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LOCAL EXISTENCE AND REGULARITY OF SOLUTIONS FOR SOME SECOND ORDER DIFFERENTIAL EQUATION WITH INFINITE DELAY

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Abstract. In this work, we study the existence and regularity of solutions for some second order differential equations with infinite delay in Banach spaces. We suppose that the undelayed part admits a cosine operator in the sense given by Da Prato and Giusi, [G. Da Prato and E. Giusi, *Una caratterizzazione dei generatori di funzioni coseno astratte*, Bollettino dell'Unione Matematica Italiana, 22, 357-362, (1967)]. The delayed part is assumed to be locally Lipschitz. Firstly, we show the existence of the mild solutions. Results are obtained by using Schauder and Banach-Piccard fixed point theorems. We also prove that the mild solution continuously depends on initial data. Secondly, we give sufficient conditions ensuring the existence of strict solutions. Last section is devoted to an application.

1. Introduction

In this work, we prove the existence and regularity of solutions for the following second order differential equations with infinite delay

$$(1.1) \qquad \begin{cases} x''(t) = Ax(t) + f(t, x_t, x'_t) \text{ for } t \geq 0 \\ \\ x_0 = \varphi \in \mathcal{B}_\alpha \\ \\ x'_0 = \varphi' \in \mathcal{B}_\alpha, \end{cases}$$

where A is the infinitesimal generator of a strongly continuous cosine family of linear operators in X, $\mathcal{B}$ is a Banach space of C^1-functions mapping $]-\infty, 0]$ to X and complying with some assumptions that will be introduced later. For $0 < \alpha < 1$, A^α is the fractional α-power of A, this operator $((-A)^\alpha, D((-A)^\alpha))$ will be given later. We assume that f is defined on a subspace $\mathcal{B}_\alpha$ of $\mathcal{B}$ with values in X, where

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$\mathcal{B}_\alpha$ is defined by

$$\mathcal{B}_\alpha = \{\varphi \in \mathcal{B};\ \varphi(\theta) \in D(A^\alpha) \text{ for } \theta < 0 \text{ and } A^\alpha\varphi \in \mathcal{B}\},$$

the function $A^\alpha\varphi$ is defined by

$$(A^\alpha\varphi)(\theta) = A^\alpha(\varphi(\theta)),$$

$\mathcal{B}_\alpha$ is endowed with the following norm $\|h\|_{\mathcal{B}_\alpha} = |h|_{\mathcal{B}_\alpha} + |h'|_{\mathcal{B}_\alpha}$ for all $h \in \mathcal{B}$, with $|\varphi|_{\mathcal{B}_\alpha} = \sup\limits_{\theta\leq 0}|\varphi(\theta)|_\alpha$, the norm $|\ .\ |_\alpha$ will be specified later. For every $t \geq 0$, x_t denotes the history function of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t+\theta) \text{ for } \ \theta \leq 0. \tag{1.2}$$

This work is motivated by [15], where the authors studied the existence and regularity of solutions for some second order partial differential equations finite delay; they assumed that $f : \mathbb{R}^+ \times \mathcal{C} \times \mathcal{C} \to X$, where $\mathcal{C}$ is the space of continuous differentiable functions from $[-r, 0]$ to X endowed with the following norm $\|h\| = |h| + |h'|$ for all $h \in \mathcal{C}$. Here by introducing α-norm and infinite delay, we obtain results which are more general than those obtained in [15].
The cosine families of operators were first extensively studied by G. Da Prato [4] and M. Sova [11, 12]. The most fundamental and extensive work on cosine families is that of H.O Fattorini in [5, 6].

