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Research Article

Pseudo-Riemannian Submanifolds with 3-planar Geodesics

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Abstract

In the present paper, we study pseudo-Riemannian submanifolds which have 3-planar geodesic normal sections. Further, we consider W-curves (helices) on pseudo-Riemannian submanifolds. Finally, we give necessary and sufficient condition for a normal section to be a W-curve on pseudo-Riemannian submanifolds.

Keywords: Pseudo-Riemannian submanifold, geodesic normal section, W-curve, planar geodesic.

3-Düzlemsel Geodezikli Yarı-Riemann Altmanifoldlar

Öz

Bu çalışmada, 3-düzlemsel geodezik normal kesitlere sahip yarı-Riemann almanifoldlar ele alınmıştır. Daha sonra, yarı-Riemann altmanifoldları üzerindeki W-eğrileri (helisler) incelenmiştir. Son olarak, yarı-Riemann altmanifoldları üzerindeki normal kesitlerin W-eğrisi olması için gerek ve yeter şartlar elde edilmiştir.

Anahtar Kelimeler : Yarı-Riemann altmanifold, geodezik normal kesit, W-eğrisi, düzlemsel geodezik

Introduction

Let M^n be an n-dimensional Riemannian manifold. A regular curve γ in M^n is called a helix if its first and second curvatures are constant and the third curvature is zero. It has been shown that every helix in a Riemannian submanifold M^n is also a helix in the ambient space [1]. For the pseudo-Riemannian manifold M_r^n, helices are defined almost the same way as the Riemannian case. The helices are characterized in Lorentzian submanifold $M_r^n \subset N_s^m$ [2].

A submanifold $M_r^n \subset N_s^m$ is said to have planar geodesics if the image of each geodesic of M_r^n lies in a 2-plane of N_s^m [3]. In the Riemannian case such submanifolds were studied in [4], [5], [6], and [7]. Recently, Kim studied minimal surfaces of pseudo-Euclidean spaces with

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geodesic normal sections [8].

In the present study, we give some results toward a characterization of 3-planar geodesic immersions $f : M_r^n \to N_s^m$ from an n-dimensional, connected pseudo-Riemannian manifold M_r^n into an m-dimensional pseudo-Riemannian manifold N_s^m. Further, we consider W-curves (helices) on pseudo-Riemannian submanifolds. Finally, we give necessary and sufficient condition for a normal section to be a W-curve on pseudo-Riemannian submanifold M_r^n.

Basic Concepts

Let $M_r^n \subset N_s^m$ be a submanifold in an m-dimensional pseudo-Riemannian manifold N_s^m of index s. Let ∇ and $\tilde{\nabla}$ denote the covariant derivatives of M_r^n and N_s^m respectively. Then, for $X, Y \in T_p(M_r^n)$ the second fundamental form *h* of M_r^n is defined by

$$h(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y. \tag{1}$$

For a normal vector field $\xi \in N(M_r^n)$ we put

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2}$$

where A_ξ is the shape operator and *D* is the normal connection of M_r^n.

The covariant derivatives of *h* is given by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{3}$$

where $X, Y, Z \in T_p(M_r^n)$ and $\overline{\nabla}$ is the Vander Waerden-Bortolotti connection [9]. Then the Codazzi equation

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z) = (\overline{\nabla}_Z h)(X,Y). \tag{4}$$

holds. If $\overline{\nabla} h = 0$, then *h* is said to be parallel [10].

The mean curvature vector field *H* of M_r^n is defined by

$$H = \frac{1}{n} \sum \langle e_i, e_i \rangle h(e_i, e_i),\ i = 1,...,n, \tag{5}$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal frame field of M_r^n. Consequently, *H* is called parallel when $DH = 0$ holds.

If the second fundamental form *h* satisfies

$$g(X,Y)H = h(X,Y), \tag{6}$$

for any $X, Y \in T_p(M_r^n)$, then M_r^n is called a totally umbilical. A totally umbilical submanifold with parallel mean curvature vector fields is said to be an extrinsic sphere [11].

