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Research Article

Frenet-Serret and Darboux Frames of A Image Curve on A Screw Surface

***Yasemin Kemer* [*a], *Erhan Ata* [a]**

[a] *Dumlupınar University, Faculty of Science and Arts, Department of Mathematics, Kütahya, Turkey*

Abstract

In literature, the action of the dual quaternion which described a rigid motion on a point and a line were investigated. Based on this idea, one can obtain a new surface by applying the rigid motion to the points of a surface. In this paper, we generate a screw surface M^v by applying the screw motion, a special case of the rigid motion, to the points of a surface M in E^3. In M^v, we obtain the frames which correspond to Serret-Frenet and Darboux frames of a curve α on M. These frames are Serret-Frenet and Darboux frames of the image curve on M^v. Then, we calculate the curvature, torsion, geodesic curvature, normal curvature and geodesic torsion of the image curve of the curve α on the screw surface M^v and obtain the relationships between these notions of the curve α and its image curve.

Key Words: Curvature, Darboux frame, Frenet frame, image curve, rigid transformation, screw surface, torsion.

Vida Yüzeyi Üzerindeki Bir Görüntü Eğrisinin Frenet-Serret ve Darboux Çatıları

Öz

Literatürde katı cisim hareketini tanımlayan dual kuaterniyonun nokta ve doğru üzerindeki etkileri incelenmiştir. Buradan hareketle bir yüzeyin noktalarına katı cisim hareketi uygulanarak yeni bir yüzey elde edilebilir. Bu çalışmada, E^3 3-boyutlu Öklid uzayında bir M yüzeyinin noktalarına katı cisim hareketinin özel hali olan vida hareketini uygulayarak bu yüzeye ait bir M^v vida yüzeyi oluşturulmuştur. M yüzeyi üzerindeki bir α eğrisinin Serret-Frenet ve Darboux çatılarına M^v vida yüzeyi üzerinde karşılık gelen çatılar elde edilmiştir. Bu çatılar α eğrisinin M^v vida yüzeyi üzerindeki görüntü eğrisine ait Serret-Frenet ve Darboux çatılarıdır. Daha sonra, görüntü eğrisine ait eğrilik, burulma, geodezik eğrilik, normal eğrilik ve geodezik burulma hesaplanıp α eğrisi ve bu eğriye ait görüntü eğrisinin hesaplanan bu kavramları arasındaki bağıntılar elde edilmiştir.

Anahtar Kelimeler: Burulma, Darboux çatısı, eğrilik, Frenet çatısı, görüntü eğrisi, katı cisim hareketi, vida yüzeyi.

[*] Corresponding author
e-mail: yasemin.kemer@dpu.edu.tr

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Introduction

Surfaces have had application areas in many areas such as mathematics, kinematics, dynamics and engineering for many years and they have been in center of interest increasingly. Mathematicians have written many articles and books by investigating surfaces as Euclidean and non-Euclidean. For these studies, one can read [1], [2], [3], [4], [5], [6], [7], [8].

Eisenhart defined parallel surfaces and their some properties in his book [3]. In [9], Ünlütürk and Özüsağlam investigated the parallel surfaces in Minkowski 3-space. In [10], Tarakçı and Hacısalihoğlu defined surfaces at a constant distance from edge of regression on a surface and gave some properties of such surfaces and then in [11], [12], [13] Sağlam and Kalkan investigated the other properties of this surface. Again Sağlam and Kalkan transported the surfaces at a constant distance from edge of regression on a surface to Minkowski 3-space and obtained their properties which they have in Euclidean space in this space.

In her Ph.D. thesis Kemer described points of a surface M in Euclidean 3-space E^3 by dual quaternions and applied screw motion, which is a special case of rigid motion, to these points and obtained a screw surface M^v [14]. The screw surfaces are a more general case of the parallel surfaces and surfaces at a constant distance from edge of regression on a surface. In special cases, one can obtain the parallel surfaces and surfaces at a constant distance from edge of regression on a surface from the screw surfaces.

In differential geometry, Serret-Frenet formulae define the kinematic properties of an object moving along a continuous, differentiable curve in Euclidean 3-space E^3 or geometric properties of the curve independent from movement. This formulae was named from two French mathematicians Jean Frederich Frenet and Joseph Alfred Serret who discovered this formulae independent from each other.

