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The Fekete-Szegö Problem for a Certain class of Analytic Functions

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Abstract: In this study, we introduce and examine a certain subclass of analytic functions in the open unit disk in the complex plane. Here, we give coefficient-bound estimates and investigate the Fekete-Szegö problem for this class. Some interesting special cases of the results obtained here are also discussed.

Analitik Fonksiyonların Belirli Bir Sınıfı İçin Fekete-Szegö Problemi Üzerine

Anahtar Kelimeler:
Katsayı tahminleri,
Fekete-Szegö problemi,
Analitik fonksiyon,

Özet: Bu çalışmada, kompleks düzlemin açık birim diskinde analitik fonksiyonların belirli bir alt sınıfı tanıtılıyor ve inceleniyor. Sonrasında tanıtılan sınıf için katsayı sınır tahminleri verilir ve Fekete-Szegö problemi incelenir. Ayrıca, bulunan sonuçların bazı ilginç özel durumları tartışılır.

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1. INTRODUCTION

In the study, we denote by A the class of all complex-valued functions f which are analytic in the open unit disk $\mathfrak{A}=\left\{t\in\mathbb{C}:|t|<1\right\}$ in the complex plane $\mathbb{C}$ and written in the form

$$f(t)=t+a_2t^2+\cdots+a_nt^n+\cdots \\ =t+\sum_{n=2}^{\infty}a_nt^n,\ t\in\mathbb{C}. \tag{1}$$

Then, the family of all univalent functions in A is denoted by S. Next, for $\alpha\in[0,1)$ $S^*(\alpha)$ denotes the starlike function classes of order α and $C(\alpha)$ denotes the convex function classes of order α in $\mathfrak{A}$. By definition, we have

$$S^*(\alpha)=\left\{f\in S:\operatorname{Re}\frac{tf'(t)}{f(t)}>\alpha,\ t\in\mathfrak{A}\right\} \text{ and}$$

$$C(\alpha)=\left\{f\in S:\operatorname{Re}\left(1+\frac{tf''(t)}{f'(t)}\right)>\alpha,\ t\in\mathfrak{A}\right\}.$$

Moreover, considering f and g analytic functions in $\mathfrak{A}$, we say f is subordinate to g and denote that condition by $f(t)\prec g(t)$ when an analytic function ω can be found such that it satisfies the conditions

$$\omega(0)=0,\ |\omega(t)|<1 \text{ and } f(t)=g(\omega(t)).$$

It can be clearly admitted by the researchers that one of the curicial subjects of the geometric function theory is the coefficient problem. Many different and interesting subclasses of analytic functions have been defined and investigated by many researchers and some estimates on the first two coefficients for the functions of these classes have been found by them (see [Brannan and Clunie, 1980; Brannan and Taha, 1986; Lewin, 1967; Netanyahu, 1969; Srivastava and et al., 2010; Zaprawa, 2014]).

It is also well known that the functional $\Delta_2(1)=a_3-a_2^2$, which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $\Delta_2(1)=a_3-\mu a_2^2$, where μ is a complex or real number (see Fekete and Szegö, 1983), is the curicial tool in analytic functions theory. In this theory, the Fekete-Szegö problem is to estimate the upper bound of $\left|a_3-\mu a_2^2\right|$ and many researchers have investigated this problem for different subclasses of analytic functions (see Mustafa 2017; Mustafa and Gündüz, 2019; Zaprawa, 2014). Very recently, the Fekete-Szegö problem for the subclass of bi-univalent functions with a shell-shaped region was studied by Mustafa and Mrugusundaramoorthy in (Mustafa and Murugusundaramoorthy, 2014) and associated with a nephroid domain in (Srivastava and et al., 2022). Also, the Fekete-Szegö problem is investigated for subclasses of bi-univalent functions with respect to the symmetric points defined by Bernoulli polynomials in (Buyankara and et al., 2022), for bi-univalent functions related to the Legendre polynomials in (Cheng and et al., 2022), for m-fold symmetric bi-univalent functions in (Oros and Cotîrlă, 2022).

2. MATERIAL AND METHOD

Now, we define some new subclasses of analytic and univalent functions as follows.

Definition 2.1. We will say a function $f\in S$ is in the class $C(\varphi)$ if it satisfies

$$1+\frac{tf''(t)}{f'(t)}\prec\varphi(t),\ t\in\mathfrak{A}.$$

In Definition 2.1, $\varphi(t)=t+\sqrt{1+t^2}$ and the branch of the square root is chosen with the initial value $\varphi(0)=1$. It can be clearly seen that by $\varphi(t)=t+\sqrt{1+t^2}$, the unit disc $\mathfrak{A}$ is mapped onto a shell-shaped region on the right half plane and φ is univalent and analytic in $\mathfrak{A}$. For the real axis, the range of φ is symmetric and φ has a positive real part in $\mathfrak{A}$ such that $\varphi(0)=\varphi'(0)=1$. Furthermore, for point $\varphi(0)=1$, φ has a star-like domain.

