Afyon Kocatepe Üniversitesi Fen ve Mühendislik Bilimleri Dergisi
Afyon Kocatepe University Journal of Science and Engineering

AKÜ FEMÜBİD **23** (2023) 041303 (933-940) AKU J. Sci. Eng. **23** (2023) 041303 (933-940)
DOI: 10.35414/ akufemubid.1231989

Araştırma Makalesi / Research Article

Shrinkage Estimation and Bootstrap Confidence Interval for Scale Parameter of Laplace Distribution

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Geliş Tarihi: 10.01.2023 Kabul Tarihi: 31.07.2023

Keywords
Biased Estimation; Bootstrap Confidence Interval; Laplace Distribution; Shrinkage Estimation.

Abstract

In this study, a biased estimator is proposed for the scale parameter of Laplace distribution. First, it is theoretically shown that the mean square error of the biased estimator is smaller than that of the maximum likelihood estimator. Then the maximum likelihood estimator is compared with the obtained biased estimator by means of a simulation study using the relative efficiency of these estimators. In addition, confidence intervals are constructed for the scale parameter of Laplace distribution with bootstrap method in order to compare them with each other in a different way.

Laplace Dağılımının Ölçek Parametresi için Daraltıcı Tahmin ve Bootstrap Güven Aralığı

Anahtar kelimeler
Yanlı Tahmin, Bootstrap Güven Aralığı; Laplace Dağılımı; Daraltıcı Tahmin.

Öz

Bu çalışmada, Laplace dağılımının ölçek parametresi için yanlı bir tahmin edici önerilmiştir. İlk olarak, yanlı tahmin edicinin hata kare ortalamasının, en çok olabilirlik tahmin edicisininkinden daha küçük olduğu teorik olarak gösterilmiştir. Daha sonra en çok olabilirlik tahmin edicisi ile elde edilen yanlı tahminci, bu tahmincilerin göreli etkinlikleri kullanılarak bir benzetim çalışması ile karşılaştırılmıştır. Ayrıca tahmin edicileri farklı bir açıdan karşılaştırmak için Laplace dağılımının ölçek parametresi için bootstrap yöntemi ile güven aralıkları oluşturulmuştur.

1. Introduction

Unbiased estimators are widely used to estimate descriptive parameters of distributions. If the unbiased estimator has high variance, it may be possible to use biased estimators which has smaller mean squared error (MSE) criterion than that of unbiased estimator. There are several studies on biased but has lower MSE estimators of unknown population parameters. Thompson (1968) suggested a shrinkage method by multiplying the best linear unbiased estimator (BLUE) by a shrinking factor to obtain an estimator with a smaller MSE than the BLUE. Shrinkage estimators are considered a lot of studies in literature as Metha and Srinivasan (1971) gave estimation of the mean by shrinkage to a point, Govindarajulu and Sahai (1972) studied on estimating parameters of normal distribution, Bhatnagar (1986) propose to use variance estimating mean, Singh and Katyar (1988) proposed a generalized class of estimators for parameters of normal distribution, Singh (1990) also studied on estimating parameters of normal distributions, Jani (1991) suggested a class of shrinkage estimators for the scale parameter of exponential distribution, Singh and Singh (1997) and Singh and Saxena (2003) studied on shrinkage estimation for the variance of a normal population, Singh and Saxena (2008) gave a family of shrinkage estimators for Weibull shape parameter, Özdemir and Ebegil (2012) proposed shrinkages estimators for the shape parameter of pareto distribution, Mehta and Singh (2014) suggested shrinkage estimators of parameters of morgenstern type bivariate logistic distribution, Singh and Mehta (2016) studied on a class of shrinkage estimators of scale parameter of uniform distribution based on k-record values, Ebegil and Özdemir (2016) proposed two different shrinkage estimator classes for the shape parameter of classical pareto distribution, Balui *et al*. (2020) gave two different shrinkage estimator classes for the scale parameter of classical rayleigh distribution and, Vishwakarma and Gupta (2022) proposed shrinkage estimator for scale parameter of gamma distribution.

