ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

2023, 11(2), pp. 148-157, DOI: 10.20290/estubtdb.1231907

RESEARCH ARTICLE

ON SOFT RING AND SOFT TOPOLOGICAL RING

Nazan ÇAKMAK POLAT [1,*], Gözde YAYLALI [2], Bekir TANAY [3]

[1,2,3] Muğla Sıtkı Koçman University, Department of Mathematics, Muğla, Turkey

ABSTRACT

The soft set theory is an affective mathematical tool to solve problems that involves uncertainties. Despite the development in the theoretical structure of soft sets, researchers did not make consensus formulation of soft element. In this study the soft ring is redefined by the help of soft operations which are based on a natural definition of soft element. This new soft ring definition is compared with the soft ring definitions in the literature. Some examples, results and theorems are given to enrich the concept of soft ring. Also soft topological ring structure which is a harmonization of soft ring and soft topology is studied with some results.

Keywords: Soft set, Soft ring, Soft topological ring

1. INTRODUCTION

Frequently solutions of real life problems are not possible with a precise and direct informational point of view. Several models have been developed to date to cope with this complexity. Often these models are not enough to identify the exact solution. The soft set theory is one of these models which was defined by Molodtsov [1]. Soft set theory has excited attention since the year it was defined, due to the freedom it gives to studies on parameters that is increase the application area of soft sets unlike other theories. Some of fundamental soft set operations such as Or, And, Union etc. were introduced by Maji et al in [2]. Soft topological structures were given in [3, 4, 5, 6, 7]. Aktaş and Çağman introduced and investigated the concept of soft group by taking universal set as a group and they also made a comparison with soft sets and other set theories in [8]. After that soft group definition extended soft ring by Acar et al in [9] and some related results about soft ring were derived by them. Another approach to soft group notion was given by Ghosh et al in [10]. Moreover soft modules and fuzzy soft modules were presented by Sun et al in [11] and Gunduz et al in [12].

In the meantime soft element and soft point structures were studied and discussed by some researchers from different perspectives. One of them is Wardowski [13] who defined soft element which provides soft topological structures resembles to pointwise topological structures. After this definition there have been studies on soft topological and soft algebraic structures from an elementary point of view such as [14] and [15] etc.

More combination of algebraic constructions and soft topological structures were studied by many researchers such as [16, 17, 18, 19, 18] and [20]. For example, as an expected extension of the familiar concepts of topological groups were given by Nazmul and Samanta in 2010 [21] and Tanay and Çakmak [16] initiated the idea of soft semi topological groups. Later some improvements were added to the notion of soft topological groups in [20].

*Corresponding Author: ncakmak@mu.edu.tr
Received: 10.01.2023 Published: 28.08.2023

Tahat et al.in [18] introduced the concept of soft topological soft rings by applying soft topological structures on a soft ring and another approach for soft topological rings was given in [17] by applying the topological structures on a soft ring. In [19], the notion soft topological ring which is linked on the soft topological structures over the rings directly, rather than on the topological structures over the subrings were introduced by Tahat et al.

In this paper unlike the studies mentioned above, firstly, definition of a soft ring will be given from the pointwise perspective. Then, definition of a soft topological ring and some theorems will be examined as a result of this study. With this approach soft topological ring structure will depend on a soft topology and a soft ring structure on a single soft set.

We referee for some basic definitions such as soft set, soft subset, intersection and union of soft sets, soft empty set and soft element from [1], [2] and [7]. An application of the these mentioned definitions is given by the following example.

Example 1.1. Mr. X and Mrs. X are deciding to move to a new city, they list the features of the city in which they want to live as follows: Economy, Health, Security, City life, Culture and Art Activities. Features of the cities give the parameter set $E = \{e_1, e_2, e_3, e_4, e_5\}$ where e_1, e_2, e_3, e_4, e_5 stands for economy, health, security, city life and culture and art activities respectively. The universe U of cities that they plan to move to are also listed as London, Paris, New York, Tokyo. Let's define soft sets (F, E) and (G, E) that describe each city with properties determined by Mr. X and Mrs. X respectively.

$(F, E) = \{(e_1, \{\text{London}\}), (e_2, \{\text{Paris}\}), (e_3, \{\text{New York, Tokyo}\})(e_4, \emptyset), (e_5, \emptyset)\}$.

$(G, E) = \{(e_1, \{\text{London, Paris}\}), (e_2, \emptyset)\ (e_3, \{\text{London, New York, Tokyo}\}), (e_4, \emptyset), (e_5, \emptyset)\}$.