Some recent contributions which used the cosine families in study of the quantitative theory of differential equations and control theory are made. We can refer the readers to works of J. M. Jeong and al [9], F.S. Acharya [1], R. Ameziane Hassani and al [8], D.N. Pandey and al [10]. In 1991, J. Bochenek [3], used the theory of cosine families to investigate the existence of solutions for the following semilinear second order differential equation initial value problem

$$\begin{cases} x''(t) = Ax(t) + g(x(t), x'(t)) \text{ for } t \in [0, T] \\ \\ x(0) = x_0 \in X \quad x'(0) = x_1 \in X. \end{cases} \tag{1.3}$$

However equation (1.3) and most other problems using cosine families theory are studied without delayed arguments.
We notice that in usual life, one reacts with a certain delay of sanding, or takes into consideration the past events to make some decisions. It is important to take into account the time-delay to study many phenomena in industrial processes, economic and biological systems. Time-delay has a major influence on the stability and asymptotic behavior of such dynamic systems, it is unavoidable to include it in the mathematical description.
To take into account the delay, the authors in [15], proved the existence and regularity of solutions for equation (1.1) with finite delay. They gave sufficient conditions ensuring the existence of strict solutions.
The present work generalizes the results obtained in [15] by using the α-norm and the infinite delay.

This work is organized as follows:
In Section 2, we collect some background materials required throughout the paper. In Section 3, we prove the existence of solutions of equation (1.1). Section

4 is devoted to the study of regularity solutions. Last Section is devoted to an application.

2. **Preliminary results**

Definition 2.1. A one parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if
$i)$ $C(s+t)+C(s-t)=2C(s)C(t)$ for all $s,t\in\mathbb{R}$,
$ii)$ $C(0)=I$,
$iii)$ $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x\in X$.

If $(C(t))_{t\in\mathbb{R}}$ is a strongly continuous cosine family in X, then $S(t)$ defined by

$$S(t)x=\int_0^t C(s)xds \text{ for } x\in X,\ t\in\mathbb{R}. \tag{2.1}$$

is a one parameter family of operators in X.

Definition 2.2. The infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ is the operator $A:X\to X$ defined by

$$Ax=\frac{d^2C(t)x}{dt^2}|_{t=0}.$$

$D(A)=\{x\in X:$ C(t)x is a twice continuously differentiable function of $t\}$.

We shall also make use of the set (called the Kisynski's Space)

$$E=\{x:\ C(t)x \text{ is a once continuously differentiable function of } t\}.$$

Proposition 1. [14] Let $(C(t))_{t\in\mathbb{R}}$ be a strongly continuous cosine family in X with infinitesimal generator A. The following are true.
$i)$ $D(A)$ is dense in X and A is a closed operator in X
$ii)$ if $x\in X$ and $r,s\in\mathbb{R}$, then $z=\displaystyle\int_r^s S(u)xdu\in D(A)$ and $Az=C(s)x-C(r)x$,
$iii)$ If $x\in X$ and $r,s\in\mathbb{R}$, then $z=\displaystyle\int_0^s\int_0^r C(u)C(v)xdudv\in D(A)$ and
$Az=\frac{1}{2}(C(r+s)x-C(s-r)x)$
$iv)$ if $x\in X$, then $S(t)x\in E$,
$v)$ if $x\in X$, then $S(t)x\in D(A)$ and $\dfrac{dC(t)x}{dt}=AS(t)x$,
$vi)$ if $x\in D(A)$, then $C(t)x\in D(A)$ and $\dfrac{d^2C(t)x}{dt^2}=AC(t)x=C(t)Ax$,
$vii)$ if $x\in E$, then $\lim\limits_{t\to 0}AS(t)x=0$,
$viii)$ if $x\in E$, then $S(t)x\in D(A)$ and $\dfrac{d^2S(t)x}{dt^2}=AS(t)x$,
$ix)$ if $x\in D(A)$, then $S(t)x\in D(A)$ and $AS(t)x=S(t)Ax$,
$x)$ $C(t+s)-C(t-s)=2AS(t)S(s)$ for all $s,t\in\mathbb{R}$.

In [6] for $0\le\alpha\le 1$, the fractional powers $(-A)^\alpha$ exist as closed linear operators in X,
(2.2)

$$D((-A)^\beta)\subset D((-A)^\alpha) \text{ for } 0\le\alpha\le\beta\le 1, \text{ and } (-A)^\alpha(-A)^\beta=(-A)^{\alpha+\beta}.$$

Throughout this paper, we assume that

($\mathbf{H_0}$) A is the infinitesimal generator of a strongly continuous cosine family of linear operators on a Banach space X.

