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SOME BOUNDS FOR THE k-GENERALIZED DIGAMMA FUNCTION

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Abstract. We presented some monotonicity properties for the k-generalized digamma function $\psi_k(h)$ and we established some new bounds for $\psi_k^{(s)}(h), s \in \mathbb{N} \cup \{0\}$, which refine recent results.

1. Introduction

The ordinary Gamma function is given by [1]:

$$\Gamma(h) = \lim_{s\to\infty} \frac{s!\, s^{h-1}}{h(h+1)(h+2)\cdots(h+(s-1))}, \qquad h > 0$$

was discovered by Euler when he generalized the factorial function to non integer values. The digamma function is the logarithmic derivative of the ordinary gamma function and is given by [1]:

$$\psi(1+h) = -\gamma + \sum_{s=1}^{\infty} \frac{h}{s(h+s)}, \qquad h > -1$$

where $\gamma = \lim_{m\to\infty}\left(\sum_{s=1}^{m} \frac{1}{s} - \log m\right) \simeq 0.577$ is the Euler-Mascheroni constant. In 2006, Kirchhoff applied the polygamma functions in the field of physics [3] and many series involving polygamma functions appeared in Feynman calculations [8]. In 2021, Wilkins and Hromadka [16] use the digamma function, as well as new

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variants of the digamma function, as a new family of basis functions in mesh-free numerical methods for solving partial differential equations. polygamma functions are used to approximate the values of many special functions and have many applications in physics, statistics and applied mathematics [14].

Many mathematicians studied the completely monotonic (CM) of some functions including the digamma function to deduce some of its bounds. An infinitely differentiable function $L(h)$ on $\mathbb{R}^+$ is CM if $(-1)^s L^{(s)}(h) \geq 0$ for $s \in \mathbb{N} \cup \{0\}$. A theorem [15, Theorem 12b] stated the sufficient condition for $L(h)$ being CM on $\mathbb{R}^+$ as:

$$L(h) = \int_0^\infty e^{-hy} dv(y),$$

where $v(y)$ is non-decreasing and the integral converges for $h \in \mathbb{R}^+$.

In 2006, Muqattash and Yahdi [10] presented the following inequality:

$$\ln h < \psi(1+h) < \ln(1+h), \qquad h \in \mathbb{R}^+. \tag{1}$$

In 2011, Batir [2] presented the following inequalities:

$$\ln\Big(h^2 + h + e^{-2\gamma}\Big) \leq 2\psi(h+1) < \ln\Big(h^2 + h + \frac{1}{3}\Big), \qquad h \in [0, \infty) \tag{2}$$

$$\ln\Big(\frac{2h+2}{e^{\frac{2}{h+1}} - 1}\Big) < 2\psi(h+1) \leq \ln\Big(\frac{2h + (e^2-1)e^{-2\gamma}}{e^{\frac{2}{1+h}} - 1}\Big), \qquad h \in [0, \infty) \tag{3}$$

and

$$\Big(\frac{1+2h}{2}\Big)e^{-2\psi(1+h)} < \psi'(1+h) < \Big(\frac{\pi^2\ e^{-2\gamma} + 6h}{6}\Big)e^{-2\psi(h+1)}, \qquad h \in (0, \infty). \tag{4}$$

In 2014, Guo and Qi [5] refined the inequality (1) by

$$\ln(h + 1/2) < \psi(1+h) < \ln\left(e^{-\gamma} + h\right), \qquad h \in \mathbb{R}^+. \tag{5}$$

Diaz and Pariguan [4] presented the $k-$generalized gamma function as:

$$\Gamma_k(h) = \lim_{s\to\infty} \frac{s!\ k^s\ (sk)^{\frac{h}{k}-1}}{h(k+h)(2k+h)\cdots((s-1)k+h)}, \qquad k, h \in \mathbb{R}^+.$$

Mansour [7] determined the Γ_k by a combination of some functional equations. The k-analogue of the digamma function is introduced by [11]

$$\psi_k(h) = \frac{-1}{k}(\gamma - \ln k) - 1/h - \sum_{s=1}^{\infty}\left(\frac{1}{sk+h} - \frac{1}{sk}\right), \qquad k, h \in \mathbb{R}^+$$

and it has the following relations for $h, k \in \mathbb{R}^+$ and $s \in \mathbb{N} \cup \{0\}$

$$k\psi_k(h) - \psi\left(\frac{h}{k}\right) = \ln k, \quad \psi_k^{(s)}(k+h) = \frac{(-1)^s\ s!}{h^{s+1}} + \psi_k^{(s)}(h)\ and\ \psi_k'(k) = \frac{\pi^2}{6k^2}. \tag{6}$$

In 2018, Nantomah, Nisar and Gehlot [12] introduced the following integral formulas:

$$\psi_k(h) = \int_0^\infty \left(\frac{2e^{-y} - e^{-ky}}{ky} - \frac{e^{-hy}}{1 - e^{-ky}} \right) dy, \qquad h, k > 0 \tag{7}$$

and

$$\psi_k^{(s)}(h) = (-1)^{s+1} \int_0^\infty y^s \left(\frac{e^{-hy}}{1 - e^{-ky}} \right) dy, \qquad h, k > 0; s \in \mathbb{N}. \tag{8}$$

Yin, Huag, Song and Dou [19] deduced the following inequality:

$$0 \le \psi_k'(h) - \frac{1}{kh} \le \frac{1}{h^2}, \qquad h, k \in \mathbb{R}^+. \tag{9}$$