The intersection of (F, E) and (G, E) is $(F, E) \tilde{\cap} (G, E) =$
$\{(e_1, \{\text{London}\}), (e_2, \emptyset), (e_3, \{\text{New York, Tokyo}\}), (e_4, \emptyset), (e_5, \emptyset)\}$.

The union of (F, E) and (G, E) is, $(F, E) \tilde{\cup} (G, E) =$
$\{(e_1, \{\text{London}, B\}), (e_2, \{\text{Paris}\}), (e_3, \{\text{London, New York, Tokyo}\}), (e_4, \emptyset), (e_5, \emptyset)\}$.

The nonempty soft elements of (F, E) are
$\{(e_1, \{\text{London}\}), (e_2, \{\text{Paris}\}), (e_3, \{\text{New York}\}), (e_3, \text{Tokyo})\}$.

Empty soft elements of (F, E) are $\{(e_4, \emptyset), (e_5, \emptyset)\}$.

Soft elements of (G, E) are,

$\{(e_1, \{\text{London}\}), (e_1, \{\text{Paris}\}), (e_2, \emptyset), (e_3, \{\text{London}\}), (e_3, \{\text{New York}\}), (e_3, \{\text{Tokyo}\}), (e_4, \emptyset), (e_5, \emptyset)\}$

$(e_2, \emptyset), (e_4, \emptyset), (e_5, \emptyset)$ are the empty soft elements of (G, E),

$(e_1, \{\text{London}\}), (e_1, \{\text{Paris}\}), (e_3, \{\text{London}\}), (e_3, \{\text{New York}\}), (e_3, \{\text{Tokyo}\})$ are nonempty soft elements of (G, E).

Complement of soft set (F, E) is $(F, E)^{\tilde{C}} =$
$\{(e_1, \{\text{Paris, New York, Tokyo}\}), (e_2, \{\text{London, New York, Tokyo}\}), (e_3, \{\text{London, Paris}\}), (e_4, U), (e_5, U)\}$.

2. SOFT RING AND SOFT TOPOLOGY

2.1.Soft Ring

Definitions such as full soft set, operations on soft sets, properties of operations on soft sets and soft group which are used in this article are taken from the paper [10]. Following the definitions above soft ring definition can be stated as in the below:

Definition 2.1. Let $(E,+_1,\cdot_1)$, $(U,+_2,\cdot_2)$ be two rings, $A \subseteq E$ and $F_A \in S_f(U)$, (F_A is a full soft set on the universe U). Consider the binary operations $\tilde{+},\tilde{\cdot}$ given on the soft set F_A below. For all $(e_1,\{u_1\}),(e_2,\{u_2\}) \tilde{\in} F_A{}^{\bullet}$.

$$(e_1,\{u_1\})\tilde{+}(e_2,\{u_2\}) = (e_1 +_1 e_2,\{u_1 +_2 u_2\})$$
$$(e_1,\{u_1\})\tilde{\cdot}(e_2,\{u_2\}) = (e_1 \cdot_1 e_2,\{u_1 \cdot_2 u_2\})$$

A soft set $(F_A,\tilde{+},\tilde{\cdot})$ over (E,U) is said to be a soft ring if the following conditions are satisfied.

i) $(F_A,\tilde{+})$ is a commutative soft group,

ii) $\alpha \tilde{\cdot} (\beta \tilde{\cdot} \gamma) = (\alpha \tilde{\cdot} \beta) \tilde{\cdot} \gamma$ for all $\alpha,\beta,\gamma \tilde{\in} F_A{}^{\bullet}$.

iii) $\alpha \tilde{\cdot} (\beta \tilde{+} \gamma) = (\alpha \tilde{\cdot} \beta) \tilde{+} (\alpha \tilde{\cdot} \gamma)$ and $(\alpha \tilde{+} \beta) \tilde{\cdot} \gamma = (\alpha \tilde{\cdot} \gamma) \tilde{+} (\beta \tilde{\cdot} \gamma)$ for all $\alpha,\beta,\gamma \tilde{\in} F_A{}^{\bullet}$.

