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Graph Surfaces Invariant by Parabolic Screw Motions with Constant Curvature in $\mathbb{H}^2 \times \mathbb{R}$

Uğur Dursun

(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

In this work we study vertical graph surfaces invariant by parabolic screw motions with pitch $\ell > 0$ and constant Gaussian curvature or constant extrinsic curvature in the product space $\mathbb{H}^2 \times \mathbb{R}$. In particular, we determine flat and extrinsically flat graph surfaces in $\mathbb{H}^2 \times \mathbb{R}$. We also obtain complete and non-complete vertical graph surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with negative constant Gaussian curvature and zero extrinsic curvature.

Keywords: *Parabolic screw motion, graph surface, Gaussian curvature, extrinsic curvature, flat surface.*
AMS Subject Classification (2020): *Primary: 53C40; Secondary: 53B25.*

1. Introduction

In [12, 17], H. Rosenberg and W. Meeks studied minimal surfaces in $M^2 \times \mathbb{R}$, where M^2 is a rounded sphere, a complete Riemannian surface with a metric of non-negative curvature, or $M^2 = \mathbb{H}^2$, the hyperbolic plane. Since then, there has been a rapid growing interest in minimal surfaces and surfaces with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, see for instance [4, 5, 9, 13, 14, 15, 18, 19, 20]. Also, surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ with constant Gaussian curvature or constant extrinsic curvature have attracted many attention in the recent years, [1, 2, 3, 6, 7, 16].

In [1], J. A. Aldeo and et al. proved that there exists a unique complete surface of positive constant Gaussian curvature in $\mathbb{H}^2 \times \mathbb{R}$ and a unique complete surface of positive constant curvature greater than 1 in $\mathbb{S}^2 \times \mathbb{R}$, up to isometries of the ambient space. These complete surfaces are precisely the revolution surfaces. Also, they proved that there is no complete immersion of constant Gaussian curvature $K < -1$ into $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. In [2] J. A. Aldeo and et al. obtained some free boundary results for compact surfaces of positive constant Gaussian curvature in $\mathbb{H}^2 \times \mathbb{R}$ and positive constant Gaussian curvature greater than 1 in $\mathbb{S}^2 \times \mathbb{R}$.

In [7], J. M. Espinar and et al. studied complete surfaces with positive extrinsic curvature in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, and they proved that every complete connected immersed surface with positive extrinsic curvature in $\mathbb{H}^2 \times \mathbb{R}$ must be properly embedded, homeomorphic to a sphere or a plane. They also showed that only complete surfaces with constant extrinsic curvature in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ are rotational sphere.

L. Belarbi [3] studied translation surfaces with constant extrinsic Gaussian curvature in the 3-dimensional Heisenberg group which are invariant under the 1-parameter groups of isometries.

In [16] R. Novais and P. D. Santos studied geometric characterizations of conformally flat and radially flat hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ are given by means of their extrinsic geometry, and in [6] Dillan and et al. classified minimal rotation hypersurfaces and flat rotation hypersurfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

Screw motion surfaces with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ were studied in [18, 19]. R. Sa Earp and E. Toubiana [19] obtained an explicit two parameter family of complete, embedded, simply connected, minimal screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with pitch ℓ, and for $\ell = 1$ each such surface has Gaussian curvature $K = -1$. In [18] R. Sa Earp studied complete minimal and surfaces with constant mean curvature invariant either by parabolic or by hyperbolic screw motions in $\mathbb{H}^2 \times \mathbb{R}$. Later, Q. Cui and et al. [4] studied the geometric behaviors of hyperbolic and parabolic screw motions surfaces immersed in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ with having

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*** Corresponding author***

constant mean curvature, where $\widetilde{PSL_2}(\mathbb{R}, \tau)$ is a homogeneous simply connected 3-manifold having isometry group of dimension 4.

The isometries of $\mathbb{H}^2$ generate isometries in $\mathbb{H}^2 \times \mathbb{R}$. In particular, a parabolic translation in $\mathbb{H}^2$ generates an isometry in $\mathbb{H}^2 \times \mathbb{R}$ that is called a parabolic isometry. In this work we only consider the parabolic isometries, and the compositions of such isometries with vertical translations which are called parabolic helicoidal-type isometries. The surfaces invariant by this kind of helicoidal isometries is called the parabolic screw motion surfaces.

Motivated by the work [18] on the parabolic screw motion surfaces with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$, we study vertical graph surfaces invariant by the parabolic screw motions in $\mathbb{H}^2 \times \mathbb{R}$ with constant Gaussian curvature or constant extrinsic curvature. We obtain the ordinary differential equations for the Gaussian curvature and extrinsic curvature of a graph surface $M(f)$ (invariant by the parabolic screw motion) in $\mathbb{H}^2 \times \mathbb{R}$ for the function of the form $f(x, y) = v(y) + \ell x$, where $v(y)$ is a $\mathcal{C}^2$ function. We prove that if a vertical graph surface $M(f)$ in $\mathbb{H}^2 \times \mathbb{R}$ for a function of the form $f(x, y) = u(x) + v(y)$ is extrinsically flat, then $u(x) = \ell x + c$, that is, $M(f)$ is a parabolic screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$, (see Sec. 3). Graph surfaces of the form $f(x, y) = u(x) + v(y)$ are also known as the translation surfaces in the literature. We determine graph surfaces $M(f)$ invariant by the parabolic screw motion (and also by parabolic translation) in $\mathbb{H}^2 \times \mathbb{R}$ with constant Gaussian curvature K and constant extrinsic curvature K_{ext}. We also obtain complete graph surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with negative constant Gaussian curvature and zero extrinsic curvature.