In 2020, Yildrim [17] deduced the following inequality:

$$-\frac{k}{12h^2} < \psi_k(h+k) - \frac{1}{k}\ln h - \frac{1}{2h} < 0, \qquad h, k \in \mathbb{R}^+. \tag{10}$$

In 2021, Moustafa, Almuashi and Mahmoud [9] presented the following asymptotic formulas for $k > 0$:

$$\psi_k(h) \sim \frac{1}{k}\ln h - \frac{1}{2h} - \sum_{m=1}^{\infty} \frac{k^{2m-1}\, B_{2m}}{(2m)\, h^{2m}}, \qquad h \to \infty \tag{11}$$

and for $s \in \mathbb{N}$,

$$\psi_k^{(s)}(h) \sim \frac{(-1)^{s-1}(s-1)!}{kh^s} - \frac{(-1)^s s!}{2h^{s+1}} + (-1)^{s+1} \sum_{m=1}^{\infty} \frac{(s+2m-1)!\, k^{2m-1}\, B_{2m}}{(2m)!\, h^{2m+s}}, \ h \to \infty \tag{12}$$

and they also deduced the inequalities:

$$\frac{1}{k}\ln h + \frac{1}{h} - \frac{k}{2}\psi_k'(h) < \psi_k(k+h) < \frac{\ln h}{k} + 1/h - \frac{k}{2}\psi_k'\Big(\frac{k+3h}{3}\Big), \qquad h, k > 0 \tag{13}$$

where the upper bound of (13) refines upper bound of (10) for all $h > \frac{k}{3}$, and for $s \in \mathbb{N}, \quad h, k \in \mathbb{R}^+$

$$\frac{(s-1)!}{kh^s} + \frac{(-1)^s k}{2}\psi_k^{(s+1)}\Big(h + \frac{k}{3}\Big) < (-1)^{s+1}\psi_k^{(s)}(h) < \frac{(s-1)!}{kh^s} + \frac{(-1)^s k}{2}\psi_k^{(s+1)}(h). \tag{14}$$

Notes: All of the $k-$digamma function results allow us to make new conclusions about the classical digamma function or new proofs for some of its established conclusions when k tends to one, and likewise for the $k-$gamma function [6, 18, 20]. For extra information about Γ_k and ψ_k functions, see [4, 7, 9, 19] and the related references therein.

We will introduce two CM functions involving $\psi_k(h)$ and $\psi_k'(h)$ functions. Some new bounds for $\psi_k^{(s)}(h)$ functions ($s \in \mathbb{N} \cup \{0\}$) will be deduced, which generalize and refine some recent results. Also, we will study the monotonicity of two functions containing the $k-$generalized digamma function and consequently, we will deduce some new best bounds for $\psi_k^{(s)}(h)$ functions $(s \in \mathbb{N} \cup \{0\})$.

2. Auxiliary Results

In [13], the following corollary was introduced:

Corollary 1. *Assume that S is a function defined on $h > h_0$, $h_0 \in \mathbb{R}$ with $\lim_{h\to\infty} S(h) = 0$. Then for $\omega \in \mathbb{R}^+$, $S(h) > 0$, if $S(h+\omega) < S(h)$ for $h > h_0$ and $S(h) < 0$, if $S(h+\omega) > S(h)$ for $h > h_0$.*

Using the monotonicity properties, we can conclude the following results:

Lemma 1.

$$\ln\left(\frac{h^2+3h+3}{3}\right) < h, \qquad \forall h \in \mathbb{R}^+, \tag{15}$$

$$\ln\left(\frac{1}{e^{-\gamma}+h}+1\right) < \frac{1}{1+h}, \qquad \forall h > \frac{e^{2\gamma}-e^{\gamma}-1}{e^{\gamma}\Big(2-e^{\gamma}\Big)} \simeq 1.00313 \tag{16}$$

and

$$\frac{2}{1+h} < \ln\left(\frac{2(1+h)}{h^2+h+e^{-2\gamma}}+1\right), \qquad \forall h > \frac{1}{\sqrt{e^{2\gamma}\Big(-3+e^{2\gamma}\Big)}} - 1 \simeq 0.352938. \tag{17}$$

Proof. Let the function $L(h) = \ln\left(\frac{h^2+3h+3}{3}\right) - h$ and then $L'(h) = \frac{-h(1+h)}{3+3h+h^2} < 0$ for all $h > 0$ and then $L(h)$ is decreasing on $(0,\infty)$ with $\lim_{h\to 0^+} L_k(h) = 0$ and then $L_k(h) < 0$ for all $h > 0$ which proves (15). Secondly, we let the function $C(h) = \ln\left(\frac{1}{e^{-\gamma}+h}+1\right) - \frac{1}{1+h}$. Then

$$C'(h) = \frac{1+e^{\gamma}-e^{2\gamma}+e^{\gamma}\Big(2-e^{\gamma}\Big)h}{(1+h)^2\Big(1+e^{\gamma}h\Big)\Big(1+e^{\gamma}(1+h)\Big)} > 0,\ \ h > h_1 = \frac{-1-e^{\gamma}+e^{2\gamma}}{e^{\gamma}\Big(2-e^{\gamma}\Big)} \simeq 1.00313.$$

Then $C(h)$ is increasing on (h_1,∞) with $\lim_{h\to\infty} C(h) = 0$ and this proves (16). By the same way, we obtain (17). □

