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Spatial Price Equilibrium, Welfare And Quadratic Programming

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This paper is an ettempt to examine the theoretical aspects of a single product spatial equilibrium analysis. In the first section of the paper spatial equilibrium analysis is defined. In the second section attention is directed to the problem of solving the equilibrium conditions for a set of known linear demand and supply functions and a given vector of transportation costs. For this purpose a quadratic programming model was utilized. In the third section the welfare implications of the spatial price equilibrium models are investigated.

I. Spatial Price Equilibrium

Introduction and Notation

Let there be an economy engaged in the production and consumption of a single commodity and let this economy consist of n regions or discrete sub-spaces denoted by k or j with k, j = 1, ... n. Furthermore, let each region be identified by both a distinct demand point and a distinct supply point at which exchange takes place.

Let $\underline{p} = (p_j)$ be the column vector of non-negative market prices at the n demand points.

Let $\underline{v} = (v_k)$ be the column vector of non-negative market prices at the n supply points.

Let $\underline{D} = (D_j)$ be the n element column vector of non-negative demand flows.

Let $\underline{Q} = (Q_k)$ be the n element column vector non-negative supply flows.

Let $\underline{X} = (X_{kj}) = (X_{11}, X_{12}, \ldots\ldots X_{1n}, \ldots. X_{n1}, \ldots\ldots X_{mm})'$ denote the n^2 column vector of non-negative flows of commodity between the supply point k and demand points.

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Let $\underline{t} = (t_{kj}) = (t_{11}, t_{12}, ... t_{1n},.... t_{n1}, ...t_{mm})'$ denote the n^2 column vector of transport costs per unit of the commodity between the supply point k and demand point j.

Let demand in region j be a function of market price in region j with form

$$D_j = h_{oj}/h_{1j} - (l/h_{1j})P_j \quad \text{if } p_j \leq h_{1j}$$

$$D_j = 0 \qquad \text{if } p_j > h_{oj} \qquad (1.1)$$

where $h_{oj}/h_{1j}>0$ and $l/h_{1j}>0$ are the coefficients of the function and the domain of the function is the set values $p_j \geq 0$.

Let supply in region k be a function of market price in region k with either

$$Qk = g_{ok}/g_{lk} + (1/g_{1k})v_k \qquad (1.2a)$$

where $g_{ok}/g_{1k} > 0$ and $l/g_{1k} > 0$ are the coefficients of the function and the domain of the function is the set values $v_k/ \geq 0$.

$$Q_k = g_{ok}/g_{lk} + (1/g_{lk})v_k \qquad \text{if } v_k > - g_{ok}$$

$$Q_j = 0 \qquad \text{if } v_k < - g_{ok} \qquad (1.2b)$$

where $g_{ok}/g_{1k} > 0$ and $l/g_{1k} > 0$ are the coefficients of the function is the set values $v_k \geq 0$.

Functions (1.1) and (1.2b) although continuous are kinked. However, future analysis requires demand functions that are differentiable at all points in their domain. This requirement can be met be defining two new variables $\tilde{p}_j$ and $\tilde{v}_k$ such that:

$$\tilde{p}_j = p_j \qquad \text{if } p_j \leq h_{oj}$$
$$\tilde{p}_j = h_{oj} \qquad \text{if } p_j < h_{oj}$$

and

$$\tilde{v}_k = v_k \qquad \text{if } v_k \leq - g_{ok}$$
$$\tilde{v}_k = - g_{ok} \qquad \text{if } v_k < - g_{1k} > 0$$

and rewriting the demand and supply functions as[1] :

$$D_j = h_{oj}/h_{1j} - (1/h_{1j})\ \tilde{p}_j \qquad (1.3)$$

1. In order to avoid excessive notation in the subsequent analysis, it is assumed that $\tilde{p}$ = and $v = \tilde{v}$.

$$Q_k = g_{ok}/g_{1k} + (1/g_{1k})\tilde{v}_k \tag{1.4}$$

Let the inverse expression for (1.3) and (1.4) be:

$$p_j = h_{oj} - h_{1j} D_j \tag{1.5}$$

$$v_k = g_{ok} + g_{1k} Q_k \tag{1.6}$$

Definition of a Spatial Price Equilibrium

A state of equilibrium exists in the spatially separated markets if, for a set of values ($\underline{D}^* \geq 0$, $\underline{Q}^* \geq 0$, $\underline{X}^* \geq 0$, $\underline{p}^* \geq 0$, $\underline{v}^* \geq 0$), the following conditions are met:

(a) Market Equilibrium:

No excess demand or excess supply possibility for all j and k. Formally,

(i) Regional consumer equilibrium

$$D_j^* - \sum_k X_{kj}^* \leq 0$$

and

$$p_j^* \,(D_j^* - \sum_k X_{kj}^* = 0 \tag{1.7}$$

for j = 1, ..., n

(ii) Regional producer equilibrium

$$-Q_k^* + \sum_k X_{kj}^* \leq 0$$

and

$$v_k^* \,(-Q_k + \sum_k X_{kj}^* \leq 0 \tag{1.8}$$

for k = 1, ..., n

From (1.7) and (1.8) it follows that :

$$p_k^* > 0 = > D_j^* \sum_k X_{kj}^*$$

$$v_k^* > 0 = > Q_k^* \sum_k X_{kj}^*$$

$$D_j^* < \sum_k X_{kj}^* => p_j^* = 0 \tag{1.9}$$

and

$$Q_k^* < \sum_k X_{kj}^* => v_k^* = 0 \tag{1.10}$$

That is, at positive demand and supply prices total demand is equal to total supply with waste of the commodity at a supply or demand point only occuring if it has a zero price.

(b) Locational Price Equilibrium:

The difference between the equilibrium demand price in region j and the equilibrium supply price in region k can not exceed the cost of transport from region j to region k and if there is a positive transhipment from j to k then the price difference must be exactly equal to the per unit transport cost.

Formally,

$$p^*j - v^*k - tkj \leq 0$$

and (1.11)

$$X^* kj\,(p^*k - v^*k - t_{kj}) = 0$$

for j, k = 1, ..., n

From (1.11) it follows that:

$$p^*j - v^*k < t_{kj} => X^*{}_{kj} = 0$$

and (1.12)

$$X^*{}_{kj} > 0 => p^*j - v^*k = t_{kj}$$

Thus, while a locational price difference less than the relevant unit transport cost is associated with a zero transhipment, a positive transhipment implies that the locational price difference is exactly equal to the unit transport cost.

II. Spatial Price Equilibrium as a Non-Linear Mathematical Programming Problem

Having identified the conditions for equilibrium in spatially separated markets, attention can now be directed at the problem of solving the equilibrium conditions for a set of known linear demand and supply curves and a given vector of transportation cost. The general problem, following Enke (1951), can be specified as:

> *......... There are three or more regions trading a homogenous good. Each region constitues a single and distinct market. The regions of each possible pair of regions are separated but not isolated by a transportation cost per physical unit which is independent of volume. There are no legal restrictions... For each region the functions which relate local production and local use to local price are known... Given these trade functions and transportation cost, we wish to ascertain:
> (1) the net price in each region;
> (2) the quantity of exports or imports for each region;
> (3) which regions export, import, or do neither;
> (4) the volume and direction of trade between each pos sible pair of regions ..."

Samuelson, in an article appearing in 1952, showed how this descriptive price behaviour could be cast, mathematically, into a maximisation problem which could be solved by iterative procedures. Subsequently, Takayama and Judge (1964) and (1971), reformulated the Samuelson model as a quadratic programming problem for the case of linear demand and supply curves.

Following Samuelson's definition of "net social pay-off of trade", which identifies the relevant value as the increas in the area under the demand curve minus the sum of the increases in transportation costs and the area under supply curve[1], Takayama and Judge (1971) formulated this descriptive problem as:

Maximise

$$NSP = \sum_j h_{oj}D_j - 1/2\sum_j h_{1j}D_j^2 - \sum_k g_{ok}Q_k - 1/2\sum_k g_{1k}Q_k^2$$
$$- \sum_k \sum_j t_{kj} - \sum_j E_j \qquad (2.1)$$

Subject to

$$D_j - \sum_j X_{1j} \leq 0 \qquad \text{for all j}$$

1. Both supply and demand curves calculated over the quantity domain.

and

$$Q_k - \sum_j X_{kj} \leq 0 \quad \text{for all } k \tag{2.2}$$

and

$D_j > 0, Q_k > 0$ and $X_{kj} > 0$ for all j and k

where (2.1) is the net social pay-off, E_j is a constant determined by intra-regional trade under autarchy and (2.2) is the constraint that there is no excess demand or supply.