Using Proposition 1, ($\mathbf{H_0}$) implies that the operator A is densely defined in X, i.e. $\overline{D(A)} = X$. We have the following result.

Proposition 2. [14] Assume that ($\mathbf{H_0}$) holds. Then there are constants $M \geq 1$ and $\omega \geq 0$ in such a way that

$$\|C(t)\| \leq Me^{\omega|t|} \text{ and } \|S(t) - S(t')\| \leq M\Big|\int_{t'}^{t} e^{\omega|s|}ds\Big| \text{ and for } t, t' \in \mathbb{R}.$$

From previous inequality, since $S(0) = 0$ we can deduce

$$\|S(t)\| \leq \frac{M}{\omega}e^{\omega t} \text{ for } t \in \mathbb{R}^+.$$

In the sequel, let us pose $M_1 = \max\Big(M, \dfrac{M}{\omega}\Big)$.

Theorem 2.3. [3] *If $g : [0, T] \times X \times X \to X$ is continuous and x is a solution of equation* (1.3)*, then x is a solution of the integral equation*

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)g(s, x(s), x'(s))ds \text{ for } t \geq 0,$$

For $0 < \alpha \leq 1$ $(-A)^\alpha$ maps onto X and is one to one, so that $D((-A)^\alpha)$ endowed with the norm $|x|_\alpha = \|(-A)^\alpha x\|$ is a Banach space. We denote by X_α this Banach space. We further assume that $(-A)^{-1}$ is compact. We require the following lemmas.

Lemma 2.4. [13] *Assume ($\boldsymbol{H_0}$) holds. Then the following are true.*
(i) For $0 < \alpha < 1$, $A^{-\alpha}$ is compact if only if $(-A)^{-1}$ is compact.
(ii) For $0 < \alpha < 1$ and $t \in \mathbb{R}$, $(-A)^\alpha C(t) = C(t)(-A)^\alpha$ and $(-A)^\alpha S(t) = S(t)(-A)^\alpha$.

Recall from [6], $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^{+\infty} t^{\alpha-1}(tI - A)^{-1}dt.$$

Lemma 2.5. [13] *Assume that ($\boldsymbol{H_0}$) holds, let $v : \mathbb{R} \to X$ such that v is continuously differentiable and let $q(t) = \int_0^t S(t-s)v(s)ds$. Then*
(i) q is twice continuously differentiable and for $t \in \mathbb{R}$,

$$q(t) \in D(A), \;\; q'(t) = \int_0^t C(t-s)v(s)ds$$

and

$$q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t).$$

(ii) For $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, $(-A)^{\alpha-1}q'(t) \in E$.

Theorem 2.6. *(Heine Theorem). Let f be a continuous function on a compact set K, then f is uniformly continuous on K.*

Theorem 2.7. *(Schauder fixed point Theorem). Let X be a locally convex topological vector space, and let $K \subset X$ be a non-empty, compact, and convex set. Then given any continuous mapping $f : K \to K$ there exists $x \in K$ such that $f(x) = x$.*

Theorem 2.8. *(Arzelà-Ascoli Theorem). Let (X, d_X) and (Y, d_Y) be compact metric spaces, $\mathcal{C}(X, Y)$ be the set of continuous functions from X to Y and let $\mathcal{F}$ be a subset of $\mathcal{C}(X, Y)$. If $\mathcal{F}$ is closed and equicontinuous then it is compact.*

3. Local existence, global continuation and blowing up of solutions

Definition 3.1. A continuous function $x :]-\infty, +\infty[\to X_\alpha$ is a strict solution of equation (1.1) if the following conditions hold
(i) $x \in C^1([0, +\infty[; X_\alpha) \cap C^2((0, +\infty[; X_\alpha)$.
(ii) x satisfies equation (1.1) on $[0, +\infty[$.
(iii) $x(\theta) = \varphi(\theta)$ for $-\infty < \theta \le 0$.