Darboux frame is a nature moving frame setting up on a surface in differential geometry of surfaces. This frame was named from French mathematician Jean Gaston Darboux. Darboux frame, as Frenet-Serret frame, is an example of a nature moving frame defined on a surface.

In this study, we obtain the image curve of a curve α on a surface M in Euclidean 3-space E^3 on a screw surface M^v. Then we calculate Serret-Frenet and Darboux frames of the image curve and we get the curvature, torsion, geodesic curvature, normal curvature and geodesic torsion of this curve. These notions are expressed in terms of the corresponding values of the the curve α. In this way, we investigate the changes in some geometric properties of a curve under the screw motion. This paper depends on the case that position vector of points of a surface and the unit normal vectors at these points are parallel.

Preliminaries

Quaternions

Let us firstly begin with Hamilton's quaternions and their connection with rotations. A rotation of angle θ about a unit vector $v = (v_x, v_y, v_z)^T$ is represented by the quaternion,

$$r = cos\frac{\theta}{2} + sin\frac{\theta}{2}(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}).$$

The conjugation

$$p' = rp\bar{r}$$

gives the action of such a quaternion on a point $p = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in space, where

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$$\bar{r} = cos\frac{\theta}{2} - sin\frac{\theta}{2}(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}).$$

The quaternions representing rotations satisfy $r\bar{r} = 1$ and also r and $\bar{r}$ represent the same rotation. The set of unit quaternions, those satisfying $r\bar{r} = 1$, comprise the group Spin(3), which is the double cover of the group of rotations $SO(3)$.

Let ε be the dual unit which satisfies the relation $\varepsilon^2 = 0$ and commutes with the quaternion units **i**, **j** and **k**. For ordinary quaternions q_0 and q_1,

$$h = q_0 + \varepsilon q_1$$

indicates a general dual quaternion. A rigid transformation is represented by a dual quaternion

$$g = r + \frac{1}{2}\varepsilon tr,$$

where r is a quaternion representing a rotation as above and $t = t_x\mathbf{i} + t_y\mathbf{j} + t_z\mathbf{k}$ is a pure quaternion representing the translational part of the transformation [15].

Points in space are represented by dual quaternions of the form,

$$\hat{p} = 1 + \varepsilon p,$$

where p is a pure quaternion as above. The action of a rigid transformation on a point is given by,

$$\begin{aligned}\hat{p}' &= (r + \frac{1}{2}\varepsilon tr)\hat{p}(r + \frac{1}{2}\varepsilon \bar{r}t)\\ &= (r + \frac{1}{2}\varepsilon tr)(1 + \varepsilon p)(r + \frac{1}{2}\varepsilon \bar{r}t)\\ &= 1 + \varepsilon(rp\bar{r} + t).\end{aligned}$$

Note that, as with the pure rotations, g and $\tilde{g}$ represent the same rigid transformation [15].

Darboux and Frenet Frames in E^3

Let M be a surface in E^3 and $\alpha(t)$ be an arbitrary speed curve on this surface. Then we call the vectors

$$\overrightarrow{T(t)} = \frac{\overrightarrow{\alpha'(t)}}{\left\|\overrightarrow{\alpha'(t)}\right\|},$$

$$\overrightarrow{N(t)} = \overrightarrow{B(t)} \wedge \overrightarrow{T(t)}$$

and

$$\overrightarrow{B(t)} = \frac{\overrightarrow{\alpha'(t)} \wedge \overrightarrow{\alpha''(t)}}{\left\|\overrightarrow{\alpha'(t)} \wedge \overrightarrow{\alpha''(t)}\right\|}$$

as the unit tangent, the unit normal and the binormal vectors of the curve α, respectively. The triple $\{\vec{T}, \vec{N}, \vec{B}\}$ defines a frame in E^3 and this frame is called Frenet frame [16].

Let $\vec{U}$ be the unit normal vector of M. Then the triple $\{\vec{T}, \vec{V}, \vec{U}\}$ defines a frame in E^3 and this frame is called Darboux frame, where $\vec{V} = \vec{U} \wedge \vec{T}$ [17].

Theorem 1.

Let M be a surface in E^3 and α be an arbitrary speed curve on this surface. Then the curvature function $\kappa(t)$ can be calculated as

$$\kappa(t) = \frac{\left\|\overrightarrow{\alpha'(t)} \wedge \overrightarrow{\alpha''(t)}\right\|}{\left\|\overrightarrow{\alpha'(t)}\right\|^3}$$

[3].

