praetz, P. D. (1972). The distribution of share price changes. Journal of business, 49-55.

Rathgeber, A. W., Stadler, J., & Stöckl, S. (2016). Modeling share returns-an empirical study on the Variance Gamma model. Journal of Economics and finance, 40(4), 653-682.

Semeraro, P. (2008). A multivariate variance gamma model for financial applications. International journal of theoretical and applied finance, 11(01), 1-18.

Stein, E. M., & Stein, J. C. (1991). Stock price distributions with stochastic volatility: an analytic approach. The review of financial studies, 4(4), 727-752.

Tankov, P. (2003). Financial modelling with jump processes. Chapman and Hall/CRC.

Tokat, H. A. (2009). Re-examination of volatility dynamics in Istanbul Stock Exchange. Investment management and financial innovations, (6, Iss. 1 (contin.)), 192-198.

Viens, F. G., Mariani, M. C., & Florescu, I. (2011). Handbook of modeling high-frequency data in finance (Vol. 4). John Wiley & Sons.

Zwillinger, D., & Jeffrey, A. (Eds.). (2007). Table of integrals, series, and products. Elsevier.

APPENDIX A: ESTIMATED OPTION PRICES

Table 4: European Call and Put Option Prices with Various Strike Prices under the VG, NIG, and BS Models Assumptions

		Call Option Prices		
Strike	**NIG**	**VG**	**BS**	**Market**
1200	169.784463	169.698553	165.800577	174.0
1250	139.206130	138.710337	134.776382	142.0
1300	113.717301	113.415851	107.869369	115.0
1350	93.152041	93.388213	85.032008	93.0
1400	76.869700	77.422520	66.050554	77.0
1450	64.051714	64.598618	50.587017	64.0
1500	53.926980	54.223663	38.226352	54.0
1550	45.862685	45.772460	28.520774	46.0
1600	39.371898	38.843751	21.026090	39.0
1650	34.089866	33.128667	15.327846	34.0
		Put Option Prices		
Strike	**NIG**	**VG**	**BS**	**Market**
1000	10.207271	10.241480	5.520252	12.0
1050	15.041657	15.264206	10.168876	15.0
1100	21.985959	22.326443	17.328258	21.0
1150	31.837063	32.099611	27.612806	30.0
1200	45.512746	45.426835	41.528859	43.0
1250	63.846424	63.350630	59.416676	61.0
1300	87.269606	86.968157	81.421675	86.0
1350	115.616358	115.852530	107.496326	115.0
1400	148.246029	148.798849	137.426883	149.0

APPENDIX B: THE PROOF OF PROPOSITIONS

Proof: Let us define log discounted stock price, $log(S(T)e^{-rT}) = X(T) - \phi(-i)$, where $X(T)$ denotes a VG process, ϕ is the log-characteristic function, and T is a finite maturity ($T < \infty$) as usual. Then, under the risk-neutral probability $\mathbb{Q}^S$, we can obtain the characteristic function of log-discounted stock price,

$$\mathbb{E}^S\left(e^{iuX(T)}\right) = S(0)\mathbb{E}\left[\frac{S(T)}{S(0)e^{rT}}e^{iuX(T)}\right] = \mathbb{E}\left[e^{X(T)(iu+1)}\right]e^{-\phi(-i)T} = \mathbb{E}\left[e^{X(T)i(u-i)}\right]e^{-\phi(-i)T}.$$

Now, by defining $u - i = v$ we obtain

$$\mathbb{E}\left(e^{ivX(T)}\right)e^{-\phi(-i)T} = \frac{\Phi(-i)}{\Phi(v)} = \frac{1-\theta\nu-0.5\sigma^2\nu}{1-i(u-i)\theta\nu+0.5(u-i)\text{^}2\sigma^2\nu} = \frac{1-\theta\nu-0.5\sigma^2\nu}{1-\theta\nu-0.5\sigma^2\nu(1-iu(\sigma^2+\theta)\frac{\nu}{1-\theta\nu-0.5\sigma^2\nu}+0.5\sigma^2u^2\frac{\nu}{1-\theta\nu-0.5\sigma^2\nu\}})}.$$

At this stage, again by defining $\kappa = (1 - \theta\nu - 0.5\sigma^2\nu)$ we end up with the following four results

$$\frac{\Phi(-i)}{\Phi(\nu)} = \left(\frac{\kappa}{\kappa\left(1 - iu(\sigma^2 + \theta)\frac{\nu}{\kappa} + 0.5(\sigma\, u)^2\frac{\nu}{\kappa}\right)}\right)^{\frac{T}{\nu}},$$

$$\Phi_X^S(u,T) = \left(\frac{1}{1 - iu\theta_{s\nu_s} + 0.5(\sigma\, u)^2\nu_s}\right)^{\frac{T}{\nu}}, \qquad (1)$$

$$\theta_s = \theta\ + \sigma^2, \qquad \nu_s = \frac{\nu}{\kappa}.$$

To derive the VG process's probability density function (pdf) of under the $\mathbb{Q}^S$ measure, (1) characteristic function under the $\mathbb{Q}^S$ is needed. After it is derived, we obtain new parameters under the $\mathbb{Q}^S$ measure. This could be obtained using density as well. However, the characteristic function is more straightforward. Then one can first represent the density under this form,

$$g^S(x,\theta_S,k,\sigma,\nu_S,\nu,T) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi\gamma}}exp\left(-0.5\left(\frac{x-\theta_S\gamma}{\sigma\sqrt{\gamma}}\right)^2\right)\frac{e^{-\frac{\gamma}{\nu_S}}\gamma^{\frac{T}{\nu}-1}\nu_S^{-\frac{T}{\nu}}}{\Gamma(\frac{T}{\nu})}d\gamma.$$