Proposition 3. Assume that $(\mathbf{H_0})$ holds. Then the solution of equation (1.1) is

$$x(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s) f(s, x_s, x'_s) ds \text{ for } t \ge 0, \tag{3.1}$$

Proof. The proof follows from Theorem 2.3. We define the function g by $g(t, x(t), x'(t)) = f(t, x_t, x'_t)$ for $t \ge 0$. Consequently, the result follows. □

Remark:

- Such solutions x satisfying equation (3.1) are called mild solutions of equation (1.1).
- The mild solutions of equation (1.1) may not be twice continuously differentiable. That is why we distinguish between mild and strict solutions.

Definition 3.2. A continuous function $x :]-\infty, +\infty[\to X_\alpha$ is a mild solution of equation (1.1) if x satisfies the following equation

$$\begin{cases} x(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \displaystyle\int_0^t S(t-s) f(s, x_s, x'_s) ds \text{ for } t \ge 0 \\ x_0 = \varphi \in \mathcal{B}_\alpha. \\ x'_0 = \varphi' \in \mathcal{B}_\alpha. \end{cases}$$

In this work, we assume that the state space $(\mathcal{B}, |.|_{\mathcal{B}})$ is a normed linear space of C^1 functions mapping $]-\infty, 0]$ into X and satisfying the following fundamental assumptions (cf [7]).

$(\mathbf{A}_1)$ There exist a positive constant H and functions $K(.), M(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x :]-\infty, a] \to X$, $x_\rho \in \mathcal{B}$, and $x(.)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold

$(\mathbf{i})$ $x_t \in \mathcal{B}$,

(**ii**) $|x(t)| \le H|x_t|_{\mathcal{B}}$, which is equivalent to $|\varphi(0)| \le H|\varphi|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$

(**iii**) $|x_t|_{\mathcal{B}} \le K(t-\sigma) \sup_{\sigma \le s \le t} |x(s)| + M(t-\sigma)|u_\sigma|_{\mathcal{B}}$.

($\mathbf{A}_2$) For the function $x(.)$ in ($\mathbf{A}_1$), $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function for $t \in [\sigma, \sigma + a]$.

(**B**) The space $\mathcal{B}$ is a Banach space.

In the following, we give a local existence of mild solutions of equation (1.1). We hold the following assumptions.

($\mathbf{H_1}$) The function $f : [0,b] \times \mathcal{B}_\alpha \times \mathcal{B}_\alpha \to X$ satisfies the following conditions

i) $f : [0,b] \times \mathcal{B}_\alpha \times \mathcal{B}_\alpha \to X$ is continuously differentiable.

ii) There exists a continuous nondecreasing function $\beta : [0,b] \to \mathbb{R}^+$ such that

$$\|f(t,\varphi,\varphi')\| \le \beta(t)|\varphi|_\alpha \quad \text{for} \quad (t,\varphi) \in [0,b] \times \mathcal{B}_\alpha.$$

($\mathbf{H_2}$) A^{-1} is compact on X.

($\mathbf{H}_3$) $A^{-\alpha}\varphi \in \mathcal{B}$ for $\varphi \in \mathcal{B}$, where the function $A^{-\alpha}\varphi$ is defined by

$$(A^{-\alpha}\varphi)(\theta) = A^{-\alpha}(\varphi(\theta)) \quad \text{for} \quad \theta \le 0.$$

Lemma 3.3. [2] *Assume that* ($\boldsymbol{H}_0$) *and* ($\boldsymbol{H}_3$) *hold. Then* $\mathcal{B}_\alpha$ *is a Banach space.*

Theorem 3.4. *Assume that* ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$) *and* ($\boldsymbol{H}_3$) *hold. Let* $\varphi \in \mathcal{B}_\alpha$ *such that* $\varphi(0) \in D(A)$ *and* $\varphi'(0) \in E$ *and assume that*

$$\|(-A)^{\alpha-1}\| \sup_{t\in[0,b]} \Big[\beta(t)(2Me^{\omega b}+1) + Me^{\omega b}\Big]\Delta_b < 1,$$

where $\Delta_b := \max_{0\le t\le b} K(t)$. *Then equation* (1.1) *has at least one mild solution on* $[0,b]$.