In this study, we focus on estimating the parameters of Laplace distribution, also known as the double exponential distribution. The probability density function of the Laplace distribution, is given as

$$f(x) = (2\beta)^{-1}e^{-\frac{|x-\alpha|}{\beta}}, -\infty < x < \infty, -\infty < \alpha < \infty, \beta > 0 \quad (1)$$

where α is the location parameter and β is the scale parameter. Govindarajulu (1996) obtained the best linear estimates under symmetric censoring for the parameters of the double exponential distribution. Some alternative estimates were also given as in Raughunandanan and Srinivasan (1971). It was considered that coefficients for computing ordered linear unbiased minimum variance estimators for the location and scale parameters of the double exponential distribution in Tiao and Lund (1970). Confidence intervals based on maximum likelihood (ML) estimators were given for the location and scale parameters of the double exponential distribution Bain and Engelhardt (1973).

In this study we obtained a biased estimator, which is adapting the shrinkage estimator supposed by Thompson (1968), for the scale parameter of the Laplace distribution using the ML estimator. Firstly, it is theoretically shown that the MSE of the biased estimator is smaller than the MSE of the ML estimator. Then the ML estimator is compared with the obtained biased estimator by means of a simulation study to show in which case the biased estimator is better than the ML estimator. At last, confidence intervals are constructed for the parameter $\boldsymbol{\beta}$ with bootstrap method using both the biased estimator and the ML estimator in order to compare them with each other.

The bootstrap method is a larger form of method class that resamples from the original dataset, hence called resampling procedures. Efron (1979) mentioned the simple nonparametric bootstrapping for independent and identically distributed observations, which "resamples the data with replacement", with previously accepted statistical tools to estimate standard errors such as the jackknife method. After the later papers by Efron and Gong (1983), Efron and Tibshirani (1986), Diaconis and Efron (1983) that the statistical and scientific community began began to take these ideas into account, to appreciate the extensions and broad applicability of the methods, and to recognize their importance (Chernick 2008). There are so many studies used the bootstrap method with a view to make statistical inference by the help of confidence intervals: see also Efron and Tibshirani (1986), Diaconis and Efron (1983), Efron (1987), Carpenter, and Bithell (2000), Park (2011), DiCiccio and Efron (1996), DiCiccio and Tibshirani (1987), Efron and Tibshirani (1993), Chesneau *et al*. (2022) and Akdoğan (2022).

The rest of the paper is organized as follows. The shrinkage estimation method is summarized in Section 2.1. A brief information about bootstrap method is given in Section 2.2. Section 3 includes a simulation study. A numerical example is given in Section 4. The paper is finalized with a conclusion section.

2. Methods

2.1. Shrinkage Estimation Method

Thompson (1968) suggested a shrinkage estimator $\hat{\theta}_s$ given as

$$\hat{\theta}_s = c(\hat{\theta}) + (1-c)\theta_0 \quad (2)$$

where $\hat{\theta}$ is the unbiased estimator, θ_0 is the prior information for parameter θ and c is a shrinking factor which is $0 \leq c \leq 1$ and also minimizes the MSE value of the proposed estimator.

To adapt the mentioned method to estimating the scale parameter of Laplace distribution, first the unbiased estimator is obtained by using the ML estimation method, which is given by

$$\hat{\beta} = \frac{\sum_{i=1}^{n}|x_i-\alpha|}{n}. \quad (3)$$

Note that $\hat{\beta}$ can be found by maximizing the likelihood function

$$L(\alpha,\beta;x) = \prod_{i=1}^{n} f(x_i) = 2^{-n}\beta^{-n}e^{-1/\beta|x_i-\alpha|}$$

and related log-likelihood function is

$$\ln L(\alpha,\beta;x) = -n\ln 2 - n\ln\beta - \frac{1}{\beta}\sum_{i=1}^{n}|x_i-\alpha|.$$