2. Preliminaries

Let $\mathbb{H}^2$ be the upper half-plane model $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ of the hyperbolic plane equipped with the hyperbolic metric $g = \dfrac{dx^2 + dy^2}{y^2}$ of constant curvature -1. We consider the product space $\widetilde{M}^3 = \mathbb{H}^2 \times \mathbb{R}$ with coordinates (x, y, t) and the metric $\tilde{g} = g + dt^2$.

Let $\widetilde{\nabla}$ denote the Riemannian connection of $\widetilde{M}^3$. The Riemannian curvature tensor $\widetilde{R}$ of $\widetilde{M}^3$ is given by

$$\widetilde{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z,$$

where X, Y, and Z are tangent vector fields on $\widetilde{M}^3$. If $X, Y \in T_p\widetilde{M}^3$ at a point $p \in \widetilde{M}^3$, then the *sectional curvature* of $\widetilde{M}^3$ for the plane spanned by X and Y in $T_p\widetilde{M}^3$ is

$$\widetilde{K}(X, Y) = -\frac{\widetilde{g}(\widetilde{R}(X, Y)X, Y)}{\widetilde{g}(X, X)\widetilde{g}(Y, Y) - \widetilde{g}(X, Y)\widetilde{g}(X, Y)}.$$

Let M be a regular surface in $\widetilde{M}^3$. Then, the Gauss equation of M in $\widetilde{M}^3$ is given by

$$\tilde{g}(\widetilde{R}(X, Y)Z, W) = \tilde{g}(R(X, Y)Z, W) + \tilde{g}(h(X, Z), h(Y, W)) - \tilde{g}(h(Y, Z), h(X, W)), \tag{2.1}$$

where $X, Y, Z, W \in TM$, h is the second fundamental form, and R is the Riemannian curvature tensor of M.

Let $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}, \partial_t = \frac{\partial}{\partial t}$ denote coordinate vector fields on $\widetilde{M}^3$. The vectors $E_1 = y\partial_x, E_2 = y\partial_y, E_3 = \partial_t$ form an orthonormal frame on $\widetilde{M}^3$, and in this frame, non-zero covariant derivatives of $\widetilde{M}^3$ are

$$\widetilde{\nabla}_{E_1} E_1 = E_2, \ \widetilde{\nabla}_{E_1} E_2 = -E_1. \tag{2.2}$$

2.1. Graph surfaces

Let Ω be an open connected region in the hyperbolic plane $\mathbb{H}^2$, and let $f : \Omega \to \mathbb{R}$ be a $\mathcal{C}^2$ function on Ω. A vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ is a set

$$M(f) = \{(x, y, f(x, y)) \in \mathbb{H}^2 \times \mathbb{R} \mid (x, y) \in \Omega\},$$

and it is called *entire* if $\Omega = \mathbb{H}^2$.

Considering the natural parameterization $\varphi(x, y) = (x, y, f(x, y))$ of $M(f)$ in $\mathbb{H}^2 \times \mathbb{R}$, the coordinate vector fields of the graph surface $M(f)$ are

$$\varphi_x(x, y) = \frac{1}{y}E_1 + f_x E_3 \quad \text{and} \quad \varphi_y(x, y) = \frac{1}{y}E_2 + f_y E_3, \tag{2.3}$$

and the coefficients of the first fundamental form induced by φ are

$$E=\tilde{g}(\varphi_x,\varphi_x)=\frac{1}{y^2}+f_x^2,\quad F=\tilde{g}(\varphi_x,\varphi_y)=f_xf_y,\quad G=\tilde{g}(\varphi_y,\varphi_y)=\frac{1}{y^2}+f_y^2. \tag{2.4}$$

Then, the determinant of the induced metric on $M(f)$ by φ is obtained as

$$EG-F^2=\frac{1+y^2(f_x^2+f_y^2)}{y^4} \tag{2.5}$$

and the graph surface $M(f)$ is regular, or φ is an immersion if $EG-F^2>0$.