Lemma 2. *For $k \in \mathbb{R}^+$, we have*

$$k\psi_k(k+h) < \gamma + \ln\left(\frac{3}{\pi^2}\right) + \ln(2h+k), \qquad \forall h > \frac{k}{2}. \tag{18}$$

Proof. Let the function $N_k(h) = \ln\left(\frac{3}{\pi^2}\right) + \gamma + \ln(2h+k) - k\psi_k(k+h)$. Then $N_k'(h) = \frac{2}{k+2h} - k\psi_k'(k+h)$ and by using (6), we obtain

$$N_k'(h) - N_k'(k+h) = \frac{k^3}{(3k+2h)(k+2h)(k+h)^2} > 0, \qquad h,k>0.$$

Using the asymptotic formula (12), we have $\lim_{h\to\infty} N_k'(h) = 0$ and then Corollary 1 gives us that $N_k'(h) > 0$ for $h > 0$ and $k > 0$. Then, we have $N_k(h)$ is increasing on $\mathbb{R}^+$ and by using (6) again, we get $N_k(h) > N_k\left(\frac{k}{2}\right) \simeq 0.0430254 > 0$ for all $h > \frac{k}{2}$ and $k > 0$. □

Lemma 3. *For $k > 0$, we have*

$$e^{k\psi_k(h+2k)} < e^{k\psi_k(h+k)} + k, \qquad \forall h > 0. \tag{19}$$

Proof. Set the function $B_k(h) = e^{k\psi_k(h+2k)} - e^{k\psi_k(h+k)} - k$. Then by using (6), we get

$$\frac{B_k(h+k) - B_k(h)}{e^{k\psi_k(h+k)}} = e^{\frac{k}{h+k} + \frac{k}{h+2k}} - 2e^{\frac{k}{h+k}} + 1 \doteqdot D_k(h).$$

Then

$$\frac{(h+k)^2}{ke^{\frac{k}{h+k}}} D_k'(h) = 2 - \frac{(5k^2+6kh+2h^2)e^{\frac{k}{h+2k}}}{(2k+h)^2} \doteqdot f_k(h).$$

Then $f_k'(h) = \frac{k^3 e^{\frac{k}{h+2k}}}{(2k+h)^4} > 0$ for all $h,k > 0$ and hence $f_k(h)$ is increasing on $\mathbb{R}^+$ with $\lim_{h\to\infty} f_k(h) = 0$. Then $f_k(h) < 0$ for $h,k > 0$ and then $D_k(h)$ is decreasing on $\mathbb{R}^+$ with $\lim_{h\to\infty} D_k(h) = 0$. Then $B_k(h+k) - B_k(h) > 0$ for $h,k > 0$. Using the asymptotic formula (11), we have $\lim_{h\to\infty} B_k(h) = 0$ and then Corollary 1 gives us that $B_k(h) < 0$ for all $h,k > 0$. □

Lemma 4. *For $k > 0$, we have*

$$e^{2k\psi_k(h+2k)} > e^{2k\psi_k(h+k)} + 2k(h+k), \qquad \forall h > 0. \tag{20}$$

Proof. Set the function $m_k(h) = e^{2k\psi_k(h+2k)} - e^{2k\psi_k(h+k)} - 2k(h+k)$. Then

$$m_k(h+k) - m_k(h) = e^{2k\psi_k(h+3k)} - 2e^{2k\psi_k(h+2k)} + e^{2k\psi_k(h+k)} - 2k^2 \doteqdot t_k(h).$$

Then by using (6), we get

$$\frac{t_k(h+k) - t_k(h)}{e^{2k\psi_k(h+k)}} = e^{\frac{2k}{h+k} + \frac{2k}{h+2k} + \frac{2k}{h+3k}} - 3e^{\frac{2k}{h+k} + \frac{2k}{h+2k}} + 3e^{\frac{2k}{h+k}} - 1 \doteqdot s_k(h).$$

Then

$$\begin{aligned}\frac{(h+k)^2 s_k'(h)}{2ke^{\frac{2k}{h+k}}} &= -\frac{(49k^4+96k^3h+72k^2h^2+24kh^3+3h^4)e^{\frac{2k}{h+2k}+\frac{2k}{h+3k}}}{(2k+h)^2(3k+h)^2} \\ &+ \frac{3(5k^2+6kh+2h^2)}{(2k+h)^2\ e^{\frac{-2k}{h+2k}}} - 3 \doteqdot u_k(h)\end{aligned}$$

Then

$$\frac{(2k+h)^4 u_k'(h)}{2k(3k^2+3kh+h^2)e^{\frac{2k}{h+2k}}} = \frac{379k^6+959k^5h+1049k^4h^2+626k^3h^3+213k^2h^4}{(3k^2+3kh+h^2)(3k+h)^4\ e^{\frac{-2k}{h+3k}}} + \frac{39kh^5+3h^6}{(3k^2+3kh+h^2)(3k+h)^4\ e^{\frac{-2k}{h+3k}}} - 3 \doteqdot w_k(h).$$