Example 2.2. Consider the soft set F_E defined by the set valued function $F: E = \mathbb{Z}_2 \to P\left(\mathbb{Z}(\sqrt{2})\right)$,

$F(\bar{0}) = \{2n + 2n\sqrt{2}: n \in \mathbb{Z}\}$,

$F(\bar{1}) = \{2n + (2n+1)\sqrt{2}: n \in \mathbb{Z}\}$,

$F(\bar{2}) = \{(2n+1) + 2n\sqrt{2}: n \in \mathbb{Z}\}$,

$F(\bar{3}) = \{(2n+1) + (2n+1)\sqrt{2}: n \in \mathbb{Z}\}$ over the rings $(E,+_1,\cdot_1) = (\mathbb{Z}_4,+,\cdot)$, $(U,+_2,\cdot_2) = (\mathbb{Z}(\sqrt{2}),+,\cdot)$. If we consider the soft elements of F_E $(\bar{1},\{2n + (2n+1)\sqrt{2}: n \in \mathbb{Z}\})$ and $(\bar{3},\{(2n+1) + (2n+1)\sqrt{2}: n \in \mathbb{Z}\})$ then $(\bar{1},\{2n + (2n+1)\sqrt{2}: n \in \mathbb{Z}\})\tilde{+}(\bar{3},\{(2n+1) + (2n+1)\sqrt{2}: n \in \mathbb{Z}\}) = (\bar{0},\{(2n+1) + 2n\sqrt{2}: n \in \mathbb{Z}\})$ which is not belong to soft set F_E. So F_E is not a soft ring over (E,U).

Example 2.3. Consider the soft set F_E defined by the set valued function $F: E = \mathbb{Z}_2 \to P(M_3(\mathbb{R}))$,

$F(\bar{0}) = \{0_{3x3}\}$,

$F(\bar{1}) = \{A: A \ upper\ triangular\ matrix\} = U_3$. If one apply the operation $\tilde{+}$ to the soft element $(\bar{1},U_3)$

$(\bar{1},U_3)\tilde{+}(\bar{1},U_3) = (\bar{0},U_3)$ that is not a soft element of the soft set F_E.

F_E is not a soft ring due to $\tilde{+}$ is not closed under the binary operation.

The soft ring definition which was given by [9] in 2010, is not related the definition stated in Definition 2.1. We can deduce this conclusion in view of the fact that a soft set F_E which is given in the above Example 2.3. is a soft ring according to the definition of soft ring stated in [9].

Example 2.4. Consider the soft set F_E defined by the set valued function $F: E = \mathbb{Z}_2 \to P(U = \mathbb{Z}_4)$,

$F(\bar{0}) = \{\bar{0},\bar{2}\}, F(\bar{1}) = \{\bar{0},\bar{2}\}$.

One can observe from the tables below that soft operations are closed and F_E is a soft ring over (E, U).

$\tilde{+}$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{2}\})$	$(\bar{1}, \{\bar{0}\})$	$(\bar{1}, \{\bar{2}\})$
$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{2}\})$	$(\bar{1}, \{\bar{0}\})$	$(\bar{1}, \{\bar{2}\})$
$(\bar{0}, \{\bar{2}\})$	$(\bar{0}, \{\bar{2}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{1}, \{\bar{2}\})$	$(\bar{1}, \{\bar{0}\})$
$(\bar{1}, \{\bar{0}\})$	$(\bar{1}, \{\bar{0}\})$	$(\bar{1}, \{\bar{2}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{2}\})$
$(\bar{1}, \{\bar{2}\})$	$(\bar{1}, \{\bar{2}\})$	$(\bar{1}, \{\bar{0}\})$	$(\bar{0}, \{\bar{2}\})$	$(\bar{0}, \{\bar{0}\})$

$\tilde{\cdot}$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{2}\})$	$(\bar{1}, \{\bar{0}\})$	$(\bar{1}, \{\bar{2}\})$
$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$
$(\bar{0}, \{\bar{2}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$
$(\bar{1}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{1}, \{\bar{0}\})$	$(\bar{1}, \{\bar{0}\})$
$(\bar{1}, \{\bar{2}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(1, \{\bar{0}\})$	$(\bar{1}, \{\bar{0}\})$

Definition 2.5. Let $(F_E, \tilde{+}, \tilde{\cdot})$ be a soft ring. If there exist an element $\tilde{1} \tilde{\in} F_E$ such that $\alpha \tilde{\cdot} \tilde{1} = \tilde{1} \tilde{\cdot} \alpha$, for all $\alpha \tilde{\in} F_E$, then F_E is called soft ring with identity.

Theorem 2.6. If E, U are rings with identities e_1 and e_2 and F_E is a soft ring that contains soft element $(e_1, \{e_2\})$, then $(e_1, \{e_2\})$ is the soft identity element.

Proof. Straightforward.

Definition 2.7. Let $(F_E, \tilde{+}, \tilde{\cdot})$ be a soft ring. If $\alpha \tilde{\cdot} \beta = \beta \tilde{\cdot} \alpha$, for all $\alpha, \beta \tilde{\in} F_E$, then F_E is called commutative soft ring .