We put $W=\sqrt{1+y^2(f_x^2+f_y^2)}$. Then, the normal vector to $M(f)$ in $\widetilde{M}^3$ is written as

$$n=\frac{1}{W}(-yf_xE_1-yf_yE_2+E_3).$$

When we evaluate the covariant derivatives of the tangent vector fields of φ we get

$$\widetilde{\nabla}_{\varphi_x}\varphi_x=\frac{1}{y^2}E_2+f_{xx}E_3,\ \widetilde{\nabla}_{\varphi_x}\varphi_y=-\frac{1}{y^2}E_1+f_{xy}E_3,\ \widetilde{\nabla}_{\varphi_y}\varphi_y=-\frac{1}{y^2}E_2+f_{yy}E_3,$$

and hence, we obtain the coefficients of the second fundamental form in the local coordinates as follows:

$$L=\tilde{g}(\widetilde{\nabla}_{\varphi_x}\varphi_x,n)=\frac{yf_{xx}-f_y}{yW},\quad M=\tilde{g}(\widetilde{\nabla}_{\varphi_x}\varphi_y,n)=\frac{yf_{xy}+f_x}{yW},\quad N=\tilde{g}(\widetilde{\nabla}_{\varphi_y}\varphi_y,n)=\frac{yf_{yy}+f_y}{yW}. \tag{2.6}$$

It is known that for surfaces in $\mathbb{R}^3$, the Gaussian (intrinsic) curvature K and extrinsic curvature K_{ext} are equal. In the following we see that the intrinsic and extrinsic curvatures differ by the sectional curvature in $\mathbb{H}^2\times\mathbb{R}$.

Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2\times\mathbb{R}$ defined by a $\mathcal{C}^2$ function f on an open connected region $\Omega\subset\mathbb{H}^2$. By using (2.3), we obtain that $\widetilde{R}(\varphi_x,\varphi_y)\varphi_x=\frac{1}{y^3}E_2$. Then, the sectional curvature of $\mathbb{H}^2\times\mathbb{R}$ for the section determined by the vectors φ_x and φ_y is

$$\widetilde{K}(\varphi_x,\varphi_y)=-\frac{\tilde{g}(\widetilde{R}(\varphi_x,\varphi_y)\varphi_x,\varphi_y)}{EG-F^2}=-\frac{1}{y^4(EG-F^2)}=-\frac{1}{1+y^2(f_x^2+f_y^2)}$$

which is bounded i.e. $-1\le\widetilde{K}<0$, and the equality case holds if and only if $f(x,y)=c$, where c is a constant. Using (2.2) and (2.3), from the Gauss equation (2.1) we have the Gaussian curvature K of $M(f)$ as

$$K=K(\varphi_x,\varphi_y)=-\frac{\tilde{g}(R(\varphi_x,\varphi_y)\varphi_x,\varphi_y)}{EG-F^2}=\widetilde{K}+K_{ext},$$

where K_{ext} is the extrinsic curvature of $M(f)$, and it is defined by $K_{ext}=(LN-M^2)/(EG-F^2)$. Thus, the Gaussian curvature K is given by

$$K=\frac{1}{EG-F^2}\Big(-\frac{1}{y^4}+(LN-M^2)\Big).$$

A vertical graph surface $M(f)$ in $\mathbb{H}^2\times\mathbb{R}$ is called *intrinsically flat* (resp., *extrinsically flat*) if $K=0$ (resp., $K_{ext}=0$) on $M(f)$.

Using (2.6), the Gaussian curvature and extrinsic curvature of $M(f)$ are obtained, respectively, as

$$K=\frac{y^2[(yf_{xx}-f_y)(yf_{yy}+f_y)-(yf_{xy}+f_x)^2]-y^2(f_x^2+f_y^2)-1}{[1+y^2(f_x^2+f_y^2)]^2} \tag{2.7}$$

and

$$K_{ext}=\frac{y^2[(yf_{xx}-f_y)(yf_{yy}+f_y)-(yf_{xy}+f_x)^2]}{[1+y^2(f_x^2+f_y^2)]^2}. \tag{2.8}$$

Also, since

$$-1\le\widetilde{K}=K-K_{ext}=-\frac{1}{1+y^2(f_x^2+f_y^2)}<0, \tag{2.9}$$

we have that

1) if $M(f)$ has constant extrinsic curvature K_{ext}, then the Gaussian curvature K is bounded, i.e., $K_{ext} - 1 \leq K < K_{ext}$;
2) if $M(f)$ has constant Gaussian curvature K, then the extrinsic curvature K_{ext} is bounded, i.e., $K < K_{ext} \leq K + 1$.

By using (2.7) and (2.8), we have the followings:

Proposition 2.1. *Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function $f : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$ defined on an open connected region Ω. Then, $M(f)$ is an intrinsically flat surface in $\mathbb{H}^2 \times \mathbb{R}$ if and only if $f(x, y)$ satisfies*

$$y^2[(yf_{xx} - f_y)(yf_{yy} + f_y) - (yf_{xy} + f_x)^2] - y^2(f_x^2 + f_y^2) - 1 = 0. \tag{2.10}$$

Proposition 2.2. *Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function $f : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$ defined on an open connected region Ω. Then, $M(f)$ is an extrinsically flat surface in $\mathbb{H}^2 \times \mathbb{R}$ if and only if $f(x, y)$ satisfies*

$$(yf_{xx} - f_y)(yf_{yy} + f_y) - (yf_{xy} + f_x)^2 = 0. \tag{2.11}$$

Proposition 2.3. *Let $v \in \mathcal{C}^2$ be defined on an open interval of $\mathbb{R}$. Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a function of the form $f(x, y) = v(y) + \ell x$, that is, $M(f)$ is invariant by the parabolic screw motion with pitch $\ell > 0$. Then, the Gaussian curvature K and the extrinsic curvature K_{ext} are given, respectively, by*

$$K = \frac{y}{2}\frac{d}{dy}\Big(\frac{1}{1 + y^2(v'^2 + \ell^2)}\Big) - \frac{1}{1 + y^2(v'^2 + \ell^2)} \tag{2.12}$$

and

$$K_{ext} = \frac{y}{2}\frac{d}{dy}\Big(\frac{1}{1 + y^2(v'^2 + \ell^2)}\Big). \tag{2.13}$$

Now, by using (2.9) we prove the following theorem.