Then

$$w_k'(h) = \frac{-2k^5\Big(129k^4+240k^3h+172k^2h^2+56kh^3+7h^4\Big)e^{\frac{2k}{h+3k}}}{(3k+h)^6(3k^2+3kh+h^2)^2} < 0, \qquad h,k>0$$

and hence $w_k(h)$ is decreasing on $(0,\infty)$ with $\lim_{h\to\infty} w_k(h)=0$. Then $w_k(h)>0$ for $h,k\in\mathbb{R}^+$ and then $u_k(h)$ is increasing on $\mathbb{R}^+$ with $\lim_{h\to\infty} u_k(h)=0$. Then $s_k(h)$ is decreasing on $\mathbb{R}^+$ with $\lim_{h\to\infty} s_k(h)=0$. Then $t_k(h+k)-t_k(h)>0$ for $h,k\in\mathbb{R}^+$. Using the asymptotic formula (11), we have $\lim_{h\to\infty} t_k(h)=0$ and then Corollary 1 gives us that $t_k(h)<0$ for all $h,k\in\mathbb{R}^+$. Then $m_k(h+k)-m_k(h)<0$ for $h,k\in\mathbb{R}^+$ with $\lim_{h\to\infty} m_k(h)=0$ and then $m_k(h)>0$ for all $h,k\in\mathbb{R}^+$. □

3. Some CM Monotonic Functions

Theorem 1. *Assume that $h,k>0$. Then the function*

$$U_{\beta,k}(h) = \psi_k'(h) - \frac{2}{kh} + \frac{2}{k^2}\ln\left(1+\frac{\beta k}{h}\right)$$

is CM on $\mathbb{R}^+$ if and only if $\beta \geq \frac{1}{2}$.

Proof.

$$U_{\beta,k}'(h) = \psi_k''(h) + \frac{2}{kh^2} + \frac{2}{k^2}\left(\frac{1}{h+\beta k} - \frac{1}{h}\right)$$

and by using (8) and the identity $\frac{1}{h^l} = \frac{1}{(l-1)!}\int_0^\infty y^{l-1}e^{-hy}dy$ for $h>0$, (see [1]), we have

$$U_{\beta,k}'(h) = \int_0^\infty \frac{2e^{-hy}}{k^2(e^{ky}-1)}\phi_k(y)dy,$$

where

$$\phi_k(y) = (e^{ky}-1)(e^{-\beta ky}-1) + ky\Big(e^{ky}-1\Big) - \frac{(ky)^2}{2}e^{ky}.$$

Let $\beta \geq \frac{1}{2}$. Then

$$e^{\frac{ky}{2}}\phi_k(y) \leq e^{ky} - e^{\frac{3ky}{2}} + e^{\frac{ky}{2}} - 1 + ky\Big(e^{\frac{3ky}{2}} - e^{\frac{ky}{2}}\Big) - 1/2(ky)^2 e^{\frac{3ky}{2}}$$

$$= \sum_{m=1}^{\infty} \frac{n(m)}{2^{m+2}\,(m+2)!}(ky)^{m+2}$$

where

$$\begin{aligned} n(m) &= 2^{m+2} - 3^{m+2} + 1 + 2(m+2)\Big(3^{m+1} - 1\Big) - 2(m+2)(m+1)3^m \\ &= -2(m+2)\Big((m-1)2^m + 1\Big) - \sum_{s=1}^{m} \frac{(m+2)(m+1)(m-s)}{(m+2-s)} \tbinom{m}{s} 2^{m+1-s} \\ &< 0 \end{aligned}$$

and then$-U'_{\beta,k}(h)$ is CM on $\mathbb{R}^+$ and hence $U_{\beta,k}(h)$ is decreasing on $\mathbb{R}^+$. Using the asymptotic formula (12), we have $\lim_{h\to\infty} U_{\beta,k}(h) = 0$ and then $U_{\beta,k}(h) > 0$. Then $U_{\beta,k}(h)$ is CM on $\mathbb{R}^+$ for $\beta \geq \frac{1}{2}$. On the other side, if $U_{\beta,k}(h)$ is CM, then by using again the asymptotic formula (12), we get $\lim_{h\to\infty} h\ U_{\beta,k}(h) = \frac{2\beta-1}{k} \geq 0$ and hence $\beta \geq \frac{1}{2}$. □

Theorem 2. *Assume that $h, k > 0$ and $\lambda \in \mathbb{R}$. Then the function*

$$F_{\lambda,k}(h) = \psi_k(h+k) - \frac{1}{k}\ln(h+\lambda k)$$

is CM on $\mathbb{R}^+$ if and only if $\lambda \leq \frac{1}{2}$. Also, the function $-F_{\lambda,k}(h)$ is CM on $\mathbb{R}^+$ if $\lambda \geq 1$.

Proof.

$$F'_{\lambda,k}(h) = -\frac{1}{k(h+\lambda k)} + \psi'_k(h+k) = \int_0^\infty \frac{e^{-hy}}{k(e^{ky}-1)}\varphi_k(y)dy,$$

where

$$\varphi_k(y) = ky - e^{-\lambda ky}\Big(e^{ky} - 1\Big).$$

Let $\lambda \leq \frac{1}{2}$, then we obtain

$$\begin{aligned} e^{\frac{ky}{2}}\varphi_k(y) &\leq 1 + ky\ e^{\frac{ky}{2}} - e^{ky} \\ &= -\sum_{l=2}^{\infty} \frac{\Big(2^l - l - 1\Big)(ky)^{1+l}}{2^l\ (1+l)!} \\ &= -\sum_{l=2}^{\infty} \frac{\Big(\sum_{s=2}^{l} \binom{l}{s}\Big)\ (ky)^{1+l}}{2^l\ (1+l)!} < 0 \end{aligned}$$