Example 2.8. The soft ring F_E given in Example 2.4. is a commutative soft ring.

Note 2.9. If E and U are commutative rings so is F_E, that is defined over (E, U).

Theorem 2.10. If $(F_E, \tilde{+}, \tilde{\cdot})$ is a soft ring with additive identity $\tilde{0}$, then for any $\gamma, \beta \tilde{\in} F_E$, we have

i) $\tilde{0} \tilde{\cdot} \gamma = \gamma \tilde{\cdot} \tilde{0} = \tilde{0}$,
ii) $\gamma \tilde{\cdot} (-\beta) = (-\gamma) \tilde{\cdot} \beta = -(\gamma \tilde{\cdot} \beta)$,
iii) $(-\gamma) \tilde{\cdot} (-\beta) = \gamma \tilde{\cdot} \beta$.

Proof: i) $\tilde{0}$ is the soft identity for the operation $\tilde{+}$ and it can be written as $\tilde{0} \tilde{\cdot} \gamma = (\tilde{0} \tilde{+} \tilde{0}) \tilde{\cdot} \gamma$. From the right cancellation law $\tilde{0} \tilde{\cdot} \gamma = \tilde{0}$ is satisfied. The other side of the equality can be done similarly.

ii) Let us prove that $\gamma \tilde{\cdot} (-\beta)$ is the inverse of $\gamma \tilde{\cdot} \beta$ according to the $\tilde{+}$.

$$\gamma \tilde{\cdot} (-\beta) \tilde{+} \gamma \tilde{\cdot} \beta = \gamma \tilde{\cdot} \big((-\beta) \tilde{+} \beta\big) = \gamma \tilde{\cdot} \tilde{0} = \tilde{0}$$

The other side of the equality can be done similarly

The last condition of the theorem can be done similarly.

Definition 2.11. Let $(F_E, \tilde{+}, \tilde{\cdot})$ be a soft ring and $G_B \tilde{\subseteq} F_E$. If G_B is closed under the operations of F_E and satisfies the conditions given in the Definition 2.1. then $(G_B, \tilde{+}, \tilde{\cdot})$ is called a soft subring of $(F_E, \tilde{+}, \tilde{\cdot})$.

Example 2.12. Consider the soft subset $B = \{\bar{0}\} \subseteq \mathbb{Z}_2$. $G_B = \{\bar{0}, \{\bar{0}, \bar{2}\}\}$ of F_E given in Example 2.4. Then $(G_B, \tilde{+}, \tilde{\cdot})$ is a soft sub ring of $(F_E, \tilde{+}, \tilde{\cdot})$.

Theorem 2.13. If $(F_A, \tilde{+}, \tilde{\cdot})$ is a soft ring over (E, U) then

i) A is a subring of E,
ii) $\bigcup_{e_i \in A} F(e_i)$ is a subring of U.

Proof. i) Ghosh et. al. proved that A is a subgroup of E, in [6]. So to show that A is a subring of E, we prove that $e_i \cdot_1 e_j \in A$, for each $e_i, e_j \in A$. Assume that $e_i, e_j \in A$ since $F_A \in S_f(U)$ there exist $u_k, u_l \in U$ such that $(e_i, \{u_k\}), (e_j, \{u_l\}) \tilde{\in} F_A$. Also $(e_i, \{u_k\}) \tilde{\cdot} (e_j, \{u_l\}) = (e_i \cdot_1 e_j, \{u_k \cdot_2 u_l\}) \tilde{\in} F_A$ which proves $e_i \cdot_1 e_j \in A$.

ii) The proof can be done similar with condition i).

Theorem 2.14. Let $(F_A, \tilde{+}, \tilde{\cdot})$ and $(G_B, \tilde{+}, \tilde{\cdot})$ be soft rings over (E, U).

i) If $F_A \tilde{\cap} G_B \in S_f(U)$ and $A \cap B \neq \emptyset$ then $(F_A \tilde{\cap} G_B, \tilde{+}, \tilde{\cdot})$ is a soft ring over (E, U).
ii) If $F_A \tilde{\cup} G_B \in S_f(U)$ and $F_A \tilde{\subseteq} G_B$ or $G_B \tilde{\subseteq} F_A$ then $(F_A \tilde{\cup} G_B, \tilde{+}, \tilde{\cdot})$ is a soft ring over (E, U).

Proof. Straightforward.