It can be said that if random variable X has the Laplace distribution with parameters α and β, then $\sum_{i=1}^{n}|x_i-\alpha|$ has the Gamma distribution with parameters n,β. So it is clear that the expected value of $\hat{\beta}$ is equal to parameter β and the variance of $\hat{\beta}$ is calculated as

$$Var(\hat{\beta}) = \frac{\beta^2}{n}. \quad (4)$$

Corollary: The shrinkage estimator of the Laplace distribution's shape parameter is proposed as

$$\hat{\beta}_s = \frac{(\beta-\beta_0)^2}{\frac{\beta^2}{n}+(\beta-\beta_0)^2}\left(\hat{\beta}-\beta_0\right)+\beta_0 \quad (5)$$

where β_0 is the prior information for parameter β. The MSE value of $\hat{\beta}_s$ is

$$MSE(\hat{\beta}_s) = \frac{\beta^2(\beta-\beta_0)^2}{\beta^2+n(\beta-\beta_0)^2}. \quad (6)$$

Proof: The shrinkage estimator of the Laplace distribution's shape parameter described as

$$\hat{\beta}_s = c\left(\hat{\beta}-\beta_0\right)+\beta_0 \quad (7)$$

which is obtained by means of Equation (2). As known, the MSE value of $\hat{\beta}_s$ is

$$MSE(\hat{\beta}_s) = E\left[\hat{\beta}_s-\beta\right]^2. \quad (8)$$

If required information is written in Equation (8), the MSE value is obtained as

$$MSE(\hat{\beta}_s) = c^2\beta^2\frac{1}{n}+(c-1)^2(\beta-\beta_0)^2. \quad (9)$$

The derivative of equation (9) with respect to c is taken and set to zero, a solution for c can be found as

$$c = \frac{(\beta-\beta_0)^2}{\frac{\beta^2}{n}+(\beta-\beta_0)^2} \quad (10)$$

It is clear that *c* given in Equation (10) minimizes the MSE of $\hat{\beta}_s$. Thus, the biased estimator $\hat{\beta}_s$ can be written as

$$\hat{\beta}_s = \frac{(\beta-\beta_0)^2}{\frac{\beta^2}{n}+(\beta-\beta_0)^2}\left(\hat{\beta}-\beta_0\right)+\beta_0. \quad (11)$$

Inserting the value of c in Equation (9), the MSE of $\hat{\beta}_s$ is

$$MSE(\hat{\beta}_s) = \frac{\beta^2(\beta-\beta_0)^2}{\beta^2+n(\beta-\beta_0)^2}. \quad (12)$$

This completes the proof. ■

Furthermore, the bias of $\hat{\beta}_s$ estimator is

$$Bias(\hat{\beta}_s) = \frac{\beta^2(\beta-\beta_0)}{\beta^2+n(\beta-\beta_0)^2}. \quad (13)$$

Equation (13) shows that $\hat{\beta}_s$is asymptotically unbiased, namely $Bias(\hat{\beta}_s)\to 0$ as $n\to\infty$.

The relative efficiency of $\hat{\beta}_s$ estimator with respect to $\hat{\beta}$ estimator is given as

$$\frac{MSE(\hat{\beta}_s)}{Var(\hat{\beta})} = \frac{n(\beta-\beta_0)^2}{\beta^2+n(\beta-\beta_0)^2}. \quad (14)$$

According to Equation (14), it can be seen that $(\hat{\beta}_s)/Var(\hat{\beta}) < 1$, and $\hat{\beta}_s$ is more efficient than $\hat{\beta}$. The shrinking parameter *c* is a function of parameter β. As our goal is to estimate parameter β, the unknown parameters are replaced by their unbiased estimators in Equation (8). Thus an estimator for c can be written as

$$\hat{c} = \frac{\left(\hat{\beta}-\beta_0\right)^2}{\frac{\hat{\beta}^2}{n}+\left(\hat{\beta}-\beta_0\right)^2}. \quad (15)$$

The MSE values of the proposed estimator, which includes the shrinking coefficient given in equation (15), compares with the variance of the unbiased estimator in the simulation study section.