Let M be a surface in E^3 and $\{\vec{T}, \vec{V}, \vec{U}\}$ be the Darboux frame of an arbitrary speed curve α on this surface. Let $v = \|\alpha'(t)\|$. The functions

$$k_g(t) = \frac{1}{v^2}\langle\overrightarrow{\alpha''(t)}, \overrightarrow{V(t)}\rangle,$$

$$k_n(t) = \frac{1}{v^2}\langle\overrightarrow{\alpha''(t)}, \overrightarrow{U(t)}\rangle,$$

$$\tau_g(t) = \frac{1}{v}\langle\overrightarrow{V'(t)}, \overrightarrow{U(t)}\rangle$$

are called the geodesic curvature, the normal curvature and the geodesic torsion of the curve α, respectively [18]. A curve for which $k_n = 0$, $k_g = 0$ or $\tau_g = 0$ at every point on the surface is called an asymptotic curve, a geodesic curve or a line of curvature, respectively [19].

Screw Surfaces

Let M and M^v be two surfaces in E^3 and $p \in M$. Let

$$r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})$$

be a rotation by an angle of θ radian about the unit vector $\vec{v} = (v_x, v_y, v_z)$ and $\vec{t} = \lambda\vec{v}$ be the translational vector. If there is a function defined as

$$f : M \to M^v$$
$$p \to f(p) = rp\bar{r} + t$$

then the surface M^v is called a screw surface of the surface M [14]. Let the rotation axis be the unit normal of the surface M at the point p, i.e. let $\vec{v} = \vec{U}$. In this case the function of a screw surface is

$$f(p) = \cos\theta p + \sin\theta\vec{U} \wedge p + 2\sin^2\frac{\theta}{2}\langle\vec{U}, p\rangle\vec{U} + \lambda\vec{U}.$$

If the position vector of the point p and the unit normal vector $\vec{U}$ are parallel then this equation becomes as

$$f(p) = \cos^2\frac{\theta}{2} p + \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\vec{U}_p.$$

This paper consists of this case. All operations will be done according to this case in the rest of the study.

Following theorem shows how to transfer tangent vectors of a surface to the screw surface of this surface:

Theorem 2.

Let M^v be the screw surface of a surface M in E^3. Let the vector fields $X \in \chi(M)$, $\overline{X} \in \chi(M^v)$ be given by the property that for $\forall p \in M$, $b_i(p) = \overline{b_i}(f(p))$, $1 \le i \le 3$ with respect to the Euclidean coordinate system $\{x_1, x_2, x_3\}$ of the space E^3, where

$$X = \sum_{i=1}^{3} b_i \frac{\partial}{\partial x_i}, \overline{X} = \sum_{i=1}^{3} \overline{b_i} \frac{\partial}{\partial x_i}.$$

Then

1. $f_*\left(X_p\right) = \cos^2\frac{\theta}{2} X_p + \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)S(X_p)$

2. $S^v(f_*(X))_{f(p)} = S(X)_p$.

[14].

Proof.

1. The points of the screw surface M^v of a surface M are obtained by the function $f : M \to M^v$ as

$$f(p) = \cos^2\frac{\theta}{2} p + \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\vec{U}_p.$$

The vectors on the surface M can be transferred to the surface M^v by the map

$$f_{*_p} \leftrightarrow \cos^2\frac{\theta}{2} I_3 + \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\left[\frac{\partial a_i}{\partial x_j}\right]_p + \frac{1}{\|p\|}\sin^2\frac{\theta}{2}\left[p_i a_j\right].$$

where $U = \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i}$ is the unit normal vector field of the surface M. Then for $\forall X_p \in T_pM$ we have (1)

$$f_{*_p}\left(X_p\right) = \cos^2\frac{\theta}{2} X_p + \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)S(X_p).$$

2. Let $U = \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i}$ and S be the unit normal vector field and shape operator of the surface M, respectively. Let us represent the unit normal vector field of M^v by U^v and the shape operator by S^v.