Definition 2.16. Let $(F_A, \tilde{+}, \tilde{\cdot})$ be a soft ring and $\tilde{\emptyset} \neq G_B \tilde{\subseteq} F_A$. If G_B satisfies the following conditions

i) for all $\gamma, \beta \tilde{\in} G_B$, $\gamma \tilde{+} \beta \tilde{\in} G_B$,
ii) for all $\beta \tilde{\in} F_A$ and for all $\gamma \tilde{\in} G_B$, $\gamma \tilde{\cdot} \beta \tilde{\in} G_B$ and $\beta \tilde{\cdot} \gamma \tilde{\in} G_B$,

then G_B is called a soft ideal of F_A. In particularly, if for all $\beta \tilde{\in} F_A$ and for all $\gamma \tilde{\in} G_B$,, then $\gamma \tilde{\cdot} \beta \tilde{\in} G_B$ is said to be a soft right ideal of F_A and $\beta \tilde{\cdot} \gamma \tilde{\in} G_B$ then G_B is said to be a soft left ideal of F_A.

Note 2.17. The soft ring $(F_A, \tilde{+}, \tilde{\cdot})$ and $G_B = \{\tilde{0}\}$ where $\tilde{0}$ is the identity of F_A according to the binary operation $\tilde{+}$ are soft ideals of F_A.

It is clear that if $(F_A, \tilde{+}, \tilde{\cdot})$ is a soft ring with an identity $\tilde{1}$ and G_B is the soft ideal of F_A then $G_B = F_A$. Every soft ideal is a soft subring but the converse side is not true in general.

Example 2.17. Consider the Example 2.4. Take the soft subset G_E of F_E where $(\bar{0}) = \{\bar{0}, \bar{2}\}$, $G(\bar{1}) = \emptyset$, is an soft ideal of F_E.

2.2. Soft Topology

Soft topological structures are studied by many authors with their own approaches. In this subsection definitions and some several properties about the soft topological spaces are reminded, which is going to be used in the third section. The essentials of the theory of soft topological structures were introduced by Roy et al. [10].

Example 2.18. [3] Let $U = \{u_1, u_2, u_3\}$, $A = \{p_1, p_2\}$ and $F_A = \{(p_1, \{u_1, u_2\}), (p_2, \{u_2, u_3\})\}$. In that case all soft subsets of F_A are listed below.

$F_{A_1}^1 = \{(p_1, \{u_1\})\}$,

$F_{A_2}^2 = \{(p_1, \{u_2\})\}$,

$F_{A_3}^3 = \{(p_1, \{u_1, u_2\})\}$,

$F_{A_4}^4 = \{(p_2, \{u_2\})\}$,

$F_{A_5}^5 = \{(p_3, \{u_3\})\}$,

$F_{A_6}^6 = \{(p_2, \{u_2, u_3\})\}$,

$F_{A_7}^7 = \{(p_1, \{u_1\}), (p_2, \{u_2\})\}$,

$F_{A_8}^8 = \{(p_1, \{u_1\}), (p_2, \{u_3\})\}$,

$F_{A_9}^9 = \{(p_1, \{u_1\}), (p_2, \{u_2, u_3\})\}$,

$F_{A_{10}}^{10} = \{(p_1, \{u_2\}), (p_2, \{u_2\})\}$,

$F_{A_{11}}^{11} = \{(p_1, \{u_2\}), (p_2, \{u_3\})\}$,

$F_{12}^{12} = \{(p_1, \{u_2\}), (p_2, \{u_2, u_3\})\}$,

$F_{A_{13}}^{13} = \{(p_1, \{u_1, u_2\}), (p_2, \{u_2\})\}$,

$F_{A_{14}}^{14} = \{(p_1, \{u_1, u_2\}), (p_2, \{u_3\})\}$,

$F_{A_{15}}^{15} = F_A$,

$F_{A_{16}}^{16} = \widetilde{\emptyset}$.

Then $\widetilde{\tau_1} = \{\widetilde{\emptyset}, F_A\}$,$\widetilde{\tau_2} = \{F_{A_1}^1, F_{A_2}^2, \ldots, F_{A_{16}}^{16}\}$, $\widetilde{\tau_3} = \{\widetilde{\emptyset}, F_A, F_{A_2}^2, F_{A_{11}}^{11}, F_{12}^{12}\}$ are soft topologies on F_A.

Definition 2.19. [14] Let $(F_A, \tilde{\tau})$ be a soft topological space and $(e_j, \{u_l\}) \widetilde{\in} F_A$. Given soft subset G_B of F_A is said to be a soft neighborhood of $(e_j, \{u_l\})$, if there exist an open soft set H_C such that $(e_j, \{u_l\}) \widetilde{\in} H_C \widetilde{\subseteq} G_B$. $N_{(e_j, \{u_l\})}$ is symbolized the all soft neighborhoods of the soft element $(e_j, \{u_l\})$.