2.2. Bootstrap Method

The basic idea behind the bootstrap is resampling the data with replacement. Suppose there are observations such that independent data points $x_1, x_2, \dots, x_n$, for convenience denoted by vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$, from this a statistic of interest $\hat{\theta}(x)$ is computed. A bootstrap sample $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ is created by choosing a random sample of n units, replacing the original data points $x_1, x_2, \dots, x_n$. The obtained estimates from the bootstrap samples are called bootstrap estimates (Efron and Tibshirani 1993). A bootstrap confidence interval can be generated using the percentile method (Efron's percentile method). Suppose that $\hat{\theta}_i$ is the *i*th bootstrap estimate from the *i*-th bootstrap sample and each bootstrap sample size is *n*. Since there is a random sampling method, it is expected that if the observations are ordered from smallest to largest, an interval that contains 90% of the $\hat{\theta}_i$ to be a 90% confidence interval for θ. The most sensible way to choose the interval is to exclude the lowest 5% and the highest 5% (Chernick 2008). In this study, it is desired to construct the 95% confidence interval, so the lowest 2.5% and the highest 2.5% are considered in simulation section.

3 Simulation Study

In this section, a data set is generated for the Laplace distribution using two independent exponential distributions with the same parameter β. Then the unbiased estimator of parameter β and its variance were calculated for different values of the parameters α and β when the sample size was n=10. Following this, we used a moment estimator of parameter β in place of the prior information β_0, so that the prior information conformed to the generated data set. Using this estimator, we obtained the shrinkage estimator mentioned in the previous section and calculated its MSE by means of Monte Carlo Simulation study where the number of replications was 75000. Then the relative efficiency is calculated by proportioning the MSE of $\hat{\beta}_s$ estimator to the variance of $\hat{\beta}$ estimator.

We carried out similar calculations for the both cases where the parameter α is known and α parameter is not known in order to estimate the parameter β.

If the parameter α is known, it can be used in Equation (3) to find the unbiased estimator the parameter β. Using different values of parameter α ranging from 0 to 4.9 with 0.1 increments and using different values of parameter β ranging from 0.1 to 5 with 0.1 increments, a data set was generated in Figure 1 which enables a visual efficiency comparison.

In Figure 1, It can be seen that the relative efficiency is smaller than 1 for all handled values of parameters α and β. Namely, the proposed shrinkage estimator has smaller MSE than that of the unbiased estimator. Also, Figure 1 shows that differences of parameters α and β have similar effects on relative efficiencies. Relative efficiencies

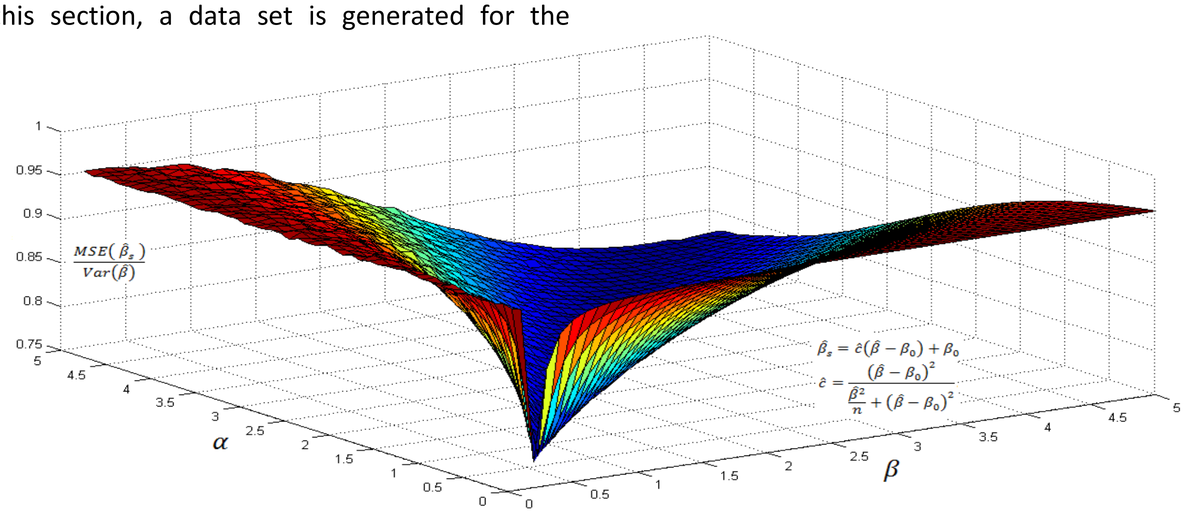

Figure 1: Relative efficiencies when parameter α is known.