Theorem 2.1. *Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function $f(x, y)$ defined on some open connected region $\Omega \subset \mathbb{H}^2$. Then, the difference between the extrinsic curvature K_{ext} and the Gaussian curvature K is a constant if and only if the function f is given by*

$$f(x, y) = \ell x \mp \Big(\sqrt{b^2 - \ell^2 y^2} + b \ln\Big(\frac{y}{b + \sqrt{b^2 - \ell^2 y^2}}\Big)\Big) + c \tag{2.14}$$

defined on the region $\Omega = \{(x, y) \in \mathbb{H}^2 |\ 0 < y < \frac{b}{\ell}\}$, where $\ell, b, c \in \mathbb{R}$ with $\ell, b > 0$. Moreover, $M(f)$ has both K_{ext} and K constant, that is, $K_{ext} = 0$ and $K = -1/(1 + b^2)$, and it is invariant by the parabolic screw motion with pitch ℓ.

Proof. Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function $f(x, y)$ defined on some open connected region $\Omega \subset \mathbb{H}^2$. From (2.9), we have $0 < K_{ext} - K \leq 1$, and $K_{ext} - K$ is a constant if and only if $f(x, y)$ satisfies

$$f_x^2 + f_y^2 = \frac{b^2}{y^2},$$

where $b = \sqrt{\frac{1}{K_{ext} - K} - 1}$. The complete solution of this partial differential equation is of the form

$$f(x, y) = \ell x \mp \int \frac{\sqrt{b^2 - \ell^2 y^2}}{y} dy + c,$$

for $0 < y < b/\ell$, where ℓ and c are integration constants with $\ell > 0$. By integration we obtain (2.14).

Let b be a positive constant. The function $f(x, y)$ given by (2.14) is of the form $f(x, y) = \ell x \mp v(y)$ with $v'(y) = \frac{\sqrt{b^2 - \ell^2 y^2}}{y}$. It can be seen easily that $1 + y^2(v'^2 + \ell^2)$ is a constant. Thus, from (2.12) and (2.13) we have $K = -1/(1 + b^2)$ and $K_{ext} = 0$, respectively. Also, for $\ell > 0$ the form of f means that $M(f)$ is a parabolic screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$ with pitch ℓ. □

2.2. Parabolic screw motion surfaces

Parabolic and hyperbolic screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ were studied in [4, 18]. For the definition of parabolic screw motion surfaces we follow [4]. We will use helicoidal-type isometries in $\mathbb{H}^2 \times \mathbb{R}$ which are the composition of isometries of $\mathbb{H}^2$ together with vertical translation in a proportional way. Let δ be the group of parabolic isometries in the half-plane $\mathbb{H}^2$, that is, the parabolic translations given by $T(x, y) = (x + c, y)$, $c \in \mathbb{R}$. This group generates helicoidal-type isometries in $\mathbb{H}^2 \times \mathbb{R}$, that is, the helicoidal isometries Γ_ℓ of pitch $\ell > 0$, generated in $\mathbb{H}^2 \times \mathbb{R}$ are given by $\widetilde{F}(x, y, t) = (T(x, y), t + \ell c)$. More precisely, for a fixed point (x_0, y_0, t_0), it is given by

$$\Gamma_\ell(x_0, y_0, t_0) = \{(x_0 + c, y_0, t_0 + c\ell) | c \in \mathbb{R}\} \subset \mathbb{H}^2 \times \mathbb{R}.$$

The surfaces invariant by this helicoidal isometry will be called the *parabolic screw motion surfaces*. If $\ell = 0$, we have surfaces invariant by parabolic translations.

In order to obtain a surface invariant by the parabolic screw motion, we consider a curve $\gamma = (0, y, v(y))$ in the yt-plane which is locally the graph of a function $v \in \mathcal{C}^2$ defined an open interval of $\mathbb{R}$. The surface $\Gamma_\ell(\gamma)$ which is invariant by this one-parameter group of helicoidal-type isometries generated by the curve γ can therefore be parameterized by

$$\varphi(x, y) = (x, y, v(y) + \ell x)$$

which is a vertical graph surface $M(f)$ defined by a function of the form $f(x, y) = v(y) + \ell x$. In the literature, a surface defined by $\varphi(x, y) = (x, y, u(x) + v(y))$ is also known as a translation surface, for instance, see [8, 11, 10] and references therein.

3. Flat and Extrinsically Flat Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

In this section we obtain intrinsically flat and extrinsically flat vertical graph surfaces invariant by the parabolic screw motions in $\mathbb{H}^2 \times \mathbb{R}$.