Example 2.20. [14] Let F_A be the soft set and $\widetilde{\tau_3}$ be the soft topology on F_A given in Example 2.18. The set of all nonempty soft elements of F_A is

$F_A^{\bullet} = \{(p_1, \{u_1\}), (p_1, \{u_2\}), (p_2, \{u_2\}), (p_2, \{u_3\})\}$. For the soft element $(p_1, \{u_1\}) \widetilde{\in} F_A$, the soft sets containing $(p_1, \{u_1\})$ are $F_A, F_{A_1}^1, F_{A_3}^3, F_{A_7}^7, F_{A_8}^8, F_{A_9}^9, F_{A_{13}}^{13}$ and $F_{A_{14}}^{14}$.

$N_{(p_1, \{u_1\})} = \{F_A, F_{A_{13}}^{13}\}$ is a set of all soft neighborhoods of $(p_1, \{u_1\})$.

$N_{(p_1, \{u_2\})} = \{F_A, F_{A_2}^2, F_{A_{11}}^{11}, F_{A_{13}}^{13}\}$ is a set of soft all neighborhoods of $(p_1, \{u_2\})$.

$N_{(p_2, \{u_2\})} = \{F_A, F_{A_{13}}^{13}\}$ is a set of all soft neighborhoods of $(p_2, \{u_2\})$.

$N_{(p_2, \{u_3\})} = \{F_A, F_{A_{11}}^{11}\}$ is a set of all soft neighborhoods of $(p_2, \{u_3\})$.

Proposition 2.21. [14] Let $(F_A, \tilde{\tau})$ be a soft topological space. A soft set $G_B \widetilde{\subseteq} F_A$ is soft open if and only if for each soft element $\alpha \widetilde{\in} G_B$ there exists a soft set $H_C \in \tilde{\tau}$ such that $\alpha \widetilde{\in} H_C \widetilde{\subseteq} G_B$.

Definition 2.22. [14] Let $(F_A, \tilde{\tau})$ be a soft topological space and $G_B \widetilde{\subseteq} F_A$. The soft topology on G_B induced by the soft topology $\tilde{\tau}$ is the family of $\tilde{\tau}_{G_B}$ of the soft subsets of G_B of the form

$\tilde{\tau}_{G_B} = \{H_C \widetilde{\cap} G_B : H_C \in \tilde{\tau}\}$.

One can prove that the family $\tilde{\tau}_{G_B}$ is a soft topology on G_B. The soft topological space $(G_B, \tilde{\tau}_{G_B})$ is called a soft topological subspace of $(F_A, \tilde{\tau})$.

Definition 2.23. [14] Let $(F_A, \tilde{\tau}_1)$ and $(G_B, \tilde{\tau}_2)$ be soft topological spaces and $\beta = \{F_{A_i} x G_{B_j} : F_{A_i} \in \tilde{\tau}_1, G_{B_j} \in \tilde{\tau}_2\}$. The collection $\tilde{\tau}$ of all arbitrary union of soft elements of β is called the soft product topology over $F_A \times G_B$.

3. SOFT TOPOLOGICAL RING

The structure of the topological ring is more improved in comparison with the concept of a topological group. Also theory of topological rings has several characteristics in common with the theory of topological groups. In the soft set theory, it would be similar. The soft topological group was defined by Polat et. al. in [18] in 2018.

After searching literature on soft rings and soft topological rings reader can deduced that the soft ring structure used the refer to a soft set F_A over a ring U such that $F(x)$ is a subring of universal set U, for every $x \in A$ and the soft topological ring studies based on this soft ring definition defined in [9]. The soft ring definition is redefined in this study. Purpose of this study is to combine soft ring and soft topological space structures on a soft set.