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Then at the point $f(p)$ the normal vector of M^v is $\overrightarrow{U^v}_{f(p)} = \sum_{i=1}^{3}\overline{a_i}\left(f(p)\right)\frac{\partial}{\partial x_i}\Big|_p = \sum_{i=1}^{3}a_i\left(p\right)\frac{\partial}{\partial x_i}\Big|_p = \vec{U}_p$, where $\overline{a_i} \circ f = a_i$.

Assume that $X_p = \sum_{i=1}^{3}b_i\left(p\right)\frac{\partial}{\partial x_i}\Big|_p$ is the unit tangent vector of a curve $\alpha : I \to M$ at the point $p \in M$. In this case the tangent vector $f_*\left(X_p\right) \in T_{f(p)}M^v$ is also the tangent vector of the curve $f \circ \alpha : I \to M^g$ at the point $f(p) \in M^g$. Thus, at the point $p = \alpha(t)$ we have

$$S(X)_p = \sum_{i=1}^{3}X_{\alpha(t)}[a_i]\frac{\partial}{\partial x_i}\Big|_{\alpha(t)} = \sum_{i=1}^{3}\frac{d(a_i \circ \alpha)}{dt}\Big|_{\alpha(t)}$$

$$= \sum_{i=1}^{3}\frac{d(\overline{a_i} \circ f \circ \alpha)}{dt}\Big|_{\alpha(t)}$$

$$= \sum_{i=1}^{3} f_*\left(X\right)\Big|_{f(p)}\left[\overline{a_i}\right]\frac{\partial}{\partial x_i}\Big|_p .$$

On the other hand, since

$$S^v(f_*(X))_{f(p)} = D_{f_*(X)_{f(p)}}U^v = \sum_{i=1}^{3}X_p[a_i]\frac{\partial}{\partial x_i}\Big|_p$$

$$= \sum_{i=1}^{3} f_*(X)\Big|_{f(p)}[\overline{a_i}]\frac{\partial}{\partial x_i}\Big|_{f(p)}$$

we get $S^v(f_*(X))_{f(p)} = S(X)_p$.

Darboux Frame of A Image Curve on Screw Surface

Let M^v be the screw surface of a surface M and $\beta = \left(f \circ \alpha\right)$ be the image of a unit speed curve α on M. Then, $\overrightarrow{\beta'} = \overrightarrow{f_*(T)}$. Since the curve β is not a unit speed curve at the point $f\left(\alpha(t)\right)$, Darboux frame of this curve is

$$\left\{\overrightarrow{T^v} = \frac{\overrightarrow{f_*(T)}}{v}, \overrightarrow{V^v} = \overrightarrow{U^v} \wedge \overrightarrow{T^v}, \overrightarrow{U^v}\right\}$$

where $v = \left\|\overrightarrow{f_*(T)}\right\|$.

Following theorem shows how to obtain Darboux frame of the surface M^v at the point $f(p) = f\left(\alpha(t)\right)$.

Theorem 3.

Let $\left\{\vec{T}, \vec{V}, \vec{U}\right\}$ be the Darboux frame of a curve α on M. Then the Darboux frame of the image curve $\beta = \left(f \circ \alpha\right)$ on M^v at the point $f(p) = f\left(\alpha(t)\right)$ is

$$\overrightarrow{T^v}_{f(p)} = \frac{1}{v}\left[\left(\cos^2\frac{\theta}{2} - k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{T_p}\right.$$

$$\left. -\tau_g\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\overrightarrow{V_p}\right]$$

$$\overrightarrow{V^v}_{f(p)} = \frac{1}{v}\left[\left(\cos^2\frac{\theta}{2} - k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{V_p}\right.$$

$$\left. +\tau_g\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\overrightarrow{T_p}\right]$$

$$\overrightarrow{U^v}_{f(p)} = \vec{U}_p$$

where k_n and τ_g are normal curvature and geodesic torsion of M, respectively.

Proof.