Considering (2.7), (2.8), and L, M, N in (2.6), for planes immersed in $\mathbb{H}^2 \times \mathbb{R}$ we have

Proposition 3.1. *Let $f(x, y) = ax + by + c$, where $a, b, c \in \mathbb{R}$. Then, the vertical graph surface $M(f)$ in $\mathbb{H}^2 \times \mathbb{R}$ is extrinsically flat if and only if $f(x, y) = c$. The graph surface $M(f)$ for $f(x, y) = c$ is an entire, complete, and totally geodesic surface invariant by the parabolic screw motions in $\mathbb{H}^2 \times \mathbb{R}$ with the intrinsic Gaussian curvature $K = -1$.*

For the vertical graph surfaces in $\mathbb{H}^2 \times \mathbb{R}$ for the function of the form $f(x, y) = u(x) + v(y)$ we have

Theorem 3.1. *Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function of the form $f(x, y) = u(x) + v(y)$ defined on some open connected region $\Omega \subset \mathbb{H}^2$.Then, $M(f)$ is extrinsically flat if and only if*

$$u(x) = \ell x + c \quad \text{and} \quad v(y) = \sqrt{b^2 - \ell^2 y^2} + b \ln\left(\frac{y}{b + \sqrt{b^2 - \ell^2 y^2}}\right) \tag{3.1}$$

on the region $\Omega = \{(x, y) \in \mathbb{H}^2 |\ 0 < y < \frac{b}{\ell}\}$, where $\ell, b, c \in \mathbb{R}$ with $\ell, b > 0$. This surface $M(f)$ is invariant by the parabolic screw motion with pitch ℓ and constant Gaussian curvature $K = -1/(1 + b^2)$.

Proof. Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function of the form $f(x, y) = u(x) + v(y)$. Then, the graph surface $M(f)$ is extrinsically flat if and only if the function f holds (2.11). That is, for $f(x, y) = u(x) + v(y)$, equation (2.11) becomes

$$u''(x) - \frac{1}{y(v' + yv'')} u'^2(x) - \frac{v'}{y} = 0. \tag{3.2}$$

This is a differential equation of the form $u''(x) + \psi_1(y)u'^2(x) + \psi_2(y) = 0$. Since ψ_1 and ψ_2 are functions of y, if $u''(x) \neq 0$, then the solution of (3.2) does not define u as a function of x, and hence there is no solution of (3.2) unless $u''(x) = 0$. So, we have $u''(x) = 0$ which implies that $u(x) = \ell x + c,\ \ell \neq 0, c \in \mathbb{R}$. Note that this result can also be followed by taking the derivative of (3.2) with respect to y. For $u(x) = \ell x + c$, we have from (3.2) that $v'v'' + \frac{v'^2}{y} + \frac{\ell^2}{y} = 0$. The solution of this differential equation gives

$$v(y) = \mp \int \frac{\sqrt{b^2 - \ell^2 y^2}}{y} dy + c,$$

where $b > 0$ and c are integration constants, and $0 < y < b/\ell$. By integrating the last integral and using a vertical translation and symmetry about the xy-plane we have (3.1). Also, from (2.12) we obtain that the Gauss curvature $K = -1/(1 + b^2)$. For the obtained functions $u(x)$ and $v(y)$, $M(f)$ is a parabolic screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$ with pitch $\ell > 0$. □

Remark 3.1. Up to a vertical translation, the vertical graph surfaces $M(f)$ in $\mathbb{H}^2 \times \mathbb{R}$ for $f(x, y) = v(y) + \ell x$ with $v(y)$ defined by the second function in (3.1) are the only surfaces invariant by the parabolic screw motion in $\mathbb{H}^2 \times \mathbb{R}$ with constant Gaussian curvature K and constant extrinsic curvature K_{ext}.

Now, by taking $\ell = 0$ in (3.1), the vertical graph surface $M(f)$ for $f(x, y) = v(y)$ is a cylinder parallel to the x-axis immersed in $\mathbb{H}^2 \times \mathbb{R}$. Such a surface is invariant by the parabolic translation. Thus, we have

Corollary 3.1. *Let $v \in \mathcal{C}^2$ be defined an open interval of $\mathbb{R}$. Up to a vertical translation and symmetry about the xy-plane, a vertical graph surface $M(f)$ in $\mathbb{H}^2 \times \mathbb{R}$ for a function of the form $f(x, y) = v(y)$ is extrinsically flat if and only if $f(x, y) = b \ln y$, $b \in \mathbb{R}_+$. Also, $M(f)$ is an entire surface invariant by the parabolic translation in $\mathbb{H}^2 \times \mathbb{R}$ with constant Gaussian curvature $K = -1/(b^2 + 1)$.*

Theorem 3.2. *Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function of the form $f(x, y) = v(y) + \ell x$ on some open connected region $\Omega \subset \mathbb{H}^2$, where ℓ is a positive constant, that is, $M(f)$ is a parabolic screw motion surface in $\mathbb{H}^2 \times \mathbb{R}$ with pitch ℓ. Then, $M(f)$ is intrinsically flat if and only if*

$$f(x, y) = \ell x \pm \int \frac{\sqrt{b - y^2 - \ell^2 y^4}}{y^2} dy \tag{3.3}$$

on the region $\Omega = \{(x, y) \in \mathbb{H}^2 |\ 0 < y < \sqrt{-1 + \sqrt{4\ell^2 b + 1}}/\sqrt{2}\ell\}$, where $b > 0$ is an integration constant. Also, the extrinsic curvature K_{ext} is given by $K_{ext} = y^2/b$.