Definition 3.1. Let $(F_A, \tilde{+}, \tilde{\cdot})$ be a soft ring and define a soft topology $\tilde{\tau}$ over F_A. If the following three conditions are satisfied then $(F_A, \tilde{+}, \tilde{\cdot}, \tilde{\tau})$ is called a soft topological ring.

i) For each soft neighborhood G_B of the soft element $(e_i, \{u_k\}) \tilde{+} \left(e_j, \{u_l\}\right)$ there exist soft neighborhoods H_C of $(e_i, \{u_k\})$ and K_D of $\left(e_j, \{u_l\}\right)$ satisfies that $H_C \tilde{+} K_D \tilde{\subseteq} G_B$.

ii) For each soft neighborhood G_B of the soft element $(e_i, \{u_k\})^{-1}$ there exist a soft neighborhood H_C of $(e_i, \{u_k\})$ such that ${H_C}^{-1} \tilde{\subseteq} G_B$.

iii) For each soft neighborhood G_B of the soft element $(e_i, \{u_k\}) \tilde{\cdot} \left(e_j, \{u_l\}\right)$ there exist soft neighborhoods H_C of $(e_i, \{u_k\})$ and K_D of $\left(e_j, \{u_l\}\right)$ respectively satisfies that $H_C \tilde{\cdot} K_D \tilde{\subseteq} G_B$.

Note 3.2. If $(F_A, \tilde{+}, \tilde{\cdot}, \tilde{\tau})$ is a soft topological ring then $(F_A, \tilde{+}, \tilde{\tau})$ is a soft topological group. Therefore, every property given for soft commutative topological groups is valid for soft topological rings.

Theorem 3.3. Let $(F_A, \tilde{+}, \tilde{\cdot})$ be a soft ring and define a soft topology $\tilde{\tau}$ over F_A. If the conditions given in below are satisfied,

i) For each soft neighborhood G_B of the soft element $(e_i, \{u_k\}) \tilde{+} \left(e_j, \{u_l\}\right)^{-1}$ there exist soft neighborhoods H_C of $(e_i, \{u_k\})$ and K_D of $\left(e_j, \{u_l\}\right)$ respectively satisfy that $H_C \tilde{+} {K_D}^{-1} \tilde{\subseteq} G_B$.

ii) For each soft neighborhood G_B of the soft element $(e_i, \{u_k\}) \tilde{\cdot} \left(e_j, \{u_l\}\right)$ there exist soft neighborhoods H_C of $(e_i, \{u_k\})$ and K_D of $\left(e_j, \{u_l\}\right)$ respectively satisfy that $H_C \tilde{\cdot} K_D \tilde{\subseteq} G_B$

then $(F_A, \tilde{+}, \tilde{\cdot}, \tilde{\tau})$ is a soft topological ring.

Proof. The proof is obvious from the continuity of composite function.

Example 3.4. [14] Let $E = \{e_1, e_2\}$, $U = \mathbb{Z}_4$ be the classes of residues of integers module 4. E is a ring defined with the operations $+, \cdot$. Tables of the operation $+, \cdot$ on E are given as in the below.

$+$	e_1	e_2
e_1	e_1	e_2
e_2	e_2	e_1

$\cdot$	e_1	e_2
e_1	e_1	e_1
e_2	e_1	e_1

Define a soft set $F: E \to P(U)$ by $F_E = \{(e_1, \{\overline{0}, \overline{2}\}), (e_2, \{\overline{1}, \overline{3}\})\}$. The table of the operations $\widetilde{+}$ and $\tilde{\cdot}$ on the soft set F_E given as;

$\widetilde{+}$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$	$(e_2, \{\overline{1}\})$	$(e_2, \{\overline{3}\})$
$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$	$(e_2, \{\overline{1}\})$	$(e_2, \{\overline{3}\})$
$(e_1, \{\overline{2}\})$	$(e_1, \{\overline{2}\})$	$(e_1, \{\overline{0}\})$	$(e_2, \{\overline{3}\})$	$(e_2, \{\overline{1}\})$
$(e_2, \{\overline{1}\})$	$(e_2, \{\overline{1}\})$	$(e_2, \{\overline{3}\})$	$(e_1, \{\overline{2}\})$	$(e_1, \{\overline{0}\})$
$(e_2, \{\overline{3}\})$	$(e_2, \{\overline{3}\})$	$(e_2, \{\overline{1}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$

$\tilde{\cdot}$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$	$(e_2, \{\overline{1}\})$	$(e_2, \{\overline{3}\})$
$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{0}\})$
$(e_1, \{\overline{2}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$	$(e_1, \{\overline{2}\})$
$(e_2, \{\overline{1}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$	$(e_2, \{\overline{1}\})$	$(e_1, \{\overline{3}\})$
$(e_2, \{\overline{3}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$	$(e_2, \{\overline{3}\})$	$(e_2, \{\overline{1}\})$

In this example one can easily prove that $(F_A, \widetilde{+}, \tilde{\cdot})$ is a commutative soft ring with a soft identity element $(e_2, \{\overline{1}\})$. Consider the soft topology $\tilde{\tau} = \{\widetilde{\emptyset}, F_A, F_{A_1}^1, F_{A_2}^2\}$ where soft subsets of F_A are given as; $F_{A_1}^1 = \{(e_1, \{\overline{0}, \overline{2}\})\}$ and $F_{A_2}^2 = \{(e_2, \{\overline{1}, \overline{3}\})\}$. Then $(F_A, \widetilde{+}, \tilde{\cdot}, \tilde{\tau})$ is a soft topological ring over (E, U).