and consequently, $-F'_{\lambda,k}(h)$ is CM on $\mathbb{R}^+$ for $\lambda \leq \frac{1}{2}$ and hence $F_{\lambda,k}(h)$ is decreasing on $\mathbb{R}^+$. Using the asymptotic formula (11), we obtain $\lim_{h\to\infty} F_{\lambda,k}(h) = 0$ and then $F_{\lambda,k}(h) > 0$. Hence $F_{\lambda,k}(h)$ is CM on $\mathbb{R}^+$ for $\lambda \leq \frac{1}{2}$. On the other hand, if $F_{\lambda,k}(h)$ is CM, then by using again the asymptotic formula (11), we obtain $\lim_{h\to\infty} h\ F_{\lambda,k}(h) = \frac{1}{2} - \lambda \geq 0$ and then $\lambda \leq \frac{1}{2}$. Now for $\lambda \geq 1$, we have $e^{ky}\varphi_k(y) \geq \sum_{l=1}^{\infty} \frac{l\ (ky)^{l+1}}{(l+1)!} > 0$

and consequently, $F'_{\lambda,k}(h)$ is CM on $\mathbb{R}^+$ for $\lambda \geq 1$ and hence $F_{\lambda,k}(h)$ is increasing on $\mathbb{R}^+$ with $\lim\limits_{h\to\infty} F_{\lambda,k}(h) = 0$ and then $F_{\lambda,k}(h) < 0$. Then $-F_{\lambda,k}(h)$ is CM on $\mathbb{R}^+$ for $\lambda \geq 1$. □

4. Some Inequalities for the ψ_k and $\psi_k^{(s)}$ Functions

Let us mention some important consequences of Theorems 1 and 2.

Corollary 2. *Let $a \in (0,\infty)$. Then we have*

$$\frac{1}{kh} - \frac{1}{k^2}\ln\left(\frac{ak+h}{h}\right) < \frac{\psi'_k(h)}{2}, \qquad k, h \in \mathbb{R}^+ \tag{21}$$

with the best possible constant $a = \frac{1}{2}$.

Proof. The inequality (21) at $a = \frac{1}{2}$ follows from $U_{\frac{1}{2},k}(h) > 0$ in Theorem 1 and the inequality (21) is equivalent that $h\ U_{a,k}(h) > 0$ which yields $a \geq \frac{1}{2}$ as stated when we proved Theorem 1. Then $a = \frac{1}{2}$ is the best in (21), since the logarithmic function is strictly increasing on $\mathbb{R}^+$. □

Remark 1. *Using the identity* $\ln(1+h) < h$ *for all* $h > -1$*, (see [1]) yields the lower bound of (21) refines the lower bound of (9) for all $h, k > 0$.*

Corollary 3. *Let $a \in (0,\infty)$ and $s = 1, 2, 3, \cdots$. Then we have*

$$\frac{2\ s!}{kh^{s+1}} + \frac{2\ (s-1)!}{k^2}\Big(\frac{1}{(h+ak)^s} - \frac{1}{h^s}\Big) < (-1)^s\psi_k^{(s+1)}(h), \qquad h, k \in (0,\infty) \tag{22}$$

with the best possible constant $a = \frac{1}{2}$.

Proof. The inequality (22) at $a = \frac{1}{2}$ follows from $(-1)^s U^{(s)}_{\frac{1}{2},k}(h) > 0$ in Theorem 1 and the inequality (22) is equivalent that $h^{s+1}\ (-1)^s U^{(s)}_{a,k}(h) > 0$. Using the asymptotic expansion (12), we have $\lim\limits_{h\to\infty} h^{s+1}\ (-1)^s U^{(s)}_{a,k}(h) = \frac{s!}{k}\Big(2a-1\Big) \geq 0$ and hence $a \geq \frac{1}{2}$. Using the decreasing property of the function $\frac{1}{h^s}$ on $(0,\infty)$ for $s = 1, 2, 3, \cdots$, we deduce that $a = \frac{1}{2}$ is the best possible constant in (22). □

Corollary 4. *Let $a \in [0,\infty)$. Then we have*

$$\ln(h+ak) < k\psi_k(k+h) < \ln(k+h), \qquad k, h \in \mathbb{R}^+ \tag{23}$$

with the best possible constant $a = \frac{1}{2}$.

Proof. The inequality (23) at $a = \frac{1}{2}$ is deduced from $F_{\frac{1}{2},k}(h) > 0$ and $F_{1,k}(h) < 0$ in Theorem 2. The left-hand side of (23) is equivalent that $h\ F_{a,k}(h) > 0$ and this gives $a \leq \frac{1}{2}$ as stated when we proved Theorem 2. Then $a = \frac{1}{2}$ is the best in (23). □

Remark 2.
- *Letting $k = 1$ and $a = 0$ in (23), we obtain (1).*

- *Using (21), we deduce that the lower bound of (23) refines the lower bound of (13) for every* $k, h \in \mathbb{R}^+$.

Corollary 5. *Let* $a \in [0, \infty)$ *and* $s = 1, 2, 3, \cdots$. *Then we have*

$$\frac{s!}{h^{s+1}}+\frac{(s-1)!}{k\ (h+k)^s} < (-1)^{s+1}\psi_k^{(s)}(h) < \frac{s!}{h^{s+1}}+\frac{(s-1)!}{k\ (h+ak)^s}, \qquad h,k \in (0,\infty) \quad (24)$$

with the best possible constant $a = \frac{1}{2}$.