Proof. Let $f(x, y) = v(y) + \ell x$. Then, from (2.12) a vertical graph surface $M(f)$ has zero Gaussian curvature, $K = 0$, if and only if the function $v(y)$ satisfies the equation

$$\frac{y}{2} \frac{d}{dv} \left(\frac{1}{1 + y^2(v'^2 + \ell^2)} \right) - \frac{1}{1 + y^2(v'^2 + \ell^2)} = 0. \tag{3.4}$$

Now we put $q(y) = 1/(1 + y^2(v'^2 + \ell^2))$. Then, we have $yq'(y) - 2q(y) = 0$, and its solution yields $q(y) = y^2/b$, where b is a non-zero integration constant. Therefore, for this $q(y)$, solving $q(y) = 1/(1 + y^2(v'^2 + \ell^2))$ for $v(y)$, and using a vertical translation, we obtain (3.3) for $b > 0$, and from (3.3) we have the region Ω in the theorem.
Now, from (2.13) and (3.4) we get $K_{ext} = \frac{1}{1+y^2(v'^2+\ell^2)} = q(y) = \frac{y^2}{b}$. □

By taking $\ell = 0$, integrating (3.3) and also considering a vertical translation and symmetry about the xy-plane, we have

Corollary 3.2. *Let $M(f)$ be a vertical graph surface (an immersed cylinder) in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function of the form $f(x, y) = v(y)$ on some open connected region $\Omega \subset \mathbb{H}^2$, that is, $M(f)$ is invariant by the parabolic translation. Then, $M(f)$ is intrinsically flat if and only if*

$$f(x, y) = \arcsin\left(\frac{y}{\sqrt{b}}\right) + \frac{\sqrt{b - y^2}}{y} \tag{3.5}$$

on the region $\Omega = \{(x, y) \in \mathbb{H}^2 |\ 0 < y < \sqrt{b}\}$, where b is a positive constant.

4. Surfaces with non-zero constant curvature

In this section we study vertical graph surfaces invariant by parabolic screw motions in $\mathbb{H}^2 \times \mathbb{R}$ with non-zero constant Gaussian curvature, and with non-zero constant extrinsic curvature.

4.1. Surfaces with non-zero constant extrinsic curvature

Theorem 4.1. *Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function of the form $f(x,y) = v(y) + \ell x$ on some open connected region $\Omega \subset \mathbb{H}^2$, where ℓ is a positive constant, that is, $M(f)$ is a parabolic screw motion surface with pitch ℓ. Then, $M(f)$ has non-zero constant extrinsic curvature K_{ext} if and only if*

$$f(x,y) = \ell x \pm \int \frac{1}{y}\sqrt{\frac{1-(1+\ell^2y^2)(b+2K_{ext}\ln y)}{b+2K_{ext}\ln y}}dy \tag{4.1}$$

on the open connected region $\Omega = \{(x,y) \in \mathbb{H}^2 |\ 0 < b + 2K_{ext}\ln y < 1$ and $(1+\ell^2y^2)(b+2K_{ext}\ln y) < 1\}$.

Proof. Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for $f(x,y) = v(y) + \ell x$. Then, $M(f)$ has non-zero constant extrinsic curvature K_{ext} if and only if

$$\frac{d}{dv}\Big(\frac{1}{1+y^2(v'^2+\ell^2)}\Big) = \frac{2K_{ext}}{y}$$

because of (2.13), which can be written as $1/(1+y^2(v'^2+\ell^2)) = b + 2K_{ext}\ln y$, where $b \in \mathbb{R}$ and $0 < b + 2K_{ext}\ln y < 1$. When we solve this equation for $v(y)$ and using a vertical translation, we obtain (4.1). □

By taking $\ell = 0$ and integrating (4.1) we have

Corollary 4.1. *Let $M(f)$ be a vertical graph surface, (an immersed cylinder) in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function of the form $f(x,y) = v(y)$ on some open connected region $\Omega \subset \mathbb{H}^2$. Then, the graph surface $M(f)$ invariant by the parabolic translation has non-zero constant extrinsic curvature K_{ext} if and only if*

$$f(x,y) = \frac{1}{2K_{ext}}\Big(\sqrt{(1-b-2K_{ext}\ln y)(b+2K_{ext}\ln y)} - \arctan\sqrt{\frac{1-b-2K_{ext}\ln y}{b+2K_{ext}\ln y}}\ \Big) \tag{4.2}$$

on the open connected region $\Omega = \{(x,y) \in \mathbb{H}^2 |\ e^{-b/2K_{ext}} < y < e^{(1-b)/2K_{ext}}\}$ for $K_{ext} > 0$, and $\Omega = \{(x,y) \in \mathbb{H}^2 |\ e^{(1-b)/2K_{ext}} < y < e^{-b/2K_{ext}}\}$ for $K_{ext} < 0$, where b is a constant.