Theorem 3.5. If $(F_A, \widetilde{+}, \tilde{\cdot}, \tilde{\tau})$ is a soft topological ring and G_B is a soft subring of F_A, so is $(G_B, \widetilde{+}, \tilde{\cdot}, \tilde{\tau}_{G_B})$.

Proof. Straightforward.

4. CONCLUSION

The soft set theory has wide field of study in different fields especially for the mathematicians in the algebraic and the topological structures. In this paper soft ring and soft topological ring structures are given from the soft element viewpoint which is very naturel approximation. For further studies the other algebraic structures can be studied by the similar viewpoint.

AUTHORSHIP CONTRIBUTIONS

Nazan ÇAKMAK POLAT: Developed the theoretical formalism, wrote the manuscript. played a key role in manuscript preparation and revision.

Gözde YAYLALI: Provided expertise in the specialized mathematical subfield related to the research. Played a crucial role in editing and refining the manuscript for clarity and coherence.

Bekir TANAY: Supervised the project. Contributed mathematical insights during regular group discussions. Contributed the final version of the manuscript.

REFERENCES

[1] Molodtsov D. Soft set theory-first result. Comput Math Appl 1999; 37: 19-31.

[2] Maji PK, Biswas R, Roy AR. Soft set theory. Comput Math Appl 2003; 45: 555-562.

[3] Çağman N, Karataş S, Enginoglu S. Soft topology. Comput. Math Appl 2011; 62(1): 351-358.

[4] Shabir M, Naz M. On soft topological spaces. Comput. Math Appl 2011; 61(7): 1786-1799.

[5] Roy S, Samanta TK. An introduction of a soft topological spaces. Proceeding of UGC sponsored national seminar on recent trends in fuzzy set theory, Rough set theory and Soft set theory at Uluberia College on 23rd and 24th September, 2011 ISBN 978-81-922305-5-9, 9-12, 2011.

[6] Roy S, Samanta TK. A note on a soft topological space. Journal of Mathematics 2014; 46(1): 19-24.

[7] Cagman N, Enginoglu S. Soft set theory and uni-int decision making. Eur J Oper Res 2010; 207: 848-855.

[8] Aktaş H, Çağman N. Soft sets and soft groups. Inf Sci 2007; 177: 2726-2735.

[9] Acar U, Koyuncu F, Tanay B. Soft sets and soft rings. Comput Math Appl 2010; 59: 3458-3463.

[10] Ghosh J, Mandal D, Samanta TK. Soft groups based on soft element. Jordan J Math Stat 2016; 9 (2): 141-159.

[11] Sun QM, Zhang ZL, Liu J. Soft sets and soft modules. Lecture Notes in Computer Science 2008; 5009: 403–409.

[12] Gunduz (Aras) C, Bayramov S, Fuzzy Soft Modules. International Mathematical Forum 2011; 6(11): 517 – 527

[13] Wardowski D. On a soft mapping and its fixed points. J Fixed Point Theory Appl 2013; 182: 1-11.

[14] Polat NÇ, Yaylalı G, Tanay B. A new approach for soft semi-topological groups based on soft element. Filomat, 2018; 32(16): 5743-5751.

[15] Polat NÇ, Yaylalı G, Tanay B. Some results on soft element and soft topological space. Math. Methods Appl Sci 2018; 42(16): 5607-5614.

[16] Tanay B, (Polat) Çakmak N. Soft Semi-Topological Groups. J Interdisc Math 2014; 17 (4): 355-363.

[17] Shah T, Shaheen S. Soft Topological Groups and Rings. Ann Fuzzy Math Inform 2013; 7 (5): 725-743.

[18] Tahat MK, Sidky F, Abo-Elhamayel A. Soft topological soft groups and soft rings. Soft computing 2018; 22: 7143-7156.

[19] Tahat MK, Sidky F, Abo-Elhamayel A. Soft topological rings. Journal of King Saud University 2019; 31(4): 1127-1136.

[20] Hida T. Soft topological group. Ann Fuzzy Math Inform 2014; 8: 1001-1025

[21] Nazmul S, Samanta SK. Soft topological groups. Kochi J Math 2010; 5: 151-161