Since the derivative of the unit normal of M at the point $p = \alpha(t)$ is $\overrightarrow{U'_p} = -k_n\overrightarrow{T_p} - \tau_g\overrightarrow{V_p}$, the tangent vector of the curve β at the point $f(p) = f(\alpha(t))$ is

$$\overrightarrow{\beta'(f(p))} = \overrightarrow{f_*\left(T_p\right)}$$

$$=\left(\cos^2\frac{\theta}{2}-k_n\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)\right)\overrightarrow{T_p}$$
$$-\tau_g\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)\overrightarrow{V_p}.$$

Then we have

$$v=\left\|\overrightarrow{f_*(T)}\right\|$$
$$=\left[\left(\cos^2\frac{\theta}{2}-k_n\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)\right)^2+\tau_g{}^2\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)^2\right]^{1/2}.$$

Therefore we get

$$\overrightarrow{T^v}_{f(p)}=\frac{1}{v}\left[\left(\cos^2\frac{\theta}{2}-k_n\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)\right)\overrightarrow{T_p}-\tau_g\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)\overrightarrow{V_p}\right].$$

Since M^v is the screw surface of M we have $\overrightarrow{U^v}_{f(p)}=\overrightarrow{U}_p$. Then

$$\overrightarrow{V^v}_{f(p)}=\left(\overrightarrow{U^v}\wedge\overrightarrow{T^v}\right)_{f(p)}$$

$$=\frac{1}{v}\left[\left(\cos^2\frac{\theta}{2}-k_n\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)\right)\overrightarrow{V_p}+\tau_g\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)\overrightarrow{T_p}\right].$$

Let α be a unit speed curve on M, $\beta=(f\circ\alpha)$ be the image of α on the screw surface M^v and $\left\{\overrightarrow{T^v},\overrightarrow{V^v},\overrightarrow{U^v}\right\}_{f(\alpha(t))}$ be the Darboux frame of the curve β at the point $f(\alpha(t))\in M^v$. Then

$$k_g^v(f(\alpha(t)))=\left\langle\overrightarrow{T^{v'}},\overrightarrow{V^v}\right\rangle_{f(\alpha(t))}=-\left\langle\overrightarrow{V^{v'}},\overrightarrow{T^v}\right\rangle_{f(\alpha(t))}$$

is called the geodesic curvature of the screw surface M^v. Let α be a unit speed curve on M, $\beta=(f\circ\alpha)$ be the image of α on the screw surface M^v and $\left\{\overrightarrow{T^v},\overrightarrow{V^v},\overrightarrow{U^v}\right\}_{f(\alpha(t))}$ be the Darboux frame of the curve β at the point $f(\alpha(t))\in M^v$. Then

$$k_n^v(f(\alpha(t)))=\left\langle\overrightarrow{T^{v'}},\overrightarrow{U^v}\right\rangle_{f(\alpha(t))}=-\left\langle\overrightarrow{U^{v'}},\overrightarrow{T^v}\right\rangle_{f(\alpha(t))}$$

is called the normal curvature of the screw surface M^v. Let α be a unit speed curve on M, $\beta=(f\circ\alpha)$ be the image of α on the screw surface M^v and $\left\{\overrightarrow{T^v},\overrightarrow{V^v},\overrightarrow{U^v}\right\}_{f(\alpha(t))}$ be the Darboux frame of the curve β at the point $f(\alpha(t))\in M^v$. Then

$$\tau_g^v(f(\alpha(t)))=\left\langle\overrightarrow{V^{v'}},\overrightarrow{U^v}\right\rangle_{f(\alpha(t))}=-\left\langle\overrightarrow{U^{v'}},\overrightarrow{V^v}\right\rangle_{f(\alpha(t))}$$

is called the geodesic torsion of the screw surface M^v.

Theorem 4.

Let α be a unit speed curve on M and $\beta=(f\circ\alpha)$ be the image of α on the screw surface M^v. Then geodesic curvature, normal curvature and geodesic torsion of $\beta=(f\circ\alpha)$ at the point $f(\alpha(t))$ are

$$k_g^v=\frac{1}{v^3}\left[\tau_g'\left(k_n\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)^2-\cos^2\frac{\theta}{2}\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)\right)-\tau_g\left(k_g v^2+\frac{\langle p,T_p\rangle}{\|p\|}\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}+k_n'\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)^2\right)\right]$$

$$k_n^v=\frac{1}{v^2}\left(k_n\cos^2\frac{\theta}{2}-\left(k_n^2+\tau_g^2\right)\left(\|p\|\sin^2\frac{\theta}{2}+\lambda\right)\right)$$

and

$$\tau_g^v=\frac{\tau_g}{v^2}\cos^2\frac{\theta}{2},$$

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respectively.

Proof.

