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Lie Algebra Contributions to Instantaneous Plane Kinematics

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Abstract — To investigate the instantaneous properties of a planar motion, Roth and Bottema [1] obtain the instantaneous invariants of a planar motion using Veldkamp's canonical frame [7]. We investigate the derivatives of time-independent planar motions with respect to the fixed frame up to thrid order and their instantaneous invariants are obtained by using the Lie algebra to the planar motion group near identity element.

Subject Classification (2020): 53Z30, 70E17, 70G65.

1. Introduction

Lie theory connects almost every branch of mathematics. It has a wide range of applications from harmonic analysis to quantum groups. In this work, our interest is Lie algebra in plane kinematics [4]. The group of rigid body motions are all related to Lie groups. Planar motion group, Spherical motion group and Spatial motion group are represented by $SE(2)$, $SO(3)$ and $SE(3)$ respectively.

Rigid body motions in $\mathbb{R}^2$ has a 3×3 homogeneous matrix representation. Any element of planar motion group is given by,

$$G=\begin{pmatrix} R & \vec{t} \\ 0 & 1 \end{pmatrix} \tag{1.1}$$

where R is the (2×2) rotation matrix and the vector $\vec{t}$ is a (2×1) translation vector. Well known Mozzi-Chasles's theorem says that each spatial motion is a screw motion. That is, any spatial displacement can be seen as a rotation about a line, the screw axis, and followed by a translation parallel to that line. In the planar case, there is a fixed point instead of a line. Any planar motion is a rotation about this fixed point.

To study planar motion, we attach a coordinate frame, M, to the moving body and a coordinate frame, F to the ground (reference frame). In plane kinematics, except pure translations, there is a single point whose coordinates are the same both in the fixed frame and in the moving frame before and after the displacement.

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This point is called the pole point of the planar displacement. The rotation angle θ and the pole points are the geometric invariants of planar kinematics.

Any rigid transformation is the combination of a rotation followed by a translation, given by,

$$\vec{X} = R\vec{x} + \vec{t} \tag{1.2}$$

where $\vec{x}$ is the coordinates of a point in the moving frame M and $\vec{X}$ is the coordinates of the point in the fixed frame F. In 1.2 the rotation matrix R and the translation vector $\vec{t}$ are given by,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \vec{t} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{1.3}$$

Then the fundamental equations of the plane kinematics can be written as;

$$X = x\cos\theta - y\sin\theta + a, \tag{1.4}$$

$$Y = x\sin\theta + y\cos\theta + b.$$

If θ, a and b are functions of a time parameter μ, one can determine a continuous motion of a point with its positions, velocity, accelaration etc. The parameters depending on μ are the concerns of *time-dependent kinematics* of the motion. But some other properties are independent of time; such as curves, tangents, poles etc. which are called the *geometric kinematics* of the motion. In this case

$$X = x\cos\theta - y\sin\theta + a(\theta), \tag{1.5}$$

$$Y = x\sin\theta + y\cos\theta + b(\theta)$$

where θ is the only parameter for the planar motion, and a, b are the functions of θ [1, 3]. In the equation 1.5 the planar motion is completely defined by the functions $a(\theta)$ and $b(\theta)$. The equation 1.5 is called *time-independent motion*.

2. Veldkamp's Canonical Frame

To discuss the instantaneous geometric invariants of a planar motion, we introduce Veldkamp's canonical frame. In his dissertation [7], for a given time-independent motion $a(\theta)$ and $b(\theta)$ are the power series of θ;

$$a(\theta) = \sum_{n=0}^{\infty} a_n(\frac{\theta^n}{n!}), \quad b(\theta) = \sum_{n=0}^{\infty} b_n(\frac{\theta^n}{n!}) \tag{2.1}$$

and the following are satisfield:

i) The moving and fixed frame are chosen such that they coincide in the "zero-position", so we have from 2.1 and 1.5

$$a_0 = b_0 = 0. \tag{2.2}$$

ii) At the moment $\theta = 0$ we place common origin of the frames at the pole, which implies

$$a_1 = b_1 = 0. \tag{2.3}$$

iii) The axes O_X and O_x are chosen along the common tangent of the pole curves (at the pole), which yields

$$a_2 = 0. \tag{2.4}$$

iv) Assuming $b_2 \neq 0$, we can take the possitive direction of the X axis as to set $b_2 > 0$. The coinciding frames O_{XY} and O_{xy} defined by i)-iv) are called canonical. Here O_{XY} and O_{xy} denote the fixed frame and the moving frame respectively. These canonical frames can be used to study instantaneous kinematics, for details see [1, 7].