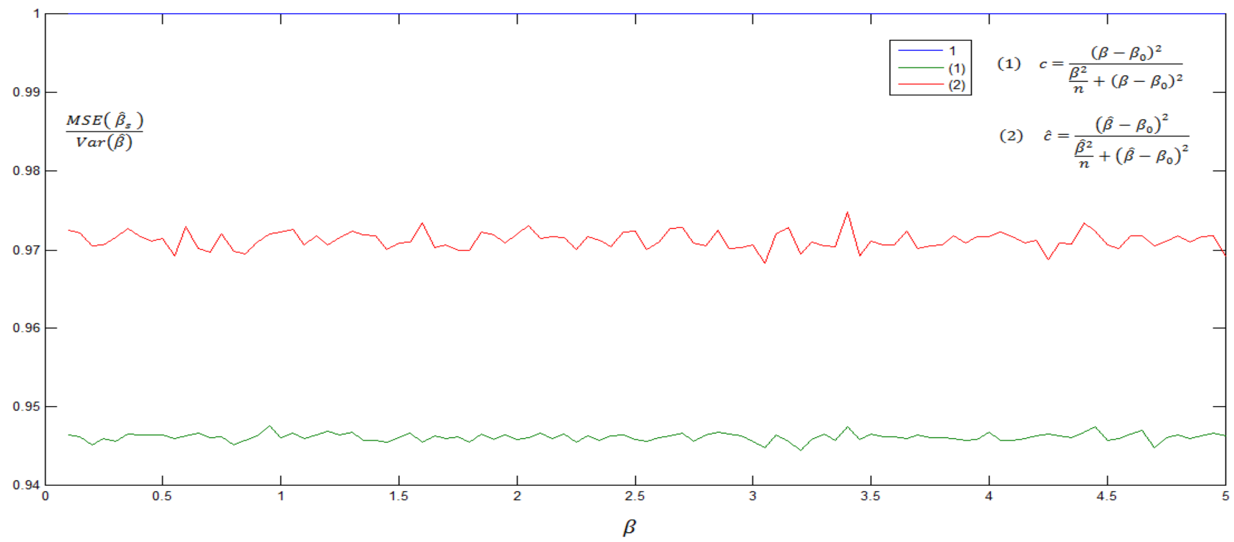

Figure 2: Relative efficiencies when the parameter α is replaced by the sample mean.

take their minimum values when parameters α and β are equal to each other.

If the parameter α isn't known, it can be replaced by an estimator as sample mean. Figure 2 shows relative efficiencies of mentioned estimators for different parameters β that take values 0.1 to 5 by 0.05 increments and sample means are used in place of parameters α. As indicated in Figure2, shrinkage estimators, which are both include parameter β and obtained by the using unbiased estimator, give relative efficiencies smaller than 1 for all handled parameters β. So, the shrinkage estimator has a smaller MSE than that of the unbiased estimator for parameter β.

Until now, we handled cases that sample size was n=10. Figure 3 shows relative efficiencies calculated for different sample sizes under the condition that parameter α isn't known. The sample size take values 2 to 50 by 1 increments. The blue line in the Figure 2 and 3 represents where the relative efficiency equals one to facilitate comparison.

It is seen that the shrinkage estimator given by Equation (5) has a smaller MSE than that of the unbiased estimator for all different sample size as indicated theoretically in Equation (14). But the shrinkage estimator, which includes the shrinkage factor given by Equation (15), takes higher values than that of unbiased estimator when $n > 22$.

There isn't a significant difference in the relative efficiencies for different values of the parameter β according to Figure 2. So we randomly handled the situation such that parameter β takes values 2 and 5 to construct the confidence intervals. Also, Figure 3 shows that $n = 22$ is a threshold for superiority of