Proof. Let $k > |\varphi|_{\mathcal{B}_\alpha}$ and Ω_k a set be defined by

$$\Omega_k = \{x \in C([0,b], X_\alpha) : \; x(0) = \varphi(0) \text{ and } |x|_\infty \le k\},$$

with $|x|_\infty = \sup_{t\in[0,b]} |x(t)|_\alpha$. For $x \in \Omega_k$, we define the function $\widetilde{x} : [0,b] \to X_\alpha$ by

$$\widetilde{x}(t) = \begin{cases} x(t) \text{ for } t \in [0,b] \\ \\ \varphi(t) \text{ for } t \in]-\infty, 0] \end{cases}$$

By virtue of assumptions ($\mathbf{A}_1 - \mathbf{i}$) and ($\mathbf{A}_2$), we deduce that the function $t \to \widetilde{x}_t$ is continuous from $[0,b]$ to $\mathcal{B}_\alpha$. Let $\mathcal{H}$ an operator on Ω_k be defined by

$$\mathcal{H}(x)(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)f(s,x_s,x'_s)ds \text{ for } t \in [0,b].$$

It is sufficient to show that $\mathcal{G}$ has a fixed point in Ω_k. We give the proof in several steps.

Step 1: There is a positive $k > |\varphi|_{\mathcal{B}_\alpha}$ such that $\mathcal{H}(\Omega_k) \subset \Omega_k$.

If not, then for each $k > |\varphi|_{\mathcal{B}_\alpha}$, there exist $x_k \in \Omega_k$ and $t_k \in [0, b]$ such that $|(\mathcal{H}x_k)(t_k)|_\alpha > k$. Then by Proposition 1 and ($\mathbf{A}_1 - \mathbf{iii}$), we have

$$\begin{aligned}
k &< |(\mathcal{H}x_k)(t_k)|_\alpha \leq |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \Big|\int_0^{t_k} S(t_k - s) f(s, \widetilde{x}_s, \widetilde{x'}_s) ds\Big|_\alpha \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \Big\| - (-A)^{\alpha-1} \int_0^{t_k} AS(t_k - s) f(s, \widetilde{x}_s, \widetilde{x'}_s) ds \Big\| \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \Big\|(-A)^{\alpha-1} \Big[\int_0^{t_k} \frac{d}{ds}\Big(C(t_k - s) f(s, \widetilde{x}_s, \widetilde{x'}_s)\Big) ds \\
&\quad - \int_0^{t_k} C(t_k - s) \frac{d}{ds}\Big(f(s, \widetilde{x}_s, \widetilde{x'}_s)\Big) ds \Big] \Big\| \\
&< |C(t_k)\varphi(0)|_\alpha + |S(t_k)\varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\| \Big(\beta(t_k) + Me^{\omega b} + 2Me^{\omega b}\beta(0)\Big) [\Delta_b |x|_\infty + M_b |\varphi|_{\mathcal{B}}].
\end{aligned}$$

where $\Delta_b := \max_{0 \leq t \leq b} K(t)$ and $\Lambda_b := \max_{0 \leq t \leq b} M(t)$. Since $|x|_\infty \leq k$, then we obtain

$$k < M_1 e^{\omega b}\Big(|\varphi(0)|_\alpha + |\varphi'(0)|_\alpha\Big) + \|(-A)^{\alpha-1}\| \sup_{t \in [0,b]} \Big[\beta(t)(2Me^{\omega b} + 1) + Me^{\omega b}\Big] [\Delta_b k + M_b |\varphi|_{\mathcal{B}}].$$

Consequently

$$1 < \frac{M_1 e^{\omega b}\Big(|\varphi(0)|_\alpha + |\varphi'(0)|_\alpha\Big)}{k} + \|(-A)^{\alpha-1}\| \sup_{t \in [0,b]} \Big[\beta(t)(2Me^{\omega b} + 1) + Me^{\omega b}\Big] \frac{[\Delta_b k + M_b |\varphi|_{\mathcal{B}}]}{k}.$$

It follows that when $k \to +\infty$ that

$$1 \leq \|(-A)^{\alpha-1}\| \sup_{t \in [0,b]} \Big[\beta(t)(2Me^{\omega b} + 1) + Me^{\omega b}\Big] \Delta_b,$$

which gives a contradiction.