Proof. The inequality (24) at $a = \frac{1}{2}$ is deduced from $(-1)^s F^{(s)}_{\frac{1}{2},k}(h) > 0$ and $(-1)^s F^{(s)}_{1,k}(h) < 0$ in Theorem 2. The right-hand side of (24) is equivalent that $h^{1+s}\ (-1)^s F^{(s)}_{a,k}(h) > 0$. Using the asymptotic expansion (12), we have

$$\lim_{h\to\infty} h^{s+1}\ (-1)^s F^{(s)}_{a,k}(h) = s!\Big(\frac{1}{2}-a\Big) \geq 0$$

and hence $a \leq \frac{1}{2}$. Then $a = \frac{1}{2}$ is the best possible constant in (24). □

Remark 3. *Using (22), we deduce that the upper bound of (24) refines the upper bound of (14) for every* $s \in \mathbb{N}$ *and* $h, k > 0$.

Lemma 5. *For* $k > 0$*, the function*

$$T_k(h) = e^{k\psi_k(h+k)} - h \quad (25)$$

is strictly decreasing convex on $(-k, \infty)$ *with* $\lim_{h\to\infty} T_k(h) = \frac{k}{2}$ *and* $\lim_{h\to 0} T_k(h) = ke^{-\gamma}$.

Proof. Using (6), we have $\lim_{h\to 0} T_k(h) = ke^{-\gamma}$. Differentiating (25) yields

$$T_k'(h) = -1 + k\psi_k'(h+k)e^{k\psi_k(k+h)}$$

and

$$\frac{T_k''(h)}{k\ e^{k\psi_k(k+h)}} = k\Big[\psi_k'(k+h)\Big]^2 + \psi_k''(k+h) \doteqdot S_k(h).$$

Applying (6), we get

$$\frac{(k+h)^2}{2k}\Big[S_k(k+h) - S_k(h)\Big] = -\psi_k'(k+h) - \frac{2h^2+4kh+2k^2-1}{2(h+k)^2} \doteqdot Q_k(h).$$

Applying (6) again, we get

$$Q_k(k+h) = Q_k(h) + \frac{A_k(k+h)}{2(k+h)^2(2k+h)^2},$$

where

$$A_k(h) = k^2 + 2kh + 2h^2 > 0, \qquad h, k \in \mathbb{R}^+$$

and then $Q_k(k+h) > Q_k(h)$ for all $h > -k$ and by using the asymptotic formula (12), we have $\lim_{h\to\infty} Q_k(h) = 0$ and then Corollary 1 gives us $Q_k(h) < 0$ for every $h > -k$. Consequently, we have $S_k(k+h) < S_k(h)$ for all $h > -k$ and by using the asymptotic expansion (12), we have $\lim_{h\to\infty} S_k(h) = 0$ and then $T_k''(h) > 0$ for every $h > -k$. Then $T_k'(h)$ is strictly increasing on $(-k,\infty)$. By using the asymptotic formulas (11) and (12), we have

$$\lim_{h\to\infty} T_k'(h) = 0 \textit{ and } \lim_{h\to\infty} T_k(h) = k/2.$$

Then $T_k'(h) < \lim_{h\to\infty} T_k'(h) = 0$ and this finishes the proof. □

And consequently, we have the following Corollary:

Corollary 6. *Set a and b be positive real numbers. Then we have*

$$\ln(h+ak) < k\psi_k(h+k) < \ln(h+bk), \qquad h,k\in\mathbb{R}^+ \tag{26}$$

where $a=\frac{1}{2}$ and $b=\frac{1}{e^\gamma}\simeq 0.56$ being the best.

Remark 4.
- *Letting $k=1$ in (26), we obtain (5).*
- *The upper bound of (26) refines the upper bound of (23) for all $h,k>0$.*

Lemma 6. *For $h\geq 0$ and $k\in\mathbb{R}^+$,*

$$\ln\left(\frac{c\,k}{e^{\frac{k}{k+h}}-1}\right) \leq k\psi_k(k+h) < \ln\left(\frac{d\,k}{e^{\frac{k}{k+h}}-1}\right), \tag{27}$$

where the constants $c = e^{-\gamma}(e-1)\simeq 0.965$ and $d=1$ are the best possible.

Proof. Set

$$f_k(h) = T_k(k+h) - T_k(h) = -k + e^{k\psi_k(k+h)}\left(e^{\frac{k}{k+h}}-1\right), \qquad h\geq 0 \textit{ and } k>0.$$

Since $T_k'(h)$ is strictly increasing on $(-k,\infty)$, then $f_k(h)$ is strictly increasing on $[0,\infty)$ and by using (6) and the asymptotic expansion (11), we get

$$f_k(0) = -k + ke^{-\gamma}(e-1) \leq -k + e^{k\psi_k(k+h)}\left(e^{\frac{k}{k+h}}-1\right) < \lim_{h\to\infty} f_k(h) = 0$$

and this gives (27). □

Remark 5. *Using (16), we deduce that the upper bound of (27) refines the upper bound of (26) for all $h > 1.00313k$ and $k>0$.*

Lemma 7. *For $h>0$ and $k>0$, we have*

$$g\ e^{-k\psi_k(h+k)} < k\psi_k'(k+h) < r\ e^{-k\psi_k(k+h)}, \tag{28}$$

where the constants $g = \frac{\pi^2\ e^{-\gamma}}{6}\simeq 0.924$ and $r=1$ are the best possible.