Helices in a Pseudo-Riemannian Manifold

Let γ be a regular curve in a pseudo-Riemannian manifold M_r^n. We denote $\gamma'(s) = X$, when $\langle X, X \rangle = \varepsilon; \varepsilon = \pm 1$ γ is called a unit speed curve. The curve γ is called a Frenet curve of rank d $(0 \le d \le n)$, if its derivatives $\gamma'(s), \gamma''(s), ..., \gamma^{(d)}(s)$ are linearly independent and $\gamma'(s), \gamma''(s), ..., \gamma^{(d+1)}(s)$ are no longer linearly independent for all $s \in I$ [7]. To each Frenet curve of order *d* we can associate an orthonormal *d* frame $\{V_1, V_2, ..., V_d\}$ along γ, called the Frenet

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frame, and $k_1, k_2, ..., k_{d-1}$ are curvature functions of γ.

We have the following result.

Proposition 1: Let $\gamma : I \to M_r^n$ be a non-null smooth curve of osculating order d in of M_r^n, and $\{V_1 = X, V_2, ..., V_d\}$ its Frenet frame. Then the following Frenet equations are hold;

$$V_1' = \nabla_X X = \varepsilon_2 k_1 V_2, \tag{7}$$

$$V_2' = \nabla_X V_2 = -\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3, \tag{8}$$

$$\vdots$$

$$V_{d-1}' = \nabla_X V_{d-1} = -\varepsilon_{d-2} k_{d-2} V_{d-2} + \varepsilon_d k_{d-1} V_d, \tag{9}$$

$$V_d' = \nabla_X V_d = -\varepsilon_{d-1} k_{d-1} V_{d-1}, \tag{10}$$

where $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1, 1 \le i \le d-1$ and k_i are curvature functions of γ.

Definition 2: A smooth curve γ of rank d on M_r^n is called a W-curve of rank d if its curvatures $k_1, k_2, ..., k_{d-1}$ are all constant and $k_d = 0$ [7].

Proposition 3: Let γ be a non-null W-curve of rank 2 in M_r^n. Then the third derivative γ''' of γ is a scalar multiple of γ'. In this case necessarily

$$\gamma'''(s) = -\varepsilon_1 \varepsilon_2 k_1^2 \gamma'(s). \tag{11}$$

Proof: By the use of (7) we have $\gamma''(s) = \varepsilon_2 k_1 V_2(s)$. Furthermore, differentiating this equation with respect to s and using (8) we obtain

$$\gamma'''(s) = -\varepsilon_1 \varepsilon_2 k_1^2 X + \varepsilon_2 k_1' V_2(s) + \varepsilon_2 \varepsilon_3 k_1 k_2 V_3(s) \tag{12}$$

Since γ is a W-curve of rank 2 then by definition k_1 is constant and $k_2 = 0$ we get the result.

Proposition 4: Let γ be a non-null W-curve of M_r^n. If γ is of osculating order 3 then

$$\gamma^{(iv)}(s) = -\varepsilon_2 (\varepsilon_1 k_1^2 + \varepsilon_3 k_2^2) \gamma''(s) \tag{13}$$

holds.

Proof: Differentiating (12) and using the fact that k_1, k_2 are constant and $k_3 = 0$ we get the result.

Planar Geodesic Immersions

Let $M_r^n \subset N_s^m$ be a submanifold in an m-dimensional pseudo-Riemannian manifold N_s^m of index s. For $p \in M_r^n$ and $X \in T_p(M_r^n)$ the vector X and the normal space $N_p(M_r^n)$ determine a $(m-n+1)$-dimensional totally geodesic submanifold Γ of N_s^n. The intersection of M_r^n with Γ gives rise a curve γ (in a neighborhood of p) called the normal section of M_r^n at p in the direction of X [12]. The submanifold M_r^n is said to have d-planar normal sections if for each normal section γ the higher order derivatives $\gamma'(s), \gamma''(s), ..., \gamma^{(d)}(s), \gamma^{(d+1)}(s), 1 \le d \le m-n+1$ are linearly dependent as vectors in Γ [12]. The submanifold M_r^n is said to have d-planar geodesic normal sections if each normal section of M_r^n is a geodesic of M_r^n. The immersion in pseudo-Euclidean space with 2-planar geodesic normal section have been studied in [3]. See also [4].