Step 2: $\mathcal{H}$ is continuous

Let $(x^n)_n \in \Omega_k$ with $x^n \to x$ in Ω_k. Then, the set

$$\Delta = \{(s, \widetilde{x}^n_s, \widetilde{x'}^n_s),\ (s, \widetilde{x}_s, \widetilde{x'}_s) :\ s \in [0, b],\ n \geq 1\}$$

is compact in $[0,b]\times\mathcal{B}_\alpha\times\mathcal{B}_\alpha$. Heine Theorem (Theorem 2.6) implies that f is uniformly continuous in Δ and

$$\begin{aligned}
|\mathcal{H}(x^n)(t)-\mathcal{H}(x)(t)|_\infty &\leq \sup_{t\in[0,b]}\Big\|-(-A)^{\alpha-1}\int_0^t AS(t-s)(f(s,x^n_s,{x'}^n_s)-f(s,x_s,x'_s))ds\Big\| \\
&\leq \sup_{t\in[0,b]}\Big\|(-A)^{\alpha-1}\Big[\int_0^t \frac{d}{ds}\Big(C(t-s)(f(s,x^n_s,{x'}^n_s)-f(s,x_s,x'_s)\Big)ds \\
&\quad -\int_0^t C(t_k-s)\frac{d}{ds}\Big((f(s,x^n_s,{x'}^n_s)-f(s,x_s,x'_s)\Big)ds\Big]\Big\| \\
&\leq \|(-A)^{\alpha-1}\|\Big((1+M_1e^{\omega b})\|f(t,x^n_t,{x'}^n_t)-f(t,x_t,x'_t)\| \\
&\quad +2M_1e^{\omega b}\|f(0,x^n_0,{x'}^n_0)-f(0,x_0,x'_0)\|\Big)\to 0 \text{ as } n\to+\infty,
\end{aligned}$$

and this yield the continuity of $\mathcal{H}$ on Ω_k.

Step 3: The set $\{\mathcal{H}(x)(t):\ x\in\Omega_k\}$ is relatively compact for each $t\in[0,b]$.

Let $t\in]0,b]$ be fixed, using the same reasoning like in the **Step 1**, we have

$$\begin{aligned}
\|(-A)^\alpha\mathcal{H}(x))(t)\| &\leq \|(-A)^{\alpha-1}\|\Big[M_1e^{\omega b}\Big(\|A\varphi(0)\|+\|A\varphi'(0)\|\Big) \\
&\quad +\sup_{t\in[0,b]}\Big[\beta(t)(2Me^{\omega b}+1)+Me^{\omega b}\Big][\Delta_b k+M_b|\varphi|_{\mathcal{B}}]\Big].
\end{aligned}$$

Then for $t\in[0,b]$ fixed, the set $\{(-A)^\alpha\mathcal{H}(x)(t):\ x\in\Omega_k\}$ is bounded in X. By $(\mathbf{H}_2)$, we deduce that $(-A)^{-\alpha}: X\to X_\alpha$ is compact. It follows that the set $\{\mathcal{H}(x)(t):\ x\in\Omega_k\}$ is relatively compact for each $t\in[0,b]$ in X_α.

Step 4: The set $\{\mathcal{H}(x):\ x\in\Omega_k\}$ is an equicontinuous family of functions.