Let, P be the set of the functions $r(t)$ analytic in $\mathfrak{A}$ and satisfying $\operatorname{Re}(r(t))>0,\ t\in\mathfrak{A}$ and $r(0)=1$ with power series

$$r(t)=1+r_1t+r_2t^2+r_3t^3+\cdots+r_nt^n+\cdots \\ =1+\sum_{n=1}^{\infty}r_nt^n,t\in\mathfrak{A}.$$

We will need the lemmas below (see Duren, 1983; Grenander, 1958) for the functions with the positive real part so that we can show our main results.

Lemma 2.2. Let $r\in\mathrm{P}$, then $|r_n|\le 2$ for $n=1,2,3,\ldots$ and

$$\left|r_2-\frac{\lambda}{2}r_1^2\right|\le 2\cdot\max\left\{1,|\lambda-1|\right\} \\ =2\cdot\begin{cases}1 & if\quad \lambda\in[0,2], \\ |\lambda-1| & elsewhere.\end{cases}$$

Lemma 2.3. Let $r \in \mathrm{P}$, then $|r_n| \le 2$ for $n = 1,2,3,...$ and

$$r_2 = \frac{r_1^2}{2} + \frac{4 - r_1^2}{2} x,$$

$$r_3 = \frac{r_1^3}{4} + \frac{\left(4 - r_1^2\right) r_1}{2} x - \frac{\left(4 - r_1^2\right) r_1}{2} x^2 + \frac{4 - r_1^2}{2} \left(1 - |x|^2\right) z$$

for some x and z with $|x| < 1$ and $|z| < 1$.

Lemma 2.4. Let $r \in \mathrm{P}$, $b \in [0,1]$ and $b(2b-1) \le d \le b$. Then,

$$\left| r_3 - 2b\, r_1 r_2 + d r_1^3 \right| \le 2.$$

Remark 2.5. As can be seen from the serial expansion the function φ given in Definition 2.1, belongs to the class P.

In this paper, we give coefficient-bound estimates and solve the Fekete-Szegö problem for the class $C(\varphi)$.

3. RESULTS AND DISCUSSION

In this section, firstly we present the below theorem on the coefficient bound estimates for the class $C(\varphi)$.

Theorem 3.1. Let the function f given by (1) be in the class $C(\varphi)$. Then,

$$|a_2| \le \frac{1}{2}, \quad |a_3| \le \frac{1}{4} \text{ and } |a_4| \le \frac{5}{24}.$$

Proof. Let $f \in C(\varphi)$. Then, according to Definition 2.1 there is an analytic function $\omega : \mathfrak{A} \to \mathfrak{A}$ with $\omega(0) = 0$ and $|\omega(t)| < 1$ satisfying the following condition

$$1 + \frac{t f''(t)}{f'(t)} = \omega(t) + \sqrt{1 + \omega^2(t)}, \; t \in \mathfrak{A}. \quad (3)$$

Let us define the function $r \in \mathrm{P}$ as follows

$$r(t) = \frac{1 + \omega(t)}{1 - \omega(t)} = 1 + r_1 t + r_2 t^2 + r_3 t^3 + \cdots + r_n t^n + \cdots. = 1 + \sum_{n=1}^{\infty} r_n t^n, \; t \in \mathfrak{A}.$$

It follows from that

$$\omega(t) = \frac{r(t) - 1}{r(t) + 1} = \frac{1}{2}\left[r_1 t + \left(r_2 - \frac{r_1^2}{2} \right) t^2 + \left(r_3 - r_1 r_2 + \frac{r_1^2}{4} \right) t^3 + \cdots \right], \quad (4)$$
$$t \in \mathfrak{A}.$$

Changing the formulation of the function $\omega(t)$ in (3) with the formulation in (4), we get

$$1 + \frac{t f''(t)}{f'(t)} = 1 + \frac{r_1}{2} t + \left(\frac{r_2}{2} - \frac{r_1^2}{8} \right) t^2 + \left(\frac{r_3}{2} - \frac{r_1 r_2}{4} \right) t^3 + \cdots, \quad (5)$$
$$t \in \mathfrak{A}.$$

Then, by equalizing the coefficients of the terms of the same degree, are obtained the following equalities for a_2, a_3 and a_4

$$2a_2 = \frac{r_1}{2}, \; 6a_3 - 4a_2^2 = \frac{r_2}{2} - \frac{r_1^2}{8},$$

$$12a_4 - 18 a_2 a_3 + 8 a_2^3 = \frac{r_3}{2} - \frac{r_1 r_2}{4}.$$

From these equalities, we get

$$a_2 = \frac{r_1}{4}, \quad (6)$$

$$a_3 = \frac{2}{3} a_2^2 + \frac{1}{12}\left(r_2 - \frac{r_1^2}{4} \right), \quad (7)$$

$$a_4 = \frac{3}{2} a_2 a_3 - \frac{2}{3} a_2^3 + \frac{1}{24}\left(r_3 - \frac{r_1 r_2}{2} \right). \quad (8)$$

By applying the Lemma 2.2, from the equality (6), we obtain immediately the first result of the theorem.