4.2. Surfaces with non-zero Constant Gaussian Curvature

Let $f(x,y) = ax + by + c$, where $a, b, c \in \mathbb{R}$. Then, from (2.7) the Gaussian curvature of the vertical graph surface $M(f)$ is obtained as

$$K = \frac{-1-2y^2(a^2+b^2)}{[1+y^2(a^2+b^2)]^2}$$

from which we can state

Proposition 4.1. *Let $f(x,y) = ax + by + c$, where $a, b, c \in \mathbb{R}$. Then, the vertical graph surface $M(f)$ in $\mathbb{H}^2 \times \mathbb{R}$ has constant negative Gaussian curvature $K = -1$ if and only if $f(x,y) = c$. The graph surface $M(f)$ invariant by the parabolic screw motions is an entire, totally geodesic and complete surface in $\mathbb{H}^2 \times \mathbb{R}$ with $K = -1$.*

Theorem 4.2. *Let $M(f)$ be a vertical graph surface in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function of the form $f(x,y) = v(y) + \ell x$ on some open connected region $\Omega \subset \mathbb{H}^2$, where ℓ is a positive constant, that is, $M(f)$ is a parabolic screw motion surface with pitch ℓ. Then, $M(f)$ has non-zero constant Gaussian curvature K if and only if the function $v(y)$ is given by*

$$f(x,y) = \ell x \pm \int \frac{1}{y}\sqrt{\frac{1-(\ell^2y^2+1)(by^2-K)}{by^2-K}}\ dy, \tag{4.3}$$

where b and K are non-zero constants; and the region Ω is given as follows:

1) *for $K > 0$, $\Omega = \Big\{(x,y) \in \mathbb{H}^2 |\ \sqrt{\frac{K}{b}} < y < \sqrt{\frac{-(b-\ell^2K)+\sqrt{(b-\ell^2K)^2+4\ell^2b(1+K)}}{2b\ell^2}}\Big\}$ if $b > 0$;*
2) *for $-1 < K < 0$, $\Omega = \Big\{(x,y) \in \mathbb{H}^2 |\ 0 < y < \sqrt{\frac{-(b-\ell^2K)+\sqrt{(b-\ell^2K)^2+4\ell^2b(1+K)}}{2b\ell^2}}\Big\}$ if $b > 0$, or*
   $\Omega = \Big\{(x,y) \in \mathbb{H}^2 |\ \sqrt{\frac{b-\ell^2K+\sqrt{(b-\ell^2K)^2+4\ell^2b(1+K)}}{2(-b)\ell^2}} < y < \sqrt{\frac{K}{b}}\Big\}$ *if $\ell^2(2\sqrt{K+1}-K-2) < b < 0$;*

3) *for* $K = -1$, $\Omega = \left\{(x, y) \in \mathbb{H}^2 | \sqrt{\frac{\ell^2+b}{-b\ell^2}} < y < \frac{1}{\sqrt{-b}}\right\}$ *if* $-\ell^2 \le b < 0$, *or* $\Omega = \left\{(x, y) \in \mathbb{H}^2 | \, 0 < y < \frac{1}{\sqrt{-b}}\right\}$ *if* $b < -\ell^2$;

4) *for* $K < -1$, $\Omega = \left\{(x, y) \in \mathbb{H}^2 | \sqrt{\frac{b-\ell^2K+\sqrt{(b-\ell^2K_0)^2+4\ell^2b(1+K)}}{2(-b)\ell^2}} < y < \sqrt{\frac{K}{b}}\right\}$ *if* $b < 0$.

Proof. Let $f(x, y) = \ell x + v(y)$. A vertical graph surface $M(f)$ has non-zero constant Gaussian curvature K if and only if the function $v(y)$ is a solution of (2.12).

Now, if we put $q(y) = 1/(1 + y^2(v'^2 + \ell^2))$, then the equation (2.12) turns to $q'(y) - \dfrac{2}{y}q(y) = \dfrac{2K}{y}$, and its solution yields $q(y) = by^2 - K$, where b is an integration constant. Therefore, for this $q(y)$ solving $q(y) = 1/(1 + y^2(v'^2 + \ell^2))$ for $v(y)$ and considering a vertical translation, we obtain (4.3). Also, from (2.5) we obtain $EG - F^2 = \frac{1}{y^4(by^2-K)}$ that implies $by^2 - K > 0$ as $M(f)$ is regular. From (4.3), the function $v(y)$ is defined if $0 < (\ell^2y^2 + 1)(by^2 - K) < 1$. Analyzing these inequalities for the values of $\ell, b,$ and K, we obtain the regions Ω stated in the theorem. ☐

Let $b = 0$ in (4.3). Then, from $EG - F^2 = \frac{1}{y^4(-K)}$, the surface $M(f)$ is regular if $K < 0$. Also, we have $K_{ext} = 0$ for the function $v(y)$ given by (4.3) because of Theorem 3.1. Thus, by integrating (4.3) and using (2.9) we have