Proof. By using the increasing property of $T_k'(h)$ on $(-k,\infty)$, we have

$$T_k'(0) = -1 + k\psi_k'(k)e^{k\psi_k(k)} < -1 + k\psi_k'(k+h)e^{k\psi_k(k+h)} < \lim_{h\to\infty} T_k'(h) = 0.$$

Using (6) yields

$$\frac{\pi^2\ e^{-\gamma}}{6} < k\psi_k'(k+h)e^{k\psi_k(k+h)} < 1,$$

which finishes the proof. □

Remark 6. *Using (23) yields* $e^{-k\psi_k(k+h)} < \frac{2}{2h+k}$ *for every* $h, k > 0$ *and then the upper bound of (28) refines the upper bound of (24) at* $s = 1$ *for all* $h, k > 0$.

Lemma 8. *For* $h > 0$ *and* $k > 0$,

$$1 - e^{-\frac{k}{h+k}} + \frac{k^2}{(h+k)^2} < k^2\psi_k'(h+k) < e^{\frac{k}{h+k}} - 1. \tag{29}$$

Proof. By applying the mean value theorem to T_k on the interval $[h, h+k]$, we obtain

$$\frac{-T_k(h) + T_k(k+h)}{k} = T_k'(h + \alpha_h), \qquad 0 < \alpha_h < k.$$

By using the increasing property of $T_k'(h)$ on $(-k, \infty)$, we obtain

$$T_k'(h) < T_k'(h + \alpha_h) < T_k'(k+h), \qquad 0 < \alpha_h < k.$$

Combining the last two relations yields

$$T_k'(h) < \frac{-T_k(h) + T_k(k+h)}{k} < T_k'(k+h)$$

and this gives us (29). □

Remark 7. *Using (19), we deduce that the upper bound of (29) refines the upper bound of (28) for every* $h, k \in \mathbb{R}^+$.

Lemma 9. *For* $k > 0$*, the function*

$$W_k(h) = e^{2k\psi_k(h+k)} - h^2 - hk \tag{30}$$

is strictly increasing concave in $(-k, \infty)$ *with* $\lim\limits_{h\to\infty} W_k(h) = \frac{k^2}{3}$ *and* $\lim\limits_{h\to 0} W_k(h) = k^2 e^{-2\gamma}$.

Proof. Using (6), we have $\lim\limits_{h\to 0} W_k(h) = k^2 e^{-2\gamma}$. Differentiating (30) yields

$$W_k'(h) = -2h - k + 2k\psi_k'(h+k)e^{2k\psi_k(k+h)},$$

$$\frac{1}{2}W_k''(h) = -1 + ke^{2k\psi_k(k+h)}\left[\psi_k''(k+h) + 2k\Big(\psi_k'(k+h)\Big)^2\right]$$

and

$$\frac{1}{2ke^{2k\psi_k(k+h)}}W_k'''(h) = \psi_k'''(k+h) + 6k\psi_k'(k+h)\psi_k''(k+h) + 4k^2\Big(\psi_k'(k+h)\Big)^3 \doteqdot V_k(h).$$

Applying (6), we get

$$\begin{aligned}\frac{(h+k)^2}{2k}\Big[V_k(k+h) - V_k(h)\Big] &= -3\psi_k''(k+h) + \frac{6(h+2k)}{(h+k)^2}\psi_k'(h+k)\\ &\quad - 6k\Big(\psi_k'(h+k)\Big)^2 - \frac{11k^2 + 12kh + 3h^2}{k(h+k)^4} \doteqdot U_k(h).\end{aligned}$$

Similarly, we get

$$\begin{aligned}\frac{(h+k)^2(h+2k)^2}{6k(3k^2+3kh+h^2)}\Big[U_k(h+k)-U_k(h)\Big] &= \psi_k'(h+k)\\ &\quad-\frac{114k^5+298k^4h+321k^3h^2}{6k(k+h)^2(2k+h)^2(3k^2+3kh+h^2)}\\ &\quad-\frac{178k^2h^3+51kh^4+6h^5}{6k(k+h)^2(2k+h)^2(3k^2+3kh+h^2)}\\ &\doteqdot H_k(h).\end{aligned}$$

And finally, we get

$$H_k(k+h)-H_k(h)=-\frac{k^4P_k(k+h)}{3(k+h)^2(2k+h)^2(3k+h)^2(3k^2+3kh+h^2)(7k^2+5kh+h^2)}$$

where

$$P_k(h)=12k^4+36k^3h+46k^2h^2+28kh^3+7h^4>0, \qquad h,k\in\mathbb{R}^+$$

and then $H_k(h+k)<H_k(h)$ for all $h>-k$ and by using the asymptotic formula (12), we have $\lim_{h\to\infty} H_k(h)=0$ and then Corollary 1 gives us $H_k(h)>0$ for every $h>-k$. Consequently, we obtain $U_k(h+k)>U_k(h)$ for all $h>-k$ with $\lim_{h\to\infty} U_k(h)=0$ and then $U_k(h)<0$ for every $h>-k$ and similarly, we get $V_k(h)>0$ for all $h>-k$. Then $W_k''(h)$ is strictly increasing on $(-k,\infty)$. By using the asymptotic formulas (11) and (12), we have

$$\lim_{h\to\infty} W_k''(h)=\lim_{h\to\infty} W_k'(h)=0 \text{ and } \lim_{h\to\infty} W_k(h)=\frac{k^2}{3}.$$

Then $W_k''(h)<0$ for all $h>-k$ and then $W_k'(h)$ is strictly decreasing on $(-k,\infty)$. Hence $W_k'(h)>\lim_{h\to\infty} W_k'(h)=0$ and this completes the proof. □

And consequently, we have the following Corollary:

Corollary 7. *Set $a,b\in\mathbb{R}^+$ and $k>0$. Then we have*

$$\frac{1}{2k}\ln\Big(h^2+hk+ak^2\Big)\le\psi_k(h+k)<\frac{1}{2k}\ln\Big(h^2+hk+bk^2\Big), \qquad h\in[0,\infty) \quad (31)$$

where the constants $a=e^{-2\gamma}\simeq 0.315$ and $b=\frac{1}{3}$ are the best possible.