Firstly using the Lemma 2.3 and then applying triangle inequality and Lemma 2.2 to the equality (7), we get

$$\left|a_3\right| \le \frac{1}{16}\tau^2 + \frac{4-\tau^2}{24}\xi,\ \xi \in (0,1)$$

with $\tau = |r_1|,\ \xi = |x| < 1$. From this, we can easily write

$$\left|a_3\right| \le \frac{1}{16}\tau^2 + \frac{4-\tau^2}{24},\ \tau \in [0,2];$$

so,

$$\left|a_3\right| \le \frac{\tau^2}{48} + \frac{1}{6},\ \tau \in [0,2].$$

By maximizing the right-hand side of the last inequality for the variable τ, we reach the second result of the theorem.

Now, let's find an upper bound estimate for the coefficient a_4. From the equalities (6)-(8), we get

$$a_4 = \frac{r_1}{32}\left(r_2 - \frac{r_1^2}{4}\right) + \frac{1}{24}\left(r_3 - \frac{r_1 r_2}{2} + \frac{r_1^3}{8}\right);$$

that is,

$$a_4 = \frac{r_1}{32}\left(r_2 - \frac{\lambda}{2}r_1^2\right) + \frac{1}{24}\left(r_3 - 2b\,r_1 r_2 + d\,r_1^3\right),$$

with $\lambda = \frac{1}{2},\ b = \frac{1}{4}$ and $d = \frac{1}{8}$.

Applying triangle equality to the last equality, we find

$$\left|a_4\right| \le \frac{|r_1|}{32}\left|r_2 - \frac{\lambda}{2}r_1^2\right| + \frac{1}{24}\left|r_3 - 2b\,r_1 r_2 + d r_1^3\right|. \tag{9}$$

Since $\lambda = \frac{1}{2} \in [0,2]$, $b = \frac{1}{4} \in [0,1]$, $d = \frac{1}{8}$ and $b(2b-1) \le d \le b$, then according to Lemma 2.2 and Lemma 2.4, we write the following inequalities

$$\left|r_2 - \frac{\lambda}{2}r_1^2\right| \le 2 \text{ and } |r_3 - 2br_1r_2 + dr_1^3| \le 2,$$

respectively. Considering these inequalities, from the inequality (9), we reach the desired estimate for the upper bound of $|a_4|$.

That is, the proof of Theorem 3.1 is done.

Now, we give the following theorem on the Fekete-Szegö problem for the class $C(\varphi)$.

Theorem 3.2. Assume that f given by (1) is in the class $C(\varphi)$ and $\mu \in \mathbb{C}$. Then,

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{12}\cdot\begin{cases} 2 & if\ \ |2-3\mu| \le 1, \\ |2-3\mu|+1 & if\ \ |2-3\mu| > 1. \end{cases}$$

Proof. Let $f \in C(\varphi)$ and $\mu \in \mathbb{C}$. Then, from the expressions for the coefficients a_2 and a_3, we write the following expression for $a_3 - \mu a_2^2$

$$a_3 - \mu a_2^2 = \frac{1}{3}(2-3\mu)a_2^2 + \frac{1}{12}\left(r_2 - \frac{r_1^2}{4}\right).$$

Considering equality (6) and applying Lemma 3.3, we write the following equality

$$a_3 - \mu a_2^2 = \frac{1}{48}\left[(2-3\mu)r_1^2 + r_1^2 + 2\left(4-r_1^2\right)x\right]$$

for some x with $|x| < 1$. From this, using triangle inequality we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{48}\left\{\left[|2-3\mu|+1\right]\tau^2 + 2\left(4-\tau^2\right)\xi\right\},$$
$$\xi \in (0,1)$$

with $\tau = |r_1|,\ \xi = |x|$. If we maximize the right-hand side of this inequality for the parameter ξ, we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{48}\left\{\left[|2-3\mu|-1\right]\tau^2 + 8\right\},\ \tau \in [0,2].$$

Since the function

$$\theta(\tau) = \left[|2-3\mu|-1\right]\tau^2 + 8,\ \tau \in [0,2]$$

is a decreasing function if $|2-3\mu| \le 1$ and an increasing function if $|2-3\mu| > 1$, from the last inequality we arrive at the result of the theorem.

Thus, the proof of the Theorem 3.2 is completed.

In the cases $\mu = 0$ and $\mu = 1$ respectively, from Theorem 3.2, we obtain the following results.

4. CONCLUSION

Corollary 4.1. Let $f \in C(\varphi)$, then $|a_3| \le \frac{1}{4}$.

Corollary 4.2. Let $f \in C(\varphi)$, then

$$\left|a_3 - a_2^2\right| \le \frac{1}{6}.$$

Remark 4.3. The result obtained in the Corollary 4.1 confirms the second inequality obtained in Theorem 3.1.

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