Then, using equation 3.471.9 from Zwillinger and Jeffrey (2007), one can further write the density,

$$g^S(x,\theta_S,k,\sigma,\nu,\nu_S) = \frac{2\exp\left(\frac{\theta_S x}{\sigma^2}\right)}{\Gamma\left(\frac{T}{\nu}\right)\sqrt{2\pi\nu^{\frac{T}{\nu}-1}}}\left(\frac{x^2}{\left(\frac{2\sigma^2}{\nu_S}+\theta_S^2\right)}\right)^{\frac{T}{2\nu}-0.25} \times\frac{K_{\frac{T}{\nu}-0.5}(\frac{x^2}{\sigma^2}(\frac{2\sigma^2}{\nu_S}+\theta_S^2))}{\sqrt{x^2+\frac{\sigma^2T^2}{\nu}}}.$$

Under the risk-neutral measure $\mathbb{Q}^S$, the result coincides with the following density function that was primarily introduced by Tankov (2003)

$$g(x,\theta,k,\sigma,\nu) = \frac{2\exp\left(\frac{\theta x}{\sigma^2}\right)}{\Gamma\left(\frac{T}{\nu}\right)\sqrt{2\pi\nu^{\frac{T}{\nu}-1}}}\left(\frac{x^2}{\left(\frac{2\sigma^2}{\nu}+\theta^2\right)}\right)^{\frac{T}{2\nu}-0.25} \times\frac{K_{\frac{T}{\nu}-0.5}(\frac{x^2}{\sigma^2}(\frac{2\sigma^2}{\nu}+\theta^2))}{\sqrt{x^2+\frac{\sigma^2T^2}{k}}}\ .\square$$

Proof: Likewise, we conduct derivations for NIG model for similar arguments; let us define log discounted stock price, $log(S(T)e^{-rT}) = Y(T) - \phi(-i)$ where $Y(T)$ is a NIG process as usual. Then, under the risk-neutral probability $\mathbb{Q}^S$, we can obtain the characteristic function of log-discounted stock price,

$$\mathbb{E}^S(e^{iuX(T)}) = S(0)\mathbb{E}[\frac{S(T)}{S(0)e^{rT}}e^{iuY(T)}] = \mathbb{E}\left[e^{Y(T)(iu+1)}\right]e^{-\phi(-i)T} = \mathbb{E}\left[e^{Y(T)i(u-i)}\right]e^{-\phi(-i)T}.$$

Given the characteristic function of NIG and defining $v = u - i$ we have

$$\frac{\Phi_{\mathrm{NIG}}(v)}{\Phi_{NIG}(-i)} = (\sqrt{\frac{1+u^2\sigma^2\,k - 2ui\sigma^2k - 2iu\theta k}{1-\sigma^2k-2\theta k}}) = (\sqrt{\frac{1+u^2\sigma^2k-2uik\sigma^2+\theta)}{1-\sigma^2k-2\theta\, k}}).$$

Now, by letting $\omega = \frac{k}{\{1-\sigma^2\, k-2\theta\, k}$ and $\theta^S = \theta + \sigma^2$ and following some algebraic manipulations, we finally obtain the characteristic function under the risk-neutral probability, $\mathbb{Q}^S$, measure as follows

$$\Phi^S(u,t) = \frac{t}{\sqrt{k\omega}}\left(1-\sqrt{1+u^2\sigma^2\omega-2ui\theta^S\omega}\right). \qquad (2)$$

To derive the probability distribution function (pdf) of NIG process under $\mathbb{Q}^S$ measure, (2) (characteristic function under $\mathbb{Q}^S$) is needed. Using 3.471.9 from Zwillinger and Jeffrey (2007), we end up with the following density,

$$f^S(x,\theta,k,\sigma) = \frac{T}{\pi}exp\left(\frac{T}{\sqrt{k\omega}}+\frac{\theta x}{\sigma^2}\right)\times\frac{K_1(\sqrt{\theta^2+\frac{\sigma^2}{k}}\frac{1}{\sigma^2}\sqrt{x^2+\frac{T^2}{\sigma^2 k}})}{\sqrt{x^2+\frac{\sigma^2T^2}{k}}}$$

under risk-neutral probability measure $\mathbb{Q}$, we used the following density function given by Tankov (2003)

$$f(x,\theta,k,\sigma) = \frac{T}{\pi}exp\left(\frac{T}{\sqrt{k}}+\frac{\theta x}{\sigma^2}\right)\times\frac{K_1(\sqrt{\theta^2+\frac{\sigma^2}{k}}\frac{1}{\sigma^2}\sqrt{x^2+\frac{T^2}{\sigma^2 k}})}{\sqrt{x^2+\frac{\sigma^2T^2}{k}}}$$

Using these densities, we prefer to calculate CDF by numerical integration. Then, it is convenient to arrive following the BS type option price formula,

$$C(S(t),K,r,T-t,\theta,\theta_S,k,\sigma) = S(t)F^S(x,\theta_S,k,\sigma) - Ke^{-r(T-t)}F(x,\theta,k,\sigma),$$

where $x = log(\frac{S(t)}{K} + (r-\phi(-i))(T-t)$. □