Since

$$\overrightarrow{\beta'(f(p))} = \left(\cos^2\frac{\theta}{2} - k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{T_p}$$

$$-\tau_g\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\overrightarrow{V_p}$$

we have

$$\overrightarrow{\beta''(f(p))} = \left(k_g\tau_g\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right) - k_n\frac{\langle p, T_p\rangle}{\|p\|}\sin^2\frac{\theta}{2}\right.$$

$$\left.-k_n'\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{T_p}$$

$$+\left(k_n\cos^2\frac{\theta}{2} - k_n^{\,2}\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right.$$

$$\left.-\tau_g^{\,2}\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{U_p}$$

$$+\left(k_g\cos^2\frac{\theta}{2} - (k_nk_g + \tau_g')\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right.$$

$$\left.-\tau_g\frac{\langle p, T_p\rangle}{\|p\|}\sin^2\frac{\theta}{2}\right)\overrightarrow{V_p}.$$

Since $\beta = (f \circ \alpha)$ is not a unit speed curve, the geodesic curvature of M^v is

$$k_g^{\,v} = \frac{1}{v^2}\left\langle\overrightarrow{\beta''}, \overrightarrow{V^v}\right\rangle_{f(p)}$$

$$= \frac{1}{v^3}\left[-\tau_g\left(\frac{\langle p, T_p\rangle}{\|p\|}\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} + k_gv^2\right.\right.$$

$$+k_n'\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)^2\right) + \tau_g'\left(k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)^2\right.$$

$$\left.\left.-\cos^2\frac{\theta}{2}\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\right].$$

The normal curvature can be obtained as

$$k_n^{\,v} = \frac{1}{v^2}\left\langle\overrightarrow{\beta''}, \overrightarrow{U^v}\right\rangle_{f(p)}$$

$$= \frac{1}{v^2}\left(k_n\cos^2\frac{\theta}{2} - \left(k_n^2 + \tau_g^2\right)\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right).$$

Finally, one can get the geodesic torsion as

$$\tau_g^{\,v} = \frac{1}{v}\left\langle\overrightarrow{V^{v\prime}}, \overrightarrow{U^v}\right\rangle_{f(p)} = -\frac{1}{v}\left\langle\overrightarrow{U'}, \overrightarrow{V^v}\right\rangle_{f(p)}$$

$$= \frac{\tau_g}{v^2}\cos^2\frac{\theta}{2}.$$

Corollary 1.

The image curve of a geodesic curve is also geodesic if and only if the curve on the main surface M is a line of curvature.

Proof.

If the curve on the main surface M is a geodesic then the geodesic curvature of this curve is $k_g = 0$. If this curve is a line of curvature then we have $\tau_g = 0$. Hence, we get $k_g^{\,v} = 0$ and this completes the proof.

Corollary 2.

The image curve of an asymptotic curve is also asymptotic if and only if the curve on the main surface M is a line of curvature.

Proof.

If the curve on the main surface M is an asymptotic curve then the normal curvature of this curve is $k_n = 0$. If this curve is a line of curvature then we have $\tau_g = 0$. Hence, we get $k_n^{\,v} = 0$ and this completes the proof.

Corollary 3.

The image curve of a line of curvature is also a line of curvature.

Proof.

Since the curve on the main surface M is a line of curvature, then we have $\tau_g = 0$. Therefore, we can obtain $\tau_g{}^v = 0$ and this completes the proof.

Frenet Frame of A Image Curve on Screw Surface

Let $\beta = (f \circ \alpha)$ be the image of α on the screw surface M^v and $\{\vec{T}, \vec{V}, \vec{U}\}$ be the Darboux frame of α on M. Since the curve β is not a unit speed curve, we have

$$\left(\overrightarrow{\beta'} \wedge \overrightarrow{\beta''}\right)|_{f(p)} = -v^2 k_n^v \tau_g \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\overrightarrow{T_p}$$
$$+v^3 k_g^v \overrightarrow{U_p} - v^2 k_n^v \left(\cos^2\frac{\theta}{2} - k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{V_p}.$$

It follows that

$$\left\|\overrightarrow{\beta'} \wedge \overrightarrow{\beta''}\right\|_{f(p)} = v^3\sqrt{\left(k_n^v\right)^2 + \left(k_g^v\right)^2}.$$