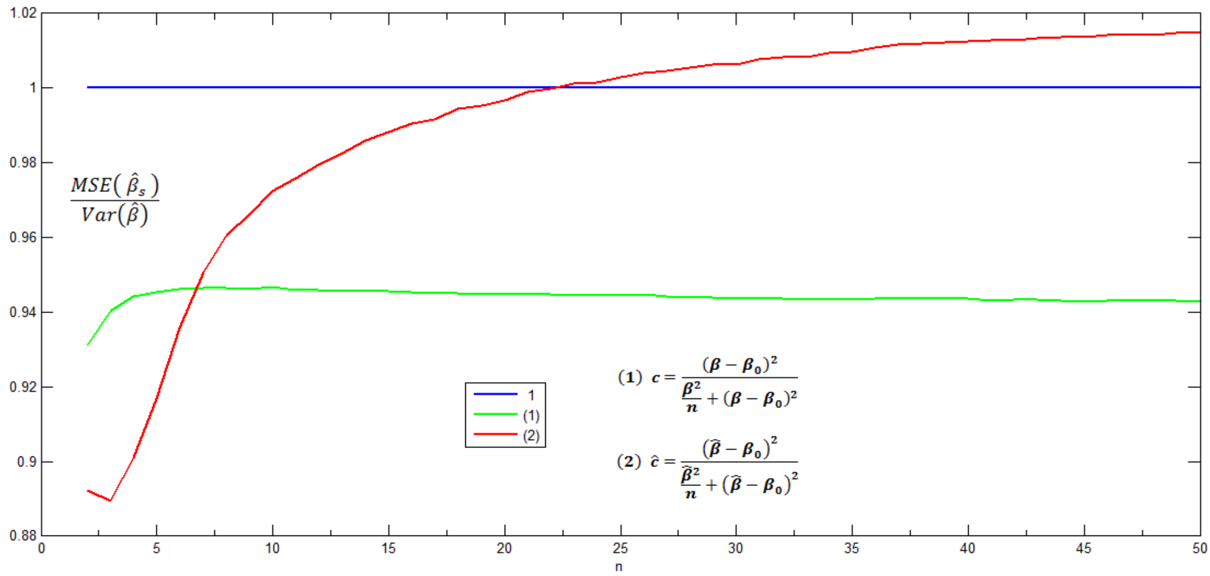

Figure 3*:* Relative efficiencies for different sample sizes.

the biased estimator. Therefore we considered four different situations for sample size ($n = 5, 10, 25, 50$) by taking into account the threshold value. The confidence intervals are obtained using the bootstrap method for all of mentioned situation. Then the method repeated 10000 times in order to calculate the confidence level which shows ratio of include parameter by confidence intervals and to obtain means of bounds of confidence intervals. The obtained means of confidence intervals, which are constructed by using both the biased estimator $\hat{\beta}_S$ estimator and the ML estimator $\hat{\beta}$, are given in Table 1. Furthermore, confidence levels for each confidence interval are in Table 1 where MLB is the mean lower bound, while MUB symbolize the mean upper bound.

Although Figure 3 shows that there is a threshold for superiority of the biased estimator such as $n = 22$, it can be seen that $\hat{\beta}_S$ estimators have smaller variances than those of $\hat{\beta}$ estimator with respect to obtained confidence intervals.

Constructed confidence intervals obtained using $\hat{\beta}_S$ estimators have narrower width than those of $\hat{\beta}$ estimator because of $\hat{\beta}_S$ have smaller MSE values than those of $\hat{\beta}$. Therefore the confidence levels of confidence intervals obtained by using $\hat{\beta}_S$ estimator are smaller than those of confidence intervals obtained by using $\hat{\beta}$ estimator.

Table 1: Confidence Intervals for parameter β using $\hat{\beta}_s$ and $\hat{\beta}$, and their confidence levels.

	$\beta = 2$				$\beta = 5$			
	$\hat{\beta}_s$		$\hat{\beta}$		$\hat{\beta}_s$		$\hat{\beta}$	
n	MLB	MUB	MLB	MUB	MLB	MUB	MLB	MUB
5	0.36	2.13	0.31	2.52	0.92	5.33	0.81	6.31
	($\hat{\alpha} = 0.52$)		($\hat{\alpha} = 0.60$)		($\hat{\alpha} = 0.51$)		($\hat{\alpha} = 0.58$)	
10	0.77	2.47	0.78	2.82	1.92	6.18	1.94	7.05
	($\hat{\alpha} = 0.73$)		($\hat{\alpha} = 0.81$)		($\hat{\alpha} = 0.71$)		($\hat{\alpha} = 0.79$)	
25	1.21	2.55	1.23	2.69	2.99	6.34	3.06	6.68
	($\hat{\alpha} = 0.84$)		($\hat{\alpha} = 0.90$)		($\hat{\alpha} = 0.82$)		($\hat{\alpha} = 0.89$)	
50	1.43	2.46	1.45	2.52	3.55	6.13	3.60	6.28
	($\hat{\alpha} = 0.89$)		($\hat{\alpha} = 0.92$)		($\hat{\alpha} = 0.89$)		($\hat{\alpha} = 0.92$)	