Roth and Bottema [1] obtained the geometry of the planar motion by differentiating the coordinate axes given in equation 1.5.

$$\begin{array}{llll} X = x & \dot{X} = -y & \ddot{X} = -x & \dddot{X} = y + a_3 \quad \ldots \\ Y = y & \dot{Y} = x & \ddot{Y} = -y + b_2 & \dddot{Y} = -x + b_3 \quad \ldots \end{array} \tag{2.5}$$

at $\theta = 0$, where dot over an alphabet denotes the derivative with respect to θ. Hence instantaneous properties of the motion depend on the constants $a_3, a_4, \ldots, a_n, \ldots$ and $b_2, b_3, b_4, \ldots, b_n, \ldots$. which are called the instantaneous invariants of the kinematics.

2.1. The Group Planar Motions and Its Lie Algebra

The (3×3) matrix G in equation 1.1 represents the planar motion group, it is the element of the Lie group $SE(2)$. The time independent motion in equation 1.5 defines a one parameter subgroup of $SE(2)$. Let $D(\theta)$ denotes the planar displacement defined in equation 1.5 in the homogeneous matrix representation,

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & a(\theta) \\ \sin\theta & \cos\theta & b(\theta) \\ 0 & 0 & 1 \end{pmatrix}.$$

If we consider an initial point $P(0)$, then the transfomed point $P(\theta)$ is written by,

$$\begin{pmatrix} P(\theta) \\ 1 \end{pmatrix} = D(\theta) \begin{pmatrix} P(0) \\ 1 \end{pmatrix}. \tag{2.6}$$

Differentiating the equation 2.6 gives,

$$\begin{pmatrix} \dot{P}(\theta) \\ 0 \end{pmatrix} = D_F(\theta) \begin{pmatrix} P(\theta) \\ 1 \end{pmatrix}, \tag{2.7}$$

where $D_F(\theta)$ is the derivative with respect to the fixed frame. The geometric velocity matrix $D_F(\theta)$ of the group element $D(\theta)$ can be found as,

$$D_F(\theta) = \dot{D}(\theta) D(\theta)^{-1} = \begin{pmatrix} 0 & -1 & \dot{a}(\theta) + b(\theta) \\ 1 & 0 & -a(\theta) + \dot{b}(\theta) \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.8}$$

Let $(X,Y,1)^t$ and $(x,y,1)^t$ be the homogeneous coordinates of $P(\theta)$ in the fixed frame and the moving frame respectively. Then the equation 2.7 at $\theta=0$ gives,

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\\ 0\end{pmatrix} = D_F(0)\begin{pmatrix}X\\ Y\\ 1\end{pmatrix} = \begin{pmatrix}0 & -1 & a_1\\ 1 & 0 & b_1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}X\\ Y\\ 1\end{pmatrix}. \tag{2.9}$$

Since the moving frame and the fixed frame are coincident at $\theta=0$, we get

$$\dot{X} = -y + a_1 \quad \dot{Y} = x + b_1.$$

In equation 2.3 a_1 and b_1 are equal to zero. Hence,

$$\dot{X} = -y \quad \dot{Y} = x.$$

The second derivatives can be obtained by differentiating the equation 2.7 ,

$$\begin{pmatrix}\ddot{P}(\theta)\\ 0\end{pmatrix} = D_{F_2}(\theta)\begin{pmatrix}P(\theta)\\ 1\end{pmatrix}, \tag{2.10}$$

where $D_{F_2}(\theta)$ denotes the second derivative matrix with respect to the fixed frame. The matrix $D_{F_2}(\theta)$ can be written as follows,

$$D_{F_2}(\theta) = \dot{D}_F(\theta) + D_F^2(\theta) = \begin{pmatrix}-1 & 0 & a(\theta)+\ddot{a}(\theta)\\ 0 & -1 & b(\theta)+\ddot{b}(\theta)\\ 0 & 0 & 0\end{pmatrix}. \tag{2.11}$$