Therefore, we get

$$\overrightarrow{B^v}|_{f(p)} = \frac{\overrightarrow{\beta'} \wedge \overrightarrow{\beta''}}{\left\|\overrightarrow{\beta'} \wedge \overrightarrow{\beta''}\right\|}$$
$$= \frac{1}{v\sqrt{\left(k_n^v\right)^2 + \left(k_g^v\right)^2}}\left[-k_n^v \tau_g \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\overrightarrow{T_p}\right.$$
$$\left.+v k_g^v \overrightarrow{U_p} - k_n^v \left(\cos^2\frac{\theta}{2} - k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{V_p}\right].$$

On the other hand, one can obtain that

$$\overrightarrow{N^v}|_{f(p)} = \left(\overrightarrow{B^v} \wedge \overrightarrow{T^v}\right)_{f(p)}$$
$$= \frac{1}{v\sqrt{\left(k_n^v\right)^2 + \left(k_g^v\right)^2}}\left[k_g^v \tau_g \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\overrightarrow{T_p}\right.$$
$$\left.+v k_n^v \overrightarrow{U_p} + k_g^v \left(\cos^2\frac{\theta}{2} - k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{V_p}\right].$$

Therefore, we can give the following theorem:

Theorem 5.

Let $\beta = (f \circ \alpha)$ be the image of α on the screw surface M^v and $\{\vec{T}, \vec{V}, \vec{U}\}$ be the Darboux frame of α on M. Then the Frenet frame of $\beta = (f \circ \alpha)$ at the point $f(p) = f(\alpha(t))$ on the screw surface M^v is

$$\overrightarrow{T^v}_{f(p)} = \frac{1}{v}\left[\left(\cos^2\frac{\theta}{2} - k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{T_p}\right.$$
$$\left.-\tau_g\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\overrightarrow{V_p}\right].$$
$$\overrightarrow{N^v}|_{f(p)} = \frac{1}{v\sqrt{\left(k_n^v\right)^2 + \left(k_g^v\right)^2}}\left[k_g^v \tau_g \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\overrightarrow{T_p}\right.$$
$$\left.+v k_n^v \overrightarrow{U_p} + k_g^v \left(\cos^2\frac{\theta}{2} - k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{V_p}\right].$$
$$\overrightarrow{B^v}|_{f(p)} == \frac{1}{v\sqrt{\left(k_n^v\right)^2 + \left(k_g^v\right)^2}}\left[-k_n^v \tau_g \left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\overrightarrow{T_p}\right.$$
$$\left.+v k_g^v \overrightarrow{U_p} - k_n^v \left(\cos^2\frac{\theta}{2} - k_n\left(\|p\|\sin^2\frac{\theta}{2} + \lambda\right)\right)\overrightarrow{V_p}\right].$$

Theorem 6.

Let $\left\{\overrightarrow{T^v}, \overrightarrow{N^v}, \overrightarrow{B^v}\right\}$ be the Frenet frame of the curve $\beta = (f \circ \alpha)$ on the screw surface M^v and Φ^v be the angle between the normal $\overrightarrow{N^v}$ of the curve β and the unit normal $\overrightarrow{U^v}$ of the surface M^v. Then the curvature of β is

$$\kappa^{v} = \sqrt{(k_n^v)^2 + (k_g^v)^2}$$

and

$$\cos\Phi^{v} = \frac{k_n^v}{K^v}.$$

Proof.

Since

$$\kappa^{v} = \frac{\left\|\overrightarrow{\beta'} \wedge \overrightarrow{\beta''}\right\|}{\left\|\overrightarrow{\beta'}\right\|^3}$$

we get

$$\kappa^{v} = \frac{v^3\sqrt{(k_n{}^v)^2 + (k_g{}^v)^2}}{v^3}$$

$$= \sqrt{(k_n^v)^2 + (k_g^v)^2}.$$

On the other hand,

$$\cos\Phi^{v} = \left\langle \overrightarrow{N^v}, \overrightarrow{U^v} \right\rangle$$

$$= \frac{1}{v\sqrt{(k_n{}^v)^2 + (k_g{}^v)^2}} v k_n{}^v \left\langle \overrightarrow{U^v}, \overrightarrow{U^v} \right\rangle$$

$$= \frac{k_n^v}{K^v}.$$

Example.