Let $x\in\Omega_k$ and $0\leq\tau_1<\tau_2\leq b$, then we have

$$\begin{aligned}
|\mathcal{H}(x)(\tau_2)-\mathcal{H}(x)(\tau_1)|_\alpha &\leq |[C(\tau_2)-C(\tau_1)]\varphi(0)|_\alpha+|[S(\tau_2)-S(\tau_1)]\varphi'(0)|_\alpha \\
&\quad +\Big|\int_0^{\tau_2} S(\tau_2-s)f(s,\widetilde{x}_s,\widetilde{x'}_s))ds-\int_0^{\tau_1} S(\tau_1-s)f(s,\widetilde{x}_s,\widetilde{x'}_s))ds\Big|_\alpha \\
&\leq |[C(\tau_2)-C(\tau_1)]\varphi(0)|_\alpha+|[S(\tau_2)-S(\tau_1)]\varphi'(0)|_\alpha \\
&\quad +\|(-A)^{\alpha-1}\|\Big(\|f(\tau_2,\widetilde{x}_{\tau_2},\widetilde{x'}_{\tau_2})-C(\tau_2-\tau_1)f(\tau_1,\widetilde{x}_{\tau_1},\widetilde{x'}_{\tau_1})\| \\
&\quad +M_1e^{\omega b}\|f(\tau_2,\widetilde{x}_{\tau_2},\widetilde{x'}_{\tau_2})-f(\tau_1,\widetilde{x}_{\tau_1},\widetilde{x'}_{\tau_1})\|+\|[C(\tau_2-\tau_1)-I]f(\tau_1,\widetilde{x}_{\tau_1},\widetilde{x'}_{\tau_1}\| \\
&\quad +\|[C(\tau_2)-C(\tau_1)]f(0,\widetilde{x}_0,\widetilde{x'}_0)\|+\|C(\tau_2)-C(\tau_1)\|\|f(\tau_1,\widetilde{x}_{\tau_1},\widetilde{x'}_{\tau_1})-f(0,\widetilde{x}_0,\widetilde{x'}_0)\|\Big)
\end{aligned}$$

Since $(-A)^{\alpha-1}$ is compact from X to X and $(C(t)_{t\in\mathbb{R}}$ is uniformly continuous on compact subsets of X, it follows that

$$|\mathcal{H}(x)(\tau_2) - \mathcal{H}(x)(\tau_1)|_\alpha \longrightarrow 0 \text{ if } \tau_1 \to \tau_2.$$

Thus, $\mathcal{H}$ maps Ω_k into an equicontinuity family of functions.
The equicontinuitie for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 < 0 < \tau_2$ are obvious.
So from **step 1** to **step 4** and Ascoli-Arzelà Theorem (Theorem 2.8), we can conclude that $\mathcal{H} : \Omega_k \to \Omega_k$ is completely continuous. Hence by Schauder fixed point Theorem (Theorem 2.7), $\mathcal{H}$ has at least one fixed point $\overline{x}$ in Ω_k which is a mild solutions of equation (1.1).□

In the following, we prove the uniqueness of local existence mild solutions of equation (1.1). In what follows, we assume that f is autonomous, that is, $f : \mathcal{B}_\alpha \times \mathcal{B}_\alpha \to X$. We make the following assumption.

($\mathbf{H}_4$) f is locally lipschitz, that is, for each $\sigma > 0$ there is a constant $k_0(\sigma) > 0$ such that if $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{B}_\alpha$ $|\varphi_1|_{\mathcal{B}_\alpha}, |\varphi_2|_{\mathcal{B}_\alpha}, |\psi_1|_{\mathcal{B}_\alpha}, |\psi_2| \leq \sigma$ then

$$\|f(\varphi_1,\varphi_2) - f(\psi_1,\psi_2)\| \leq k_0(\sigma)(|\varphi_1 - \psi_1|_{\mathcal{B}_\alpha} + |\varphi_2 - \psi_2|_{\mathcal{B}_\alpha}).$$

($\mathbf{H}_5$) The function $t \to AC(t)$ is locally bounded.

Theorem 3.5. *Assume that ($\boldsymbol{H_0}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H}_3$), ($\boldsymbol{H_4}$) and ($\boldsymbol{H_5}$) hold. Let $\varphi \in \mathcal{B}_\alpha$ such that $\varphi(0) \in D(A)$ and $\varphi'(0) \in E$. Then, equation (1.1) has one and only one solution $x(.,\varphi)$ defined on a maximal interval $]-\infty, b_\varphi[$ and*

$$b_\varphi = +\infty \text{ or } \overline{\lim}_{t\to b_\varphi^-}(|x(t,\varphi)|_\alpha + |x'(t,\varphi)|_\alpha) = +\infty$$