Then the equation 2.10 at $\theta=0$ is,

$$\begin{pmatrix}\ddot{X}\\ \ddot{Y}\\ 0\end{pmatrix} = D_{F_2}(0)\begin{pmatrix}X\\ Y\\ 1\end{pmatrix} = \begin{pmatrix}-1 & 0 & a_2\\ 0 & -1 & b_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}X\\ Y\\ 1\end{pmatrix}. \tag{2.12}$$

Since $a_2=0$ in 2.4 and the frames are coincident at $\theta=0$,

$$\ddot{X} = -x \quad \ddot{Y} = -y + b_2.$$

Finally, the third derivative of the equation 2.6 can be found as,

$$\begin{pmatrix}\dddot{P}(\theta)\\ 0\end{pmatrix} = D_{F_3}(\theta)\begin{pmatrix}P(\theta)\\ 1\end{pmatrix}, \tag{2.13}$$

where $D_{F_3}(\theta)$ denotes the third derivative with respect to the fixed frame. The third order derivative matrix $D_{F_3}(\theta)$ can be found as follows,

$$\begin{aligned} D_{F_3}(\theta) &= \ddot{D}_F(\theta) + 2\dot{D}_F(\theta)D_F(\theta) + D_F(\theta)\dot{D}_F(\theta) + D_F^3(\theta) \\ &= \begin{pmatrix} 0 & 1 & \ddot{a}(\theta) - b(\theta) \\ -1 & 0 & a(\theta) + \dddot{b}(\theta) \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.14}$$

Then the equation 2.13 at $\theta = 0$ is,

$$\begin{pmatrix} \dddot{X} \\ \dddot{Y} \\ 0 \end{pmatrix} = D_{F_3}(0) \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & a_3 \\ -1 & 0 & b_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}. \tag{2.15}$$

That is,

$$\dddot{X} = y + a_3 \quad \dddot{Y} = -x + b_3.$$

Hence the kinematic invariants of time-independent planar motion are obtained by using the elements of Lie algebra, $se(2)$ to the planar motion group, $SE(2)$. Similarly, higher order terms in the equation 2.5 can be obtained by using the higher order derivatives of equation 2.6.

3. Conclusion

Instantaneous properties of a planar motion are obtained by Bottema and Roth [1] using canonical frame which was introduced by Veldkamp [7]. In this study, the derivatives of time-independent planar motions with respect to the fixed frame are given and the instantaneous invariants of planar motions are obtained by using the Lie algebra to $SE(2)$.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Bottema, O. and Roth, B. (1990) Theoretical kinematics. Dover Publications.

[2] Hausdorff, F. (1906). Die symbolische Exponentialformel in der Gruppentheorie. Ber. Verh. Kgl. SËA chs. Ges.Wiss. Leipzig., Math.-phys. Kl., 58, 19-48.

[3] McCarthy, J. M., Soh, G. S. (2010) Geometric design of linkages (Vol. 11). Springer Science and Business Media.

[4] Selig, J. M. (2005) Geometric fundamentals of robotics (Vol. 128). New York: Springer.

[5] Selig, J. M. (2007) Cayley maps for SE (3). In 12th International Federation for the Promotion of Mechanism and Machine Science World Congress (p. 6). London South Bank University.

[6] Selig, J. M. (2007) Centrodes and Lie algebra. In 2th IFT oMM World Congress (p. 6). London South Bank University.

[7] Veldkamp, G. R. (1963) Curvature theory in plane kinematics. JB Wolters.