Example 5: [3] Pseudo-Riemannian sphere

$$S^n_r(c)=\left\{p\in E^{n+1}_r:\langle p-a,p-a\rangle=\frac{1}{c}\right\},c>0, \quad (14)$$

and pseudo-Riemannian hyperbolic space

$$H^n_r(c)=\left\{p\in E^{n+1}_{r+1}:\langle p-a,p-a\rangle=\frac{1}{c}\right\},c<0, \quad (15)$$

both have 2-planar geodesic normal sections.

We have the following result.

Proposition 6: Let γ be a non-null geodesic normal section of $M^n_r\subset N^m_s$. If $\gamma'(s)=X(s)$, then we have

$$\gamma''(s)=h(X,X), \quad (16)$$

$$\gamma'''(s)=-A_{h(X,X)}X+(\overline{\nabla}_X h)(X,X), \quad (17)$$

$$\begin{aligned}\gamma^{(iv)}(s)=&-\nabla_X(A_{h(X,X)}X)-h(A_{h(X,X)}X,X)\\&-A_{(\overline{\nabla}_X h)(X,X)}X+(\overline{\nabla}_X\overline{\nabla}_X h)(X,X),\end{aligned} \quad (18)$$

Definition 7: The submanifold M^n_r (or the isometric immersion *f*) is said to be pseudo-isotropic at *p* if

$$L=\langle h(X,X),h(X,X)\rangle,$$

is independent of the choice of unit vector *X* tangent to M^n_r at *p*. In particular if *L* is independent of the points then M^n_r is said to be constant pseudo-isotropic.

The submanifold M^n_r is pseudo-isotropic if and only if

$$\langle h(X,X),h(X,Y)\rangle=0,$$

for any orthonormal vectors *X* and *Y* [3].

The following results are well-known.

Theorem 8: [3] If a submanifold $M^n_r\subset E^m_s$ has 2-planar geodesic normal sections, then it is a submanifold with zero mean curvature in a hypersphere S^{m-1}_{s-1} or H^{m-1}_{s-1} if and only if *L* is a non-zero constant.

Theorem 9: [8] The surface $M^2_r\subset E^m_s$ with 2-planar geodesic normal sections is constant pseudo-isotropic.

Theorem 10: [13] Let M^n_r be a pseudo-Riemannian submanifold of index *r* of a pseudo-Euclidean space E^m_s of index *s* with geodesic normal sections. Then

$$\langle(\overline{\nabla}_X h)(X,X),(\overline{\nabla}_X h)(X,X)\rangle \quad (19)$$

is constant on the their tangent bundle *UM* of M^n_r.

Theorem 11: [13] Let M^2_r be a minimal surface of E^5_s with geodesics normal sections. Then we have

i) M^2_r has parallel second fundamental form and 0-pseudo isotropic (i.e. L=0),

ii) M^2_r has 2-planar geodesic normal sections,

iii) M^2_r is flat.

Main Results

Submanifolds M^n in E^{n+d} with 3-planar normal sections have been studied by S.J. Li for the case M^n is isotropic [14] and sphered [15]. See also [16] for the case M^n is a product manifold in E^{n+d}. In [17] the authors consider submanifolds in a real space form $N^{n+d}(c)$ with 3-planar geodesic normal sections.

We proved the following results.

Proposition 12: Let $M_r^n \subset N_s^m$ be a submanifold with 3-planar geodesic normal sections then M_r^n is constant pseudo-isotropic.

Proof: Similar to the proof of Lemma 4.1 in [18].