The Lagrangean function appropriate to the problem is:[1]

$$L = L(D_j, Q_k, X_{kj}, \gamma_j, \lambda_k) = NSP + \sum_j \gamma_j (-D_j + \sum_k X_{kj})$$
$$+ \gamma_k (-Q_k + \sum_j X_{kj}) \tag{2.3}$$

where lj ≥ 0 for j = 1, ..., n and k ≥ 0 for lk = 1, ..., n are the Lagrangean multipliers.

The necesary (first order) conditions for D^*_j, Q^*_k, X^*_{jk} to be a maximum to the problem are[2]:

(a) $D^*_j(\partial L/\partial D_j) = p^*_j - \gamma^*_j \leq 0$
and $D^*_j (\partial L/\partial D_j) = 0$
for j = 1, ..., n

(b) $\partial L/\partial Q_k = -v^*_k + \lambda^*_k \leq 0$
and $Q^*_k (\partial L/\partial Q_k) = 0$
for k = 1, ..., n

(c) $\partial L/\partial X_{kj} = -\gamma^*_k - \lambda^*_k - t_{kj} \leq 0$
and $X^*_{kj} (\partial L/\partial X_{kj}) = 0$
for j, k = 1, ..., n

1. The constant term $\sum_j E_j$ is omitted because it does not influence the solution.
2. Form of the problem implies that the necessary conditions are also sulfficient. Furthemore, it can be shown that the problem has a finite maximum solution. However, while the D^*_j and Q^*_k values implied by the solution are unique X^*_{kj} may be non-unique.

(d) $\partial L/\partial g_j = -D^*_j + \sum_k X_{kj} \geq 0$

and $\gamma^*_j(-D^*_j + \sum X^*_{kj}) = 0$

for $j = 1, \ldots\ldots\ldots, n$

(e) $\partial L/\partial \lambda_j = Q^*k - \sum_j X_{kj} \geq 0$

and $\lambda_k (Q^*_k - \sum_j X_{kj}) = 0$

for $k = 1, \ldots\ldots\ldots, n$

Evaluation of the Programming Problem

Let $D^*_j > 0$ and $Q^*_k > 0$ for all j, k = 1, ... n be the solution of the problem[1]. Then form (2.4a) and (2.4b) it follows that

$$\begin{aligned} &(a) \quad p^*_j = \gamma^*_j \\ & \qquad\qquad\qquad\qquad \text{for all } j, k = 1, \ldots\ldots, n \qquad\qquad (2.5) \\ &(b) \quad v^*_k = \lambda^*_k \end{aligned}$$

That is, the Lagrangean multipliers can be interpreted as the equilibrium market prices. Consequently, (2.4d) and (2.4e) are identical to the market equilibrium conditions, (1.7) and (1.8) Furthemore, (2.4c) is identical to the locational price equilibrium condition, (1.11). Thus, there exists a formal equivalence between the conditions for spatial price equilibrium, as set out in the previous section, and the quadratic programming problem specified above.

III. Welfare Implications of the Spatial Price Equilibrium Models

The concept of social pay-off as defined by Samuelson (1952) has welfare connotations, However, both Samuelson, Takayama and Judge deliberately abstained from attaching such an implication to the results of the spatial price equilibrium models. Samuelson's reasons, which were later reiterated by Takayama and Judge, were as follows:

> "This magnitude is artificial in the sense that no competitor in the market will be aware of, or concerned with it. It is artificial in the sense that after an Invisible Hand has led to its maximisation, we need not necessarily attach any welfare significance to the result."

1. It is assumed thet the spatial price equilibrium model is regular, i.e. $p^*_j>0$ and $v^*_k>0$ for all j, k = 1, ..., n.

However, if the system is constained by intevention and if there exists some constained optimum solution then even though the competitive process will still achieve that solution, the net social pay-off will be affected. Furthemore, existence of an agency, i.e.. government, which puts public policies into practice, implies "awareness". Consequently, in the following paragraphs, the tools of partial equilibrium welfare economics, under certain simplifying assumptions, are utilised to provide an evaluation of the welfare effects of the relevant goverment policies. This is subject to the usual ceveats that (a) spatial equilibrium analysis is strictly partial as the regional demand and supply curves are specified on the ceteris paribus assumption that other prices are constant, (b) the social and private valuation of the benefit and costs of the commodity are assumed to coincide.