Remark 8.
- *Putting $k=1$ in (31) yields (2).*
- *Using (15), we deduce that the upper bound of (31) refines the upper bound of (10) for $h,k>0$.*
- *For $k>0$, the upper and lower bounds of (31) refine the upper and lower bounds of (26) for $h>\left(\frac{\frac{1}{3}-e^{-2\gamma}}{2e^{-\gamma}-1}\right)k\simeq 0.147224\ k$ and $h>0$ respectively.*

Lemma 10. *For $h \geq 0$ and $k > 0$, we have*

$$\frac{1}{2k} \ln\left(\frac{2hk + ck^2}{e^{\frac{2k}{h+k}} - 1}\right) < \psi_k(h+k) \leq \frac{1}{2k} \ln\left(\frac{2hk + dk^2}{e^{\frac{2k}{h+k}} - 1}\right), \tag{32}$$

where $c = 2$ and $d = e^{-2\gamma}(e^2 - 1) \simeq 2.014$ are the best possible.

Proof. Set

$$M_k(h) = W_k(k+h) - W_k(h) = e^{2k\psi_k(k+h)}\left(e^{\frac{2k}{h+k}} - 1\right) - 2hk - 2k^2, \qquad h \geq 0, \qquad k > 0.$$

Since $W_k'(h)$ is strictly decreasing on $(-k, \infty)$, then $M_k(h)$ is strictly decreasing on $[0, \infty)$ and by using (6) and the asymptotic expansion (11), we get

$$M_k(0) = k^2 e^{-2\gamma}(e^2 - 1) - 2k^2 \geq e^{2k\psi_k(h+k)}\left(e^{\frac{2k}{h+k}} - 1\right) - 2hk - 2k^2 > \lim_{h\to\infty} M_k(h) = 0$$

and this gives (32). □

Remark 9.
- *Letting $k = 1$ in (32), we obtain (3).*
- *Using (17), we deduce that the lower bound of (32) refines the lower bound of (31) for $h > 0.352938k$.*

Lemma 11. *For $h > 0$ and $k > 0$, we have*

$$\left(\frac{h}{k} + a\right) e^{-2k\psi_k(h+k)} < \psi_k'(h+k) < \left(\frac{h}{k} + b\right) e^{-2k\psi_k(h+k)}, \tag{33}$$

where the constants $a = \frac{1}{2}$ and $b = \frac{\pi^2\ e^{-2\gamma}}{6} \simeq 0.519$ are the best possible.

Proof. Using the decreasing property of $W_k'(h)$ on $(-k, \infty)$ yields

$$W_k'(0) > 2k\psi_k'(h+k)e^{2k\psi_k(h+k)} - 2h - k > \lim_{h\to\infty} W_k'(h) = 0.$$

Using(6), we have

$$\frac{\pi^2\ e^{-2\gamma}\ k}{3} > 2k\psi_k'(h+k)e^{2k\psi_k(h+k)} - 2h > k,$$

which finishes the proof.

□

Remark 10.
- *Putting $k = 1$ in (33) gives (4).*
- *Using (18), we deduce that the lower bound of (33) refine the lower bound of (28) for $h > \frac{k}{2}$ and $k > 0$.*

Lemma 12. *For $h > 0$ and $k > 0$,*

$$\begin{aligned} \frac{1}{2k^2}\left(e^{\frac{2k}{h+k}} - 1 - k^2 e^{-2k\psi_k(h+k)}\right) &< \psi_k'(h+k) \\ &< \frac{1}{2k^2}\left(\frac{2k^2}{(h+k)^2} + 1 - e^{-\frac{2k}{h+k}} + k^2 e^{-2k\psi_k(h+2k)}\right). \end{aligned} \tag{34}$$

Proof. By applying the mean value theorem to W_k on the interval $[h, h+k]$, we get

$$\frac{W_k(h+k) - W_k(h)}{k} = W_k'(h + \beta_h), \qquad 0 < \beta_h < k.$$

Using the decreasing property of $W_k'(h)$ on $(-k, \infty)$ yields

$$W_k'(h+k) < \frac{W_k(h+k) - W_k(h)}{k} < W_k'(h)$$

and this gives us (34). □

Remark 11. *Using (20), we deduce that the lower bound of (34) refine the lower bound of (33) for* $h, k \in \mathbb{R}^+$.

5. Conclusion

The main conclusions of this paper are stated in Theorems 1 and 2 and Lemmas 5 and 9. The authors proved the CM and the monotonicity properties of four functions containing the $k-$generalized digamma and polygamma functions, derived some new bounds for $\psi_k^{(s)}(h)$ functions ($s \in \mathbb{N} \cup \{0\}$). These bounds refine some recent results.

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