Proposition 13: Let $M_r^n \subset N_s^m$ be a submanifold with 3-planar geodesic normal sections then we have

$$(\overline{\nabla}_X h)(X,X) = \varepsilon_2 (Xk_1)V_2 + \varepsilon_2\varepsilon_3 k_1 k_2 V_3, \quad (20)$$

and

$$A_{h(X,X)}X = \varepsilon_1\varepsilon_2 k_1^2 X, \quad (21)$$

hold.

Proof: Let γ be a normal section of M_r^n at point $p=\gamma(s)$ in the direction of X. Further, we suppose that $k_1(s)$ is positive. Then k_1 is also smooth and there exists a unit vector field V_2 along γ normal to M_r^n such that

$$h(X,X) = \langle V_2, V_2\rangle k_1 V_2. \quad (22)$$

Since $\overline{\nabla}_X V_2$ is also tangent to M_r^n, there exists a vector field V_3 normal to M_r^n and mutually orthogonal to X and V_2 such that

$$\tilde{\nabla}_X V_2 = -\langle X, X\rangle k_1 X + \langle V_3, V_3\rangle k_2 V_3. \quad (23)$$

Differentiating (22) covariantly and using (23) we get

$$(\overline{\nabla}_X h)(X,X) = -\varepsilon_1\varepsilon_2 k_1^2 X + \varepsilon_2 (Xk_1)V_2 + \varepsilon_2\varepsilon_3 k_1 k_2 V_3, \quad (24)$$

where $\langle V_i, V_i\rangle = \varepsilon_i = \pm 1$. Comparing (24) with (17) we get the result.

Proposition 14: Let γ be a normal section of M_r^n at point $p=\gamma(s)$ in the direction of X. γ is a non-null W-curve of rank 2 in M_r^n if and only if

$$\nabla_X \nabla_X X + g(\nabla_X X, \nabla_X X) g(X,X) X = 0 \quad (25)$$

holds.

Proof: Since $\gamma'(s) = X(s)$, $\gamma''(s) = \nabla_X \nabla_X X$ and

$$g(X,X) = \varepsilon_1, \; g(\nabla_X X, \nabla_X X) = \varepsilon_2 k_1^2.$$

So, by the use of the equality $\gamma''(s) = \varepsilon_2 k_1 V_2(s)$ we get the result.

Theorem 15: Let M_r^n be a totally umbilical submanifold of N_s^m with parallel mean curvature vector field . If the normal section γ is a W- curve of osculating order 2. Then γ is also a W-curve of N_s^m with the same order.

Proof: Suppose γ is a W-curve of rank 2 in M_r^n then it satisfies the equality (25). Further, by the use of (1) we get

$$\gamma'' = \tilde{\nabla}_X X = \nabla_X X + h(X,X). \quad (26)$$

Since M_r^n is totally umbilical then $g(X,X)H = h(X,X)$. So, the equation (26) reduces to

$$\gamma'' = \tilde{\nabla}_X X = \nabla_X X + g(X,X)H. \quad (27)$$

Differentiating the equation (27) with respect to X, we obtain

$$\gamma''' = \tilde{\nabla}_X \tilde{\nabla}_X X = \nabla_X \nabla_X X + g(X, \nabla_X X)H + g(X, X)(-A_H X + D_X H). \quad (28)$$

Further, taking use of $DH = 0$ and (26)-(28) get

$$\begin{aligned} &\tilde{\nabla}_X \tilde{\nabla}_X X + g(\tilde{\nabla}_X X, \tilde{\nabla}_X X) g(X, X) X \\ &= \nabla_X \nabla_X X - g(H, H) g(X, X) X \\ &+ \{g(\nabla_X X, \nabla_X X) g(X, X)\} g(X, X) X \\ &= \nabla_X \nabla_X X + g(\nabla_X X, \nabla_X X) g(X, X) X. \end{aligned}$$

So, by previous proposition γ is a W-curve of rank 2 in N_s^m .

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