The net social pay-off function has been defined in the conventional manner as:

$$NSP = \sum_j h_{oj}D_j - 1/2\sum_j h_{1j}D_j^2 - \sum_k g_{ok}Q_k - 1/2\sum_k g_{1k}Q_k^2 - \sum_k \sum_j t_{kj} - \sum_j E_j \qquad (3.1)$$

where Ej is a constant determined by interaregional trade, demand and supply flows in the autarchy situations. This form is assumed to hold whether or not there is intervention. Thus, effect on the net social pay,off the government policies can be expressed as:

$$\Delta\, NSP = (Z^*_1 - \widehat{Z}_1) - (Z^*_2 - \widehat{Z}_2) - (Z^*_3 - \widehat{Z}_3) \qquad (3.2)$$

where

$$Z^*_1 = \sum_j h_{oj}\, D^*_j - 1/2 \sum_j h_{1j}\, D^{*2}_j$$

$$\widehat{Z}^*_1 = \sum_j h_{oj}\, \widehat{D}^*_j - 1/2 \sum_j h_{1j}\, \widehat{D}^{*2}_j$$

$$Z^*_2 = \sum_k g_{ok}\, Q^*_k - 1/2 \sum_k g_{1k}\, Q^{*2}_k$$

$$\widehat{Z}^*_2 = \sum_k g_{ok}\, \widehat{Q}^*_k - 1/2 \sum_k g_{1k}\, \widehat{Q}^{*2}_k$$

$$Z^*3 = \sum_k \sum_j t_{kj}\, X^{*2}_{kj}$$

$$\widehat{Z}^*3 = \sum_k \sum_j t_{kj}\, \widehat{X}^{*2}_{kj}$$

D^*, Q^*_k, X^*_{kj}	-the level of demand, supply and intra-regional trade flows under competitive conditions,
$\hat{D}^*_j, \hat{Q}^*_k, \hat{X}^*_{kj}$,	- the level of demand, supply and intra-regional trade flows under government policy.

The conventional analysis procees by assuming,

.(a) the increase (decrease) in the area under the demand curve, over the domain of D^*j to $\hat{D}_j$ ($\hat{D}_j$ to D^*_j) represent the maximum (minimum) amuont of money consumers in the jth region are prepared to pay (accept) for the increase (decrease) in consumption, D^*j minus $\hat{D}_j$ ($\hat{D}_j$ minus D^*) Therefore, the change in the area under the demand curve in the jth region as a result of goverment policy may be described as the valuation placed on government policy by consumers in that region. This assumption will be a reasonable one if the demand curve in the jth region approximates a Hicksian Compensated Demand curve; i.e. if the income effect of a change in the demand price over the range of the function P^*j to $\hat{P}_j$ is negligible.

(b) at least over the domain Q^*_k to $\hat{Q}_k$ the supply curve in the kth region represent a maginal (opportunity) cost curve excluding intra-marginal rent paymerts. The supply curve thus indicates the minimum amount of inducement required by producers to supply each additional unit, and hence an increase (decrease) in the area under the supply curve in the k^{th} region will represent the increase (decrease) in the total cost of production brought about by intervention.

(c) transportation services are supplied at constant unit cost i.e. they are not related to the volume of transportation activity, Thus (Z^*_3 $\hat{Z}_3$) represent the increased (decreased) cost to society which is incurred (enjo6yed) by the expanded (diminished) use of transportation activities.

(d) assumptions (a), (b) and (c) respectively hold for all j = 1, ... n, k = 1,... n and j, k = 1, ... n

In view of the assumptions (a) and (d) (Z^*_1 $\hat{Z}_1$) can be indentified as the gross gain to society of the government policy. Furthermore, under assumtions (b), (d) and (c), (d) respectively (Z^*_2 $\hat{Z}_2$) and (Z^*_3 $\hat{Z}_3$) represent the increased cost to society incurred by the changes in the levels of production and transportation services (which are brought abut by the government policies). Concequently, net additional social pay - off, NSP, may be described as the net welfare gain (loss) to society of goverment intercention.

Finally, it should be noted they government involvement generally impylies a deviation from competitve conditions. Thus one would expect a fall in NSP. The magnitude of this fall is one of the criteria by which government policies may be assesed.

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