4. Numerical Example

We use a data set consisting exchange rates (EURO/DOLAR) between the years 1999-2020. The data is given in Table 2 and also available at https://ec.europa.eu/eurostat/databrowser/view/ert_bil_eur_a/default/table?lang=en.

Table 2: Yearly average exchange rates (1 UNIT of EUR = X UNITS of USD)

Year	Rate	Year	Rate	Year	Rate	Year	Rate
1999	1.07	2005	1.25	2011	1.39	2017	1.13
2000	0.92	2006	1.26	2012	1.29	2018	1.18
2001	0.90	2007	1.37	2013	1.33	2019	1.12
2002	0.94	2008	1.47	2014	1.33	2020	1.14
2003	1.13	2009	1.39	2015	1.11		
2004	1.24	2010	1.33	2016	1.11		

Using Equation (3) the ML estimates for the data is obtained as $\hat{\beta} = 0.1318$. The shrinkage estimation, which is given in Equation (7), is calculated as $\hat{\beta}_S = 0.1182$. It can be said that using ML estimation the data fits the Laplace distribution according to Kolmogorov-Smirnov test since the test statistic and p-value are 0.1345 and 0.8998, respectively. Since the Kolmogorov-Smirnov test statistics for the data using shrinkage estimation is 0.1545 and related p-value is 0.7822, the data can be considered to fit the Laplace distribution. Log-likelihood values using ML and shrinkage estimates are 1.333 and 1.442, respectively. So, the shrinkage estimator is more preferable than ML because of its greater log-likelihood value for this numerical example. Further, AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) values are calculated and given in Table 3. Similar to comment based on log-likelihood values, it is more convenient to use the shrinkage estimator according to AIC and BIC values, since interested value for the shrinkage estimate are smaller than those of ML estimate.

Table 3: Log-likelihood, AIC and BIC values for yearly average exchange rates using the ML and shrinkage estimates

	ML	Shrinkage
Log-Likelihood	1.333	1.442
AIC	1.334	1.115
BIC	4.902	4.683

5. Conclusions

The biased estimators sometimes give a smaller MSE than unbiased estimators. In such cases, it may be preferable to use biased estimators instead of unbiased estimators. In this study, the biased estimator for the scale parameter of Laplace distribution was obtained by using the shrinkage estimation method proposed by Thompson (1968). The estimator obtained in this way gives a lower MSE than the unbiased estimator. It was theoretically shown that the shrinkage estimator has smaller MSE than that of the unbiased estimator by help of Equation (14). But in this equation, the shrinkage estimator depended on the scale parameter. Since this is inconvenient in practice, a different shrinkage estimator was obtained by replacing the scale parameter with its unbiased estimator and the efficiency of this shrinkage estimator is calculated by means of simulation study. In the simulation study, we generated data sets using double exponential distribution for different α and β parameters. The unbiased estimator, the moment estimator which refers to prior information, and shrinkage estimators are obtained for mentioned data sets. After that, we calculated relative efficiencies of estimators using their MSE values. As indicated in the first figure and the second one, relative efficiencies take smaller values than 1 for all handled situations. Constructed confidence intervals also showed that confidence intervals obtained using the $\hat{\beta}_s$ estimator have narrower width than those of $\hat{\beta}$ estimator. This indicates that the shrinkage estimator has a smaller MSE than that of the unbiased estimator. So it can be said that the shrinkage estimator more efficient than the unbiased estimator for the scale parameter of Laplace Distribution when $n < 22$. It may seem like a restricted frame, as the proposed estimator is preferable for small samples. However, it should be kept in mind that many studies work with small samples in cases where the experiments cannot be repeated or in environments where data acquisition is difficult.

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