Consider the surface

$$M = \left\{\varphi(u,v) = (\cos u\cos v, \cos u\sin v, \sin u) \mid -\pi/2 \le u \le 0, -\pi/2 \le v \le 0\right\}.$$

The unit normal of the surface is $\vec{U} = (-\cos u\cos v, -\cos u\sin v, -\sin u)$.

Let the rotation axis be $\overrightarrow{U_p}$, rotation angle be $\pi/2$ and the translation vector be $2\overrightarrow{U_p}$ for $\forall p = \phi(u,v) \in M$. Then the screw motion rotates the points of M about $\overrightarrow{U_p}$ by an angle of $\pi/2$ radian and translates them two units in direction of $\overrightarrow{U_p}$. After the necessary calculations we get the screw surface

$$M^{v} = \left\{(-2\cos u\cos v, -2\cos u\sin v, -2\sin u) \mid -\pi/2 \le u \le 0, -\pi/2 \le v \le 0\right\}$$

of the surface M.

Let us consider the curve

$$C = \left\{\varphi(u,0) = (0, -cosu, sinu) \mid -\pi/2 \le u \le 0, -\pi/2 \le v \le 0\right\}.$$

Darboux frame of this curve at the point $p_0 = (0,-1,0) \in M$ is

$$\vec{T}(p_0) = (0,0,1),$$
$$\vec{V}(p_0) = (1,0,0),$$
$$\vec{U}(p_0) = (0,1,0).$$

One can easily calculate the geodesic curvature, the normal curvature and the geodesic torsion of the curve at the point p_0 as $k_g(p_0) = 0$, $k_n(p_0) = 1$ and $\tau_g(p_0) = 0$, respectively. Frenet frame of the curve C can be obtained as

$$\vec{T}(p_0) = (0,0,1),$$
$$\vec{N}(p_0) = (0,1,0),$$
$$\vec{B}(p_0) = (-1,0,0).$$

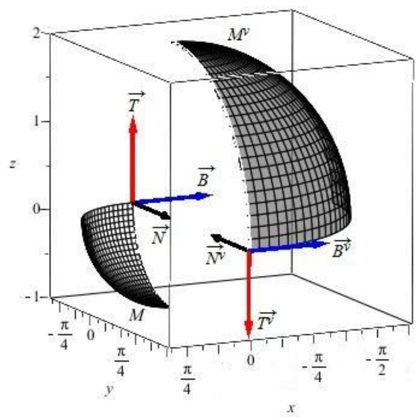

Figure 1: Serret-Frenet Frame of A Image Curve

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On the other hand, image of the point p_0 on the screw surface will be the point $g_0 = f(p_0) = (0,2,0)$. Then, by Theorem 5, we get the Frenet frame of the image curve C' on the screw surface at the point g_0 as (Figure 1)

$$\overrightarrow{T^v}_{(f(p_0))} = (0,0,-1),$$
$$\overrightarrow{N^v}_{(f(p_0))} = (0,-1,0),$$
$$\overrightarrow{B^v}_{(f(p_0))} = (-1,0,0)$$

and by Theorem 4 Darboux frame of the image curve C' on the screw surface at the point g_0 as (Figure 2)

$$\overrightarrow{T^v}_{(f(p_0))} = (0,0,-1),$$
$$\overrightarrow{V^v}_{(f(p_0))} = (-1,0,0),$$
$$\overrightarrow{U^v}_{(f(p_0))} = (0,1,0).$$

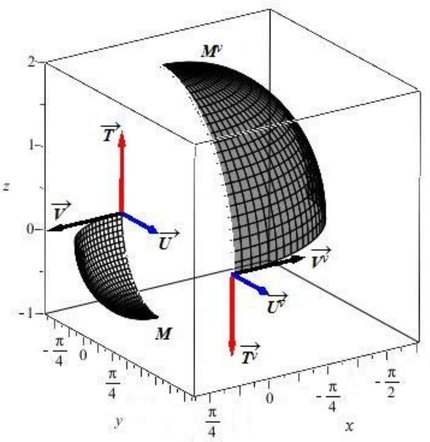

Figure 2: Darboux Frame of A Image Curve

By Theorem 4, the geodesic curvature, the normal curvature and the geodesic torsion of the curve C' at the point g_0 can be calculated as $k_g^v(p_0) = 0$, $k_n^v(p_0) = -1/2$ and $\tau_g^v(p_0) = 0$, respectively.

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