Corollary 4.2. *The vertical graph surface $M(f)$ invariant by a parabolic screw motion has negative constant Gaussian curvature with $-1 < K < 0$ for the function*

$$f(x, y) = \ell x \pm \left(\sqrt{\lambda^2 - \ell^2y^2} + \lambda \ln\left(\frac{y}{\lambda + \sqrt{\lambda^2 - \ell^2y^2}}\right)\right) \tag{4.4}$$

defined on the region $\Omega = \left\{(x, y) \in \mathbb{H}^2 | \, 0 < y < \frac{\lambda}{\ell}\right\}$, *where* $\lambda = \sqrt{\frac{1+K}{-K}}$.

Now, by taking $\ell = 0$ in (4.3), and considering a vertical translation and symmetry about the xy-plane, we have

Corollary 4.3. *Let $M(f)$ be a graph surface (immersed cylinder) in $\mathbb{H}^2 \times \mathbb{R}$ for a $\mathcal{C}^2$ function of the form $f(x, y) = v(y)$ on some open connected region $\Omega \subset \mathbb{H}^2$. Then, $M(f)$ invariant by the parabolic translation has non-zero constant Gaussian curvature K if and only if the function f is given by*

1)

$$f(x, y) = \sqrt{\frac{1+K}{K}} \tan^{-1}\left(\sqrt{\frac{1+K}{K}}\sqrt{\frac{by^2-K}{1+K-by^2}}\right) - \sin^{-1}\sqrt{by^2-K} \tag{4.5}$$

defined on the region $\Omega = \left\{(x, y) \in \mathbb{H}^2 | \sqrt{\frac{K}{b}} < y < \sqrt{\frac{1+K}{b}}\right\}$ *for* $K > 0$ *and* $b > 0$, *or* $\Omega = \left\{(x, y) \in \mathbb{H}^2 | \sqrt{\frac{1+K}{b}} < y < \sqrt{\frac{K}{b}}\right\}$ *for* $K \le -1$ *and* $b < 0$;

2)

$$f(x, y) = \sqrt{\frac{1+K}{-K}} \tanh^{-1}\left(\sqrt{\frac{1+K}{-K}}\sqrt{\frac{by^2-K}{1+K-by^2}}\right) + \sin^{-1}\sqrt{by^2-K} \tag{4.6}$$

defined on the region $\Omega = \left\{(x, y) \in \mathbb{H}^2 | \, 0 < y < \sqrt{\frac{1+K}{b}}\right\}$ *for* $-1 < K < 0$ *and* $b > 0$, *or* $\Omega = \left\{(x, y) \in \mathbb{H}^2 | \, 0 < y < \sqrt{\frac{K}{b}}\right\}$ *for* $-1 \le K < 0$ *and* $b < 0$;

3) $f(x, y) = \sqrt{\frac{1+K}{-K}} \ln y$ *defined on the region* $\Omega = \mathbb{H}^2$ *for* $-1 < K < 0$ *and* $b = 0$.

When we evaluate the geodesics of the surface $M(f)$ for the function $f(x, y) = a \ln y$ on the region $\Omega = \mathbb{H}^2$ we obtain the geodesics parametrized by arc length parameter as follows:

$$\gamma_1(s) = \left(x_0, y_0e^{s/\sqrt{1+a^2}}, a \ln\left(y_0e^{s/\sqrt{1+a^2}}\right)\right), \; s \in \mathbb{R}$$

and

$$\gamma_2(s)=\Big(\frac{\sqrt{1+a^2}}{x_0}\tanh\Big(\frac{s-y_0}{\sqrt{1+a^2}}\Big)+x_1,\frac{1}{x_0}\text{sech}\Big(\frac{s-y_0}{\sqrt{1+a^2}}\Big),a\ln\Big(\frac{1}{x_0}\text{sech}\Big(\frac{s-y_0}{\sqrt{1+a^2}}\Big)\Big)\Big),$$

$s\in\mathbb{R}$, which are complete, where x_0, x_1, y_0 are integration constants. Therefore, the surface $M(f)$ is complete with constant negative Gaussian curvature K with $-1<K<0$.

By Proposition 4.1 and Corollary 4.3 it is seen that the vertical graph surfaces $M(f)$ defined by $f(x,y)=c=constant$ and $f(x,y)=a\ln y$ are the only complete and entire surfaces invariant by parabolic translation in $\mathbb{H}^2\times\mathbb{R}$ with constant negative Gaussian curvature. For $f(x,y)=c$, $M(f)$ has $K_{ext}=0$ and $K=-1$, and for $f(x,y)=a\ln y$, $M(f)$ has $K_{ext}=0$ and $K=-1/(1+a^2)$.

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Affiliations

UĞUR DURSUN
ADDRESS: Işık University, Faculty of Engineering and Natural Sciences, Department of Mathematics, 34980 Şile, Istanbul, Türkiye
E-MAIL: ugur.dursun@isikun.edu.tr
ORCID ID: 0000-0002-5225-186X