Proof. Let $\varphi \in \mathcal{B}$ with $\|\varphi\| < \sigma$, using ($\mathbf{H_4}$) we have

$$\|f(\varphi,\varphi')\| \leq k_0(\sigma)\|\varphi\|_{\mathcal{B}_\alpha} + \|f(0,0)\| \leq k_0(\sigma)\sigma + \|f(0,0)\|,$$

For a given $\varphi \in \mathcal{B}_\alpha$, let us pose $\sigma = \|\varphi\|_{\mathcal{B}_\alpha} + 1$ and $k_1(\sigma) = k_0(\sigma)\sigma + \|f(0,0)\|$. We define the function z by

$$z(t) = \begin{cases} C(t)\varphi(0) + S(t)\varphi'(0) \text{ for } t \geq 0 \\ \\ \varphi(t) \qquad \text{for } t \leq 0. \end{cases}$$

Assumptions ($\mathbf{A}_1 - \mathbf{i}$) and ($\mathbf{A}_2$) imply that $z_t \in \mathcal{B}_\alpha$ and the function $t \mapsto z_t$ is continuous. Consequently for $b_1 \in]0,1[$, we can find $b_2 \in]0,1[$ such that

$$\|z_t - \varphi\|_{\mathcal{B}_\alpha} \leq b_1 \text{ for } \ t \in [0, b_2].$$

For $0 \leq b \leq b_2$ be such that

$$\max\Big(\Delta_b\|(-A)^{\alpha-1}\|\mu_0 k_1(\sigma)b, \ \ \Delta_b\|(-A)^{\alpha-1}\|\mu_0 k_1(\sigma)b^2\Big) < \frac{1-b_1}{2}, \tag{3.2}$$

where $\Delta_b := \max\limits_{0\leq t\leq b} K(t)$ and μ_0 is a positive real number such that $\|AC(t)\| \leq \mu_0$ for all $t \in [0, b_1]$. For $x \in C([0,b]; X_\alpha)$, such that $x(0) = \varphi(0)$, we define its extension $\widetilde{x}$ on $]-\infty, b]$ by

$$\widetilde{x}(t) = \begin{cases} x(t) \text{ for } t \in [0,b] \\ \\ \varphi(t) \text{ for } t \in]-\infty, 0]. \end{cases}$$

Consider the following set

$$\Gamma_b = \left\{ \ x \in C^1(]-\infty,b];X_\alpha) : x(s)=\varphi(s),\ x'(s)=\varphi'(s) \text{ if } s\in]-\infty,0] \ \text{ and } \sup_{s\in[0,b]}(\|\widetilde{x}_s-\varphi\|_{\mathcal{B}_\alpha})\le 1 \ \right\}$$

and we define the $\mathcal{F}$ on Γ_b by

$$\begin{cases} \mathcal{F}(x)(t) = C(t)\varphi(0)+S(t)\varphi'(0)+\displaystyle\int_0^t S(t-s)f(x_s,x'_s)ds \text{ for } t\in[0,b] \\ \\ \mathcal{F}(x_0)(t)=\varphi(t) \ \text{ for } t\in]-\infty,0] \\ \\ (\mathcal{F}(x_0))'(t)=\varphi'(t) \ \text{ for } t\in]-\infty,0]. \end{cases}$$

We give the proof in several steps.

Step 1: $\mathcal{F}(\Gamma_b)\subset\Gamma_b$

Using $(\mathbf{H}_4)$ and assumption $(\mathbf{A}_1)$, for $x\in\Gamma_b$, $s\mapsto f(\widetilde{x}_s,\widetilde{x}'_s)$ is continuous function on $[0,b]$. Then

$$s\mapsto\int_0^t S(t-s)f(\widetilde{x}_s,\widetilde{x}'_s)ds$$

is continuous on $[0,b]$, consequently $v=\mathcal{F}x$ is continuous on $[0,b]$. It remains to show that $v\in\Gamma_b$. For $t\in[0,b]$, we have

$$\|\widetilde{v}_t-\varphi\|_{\mathcal{B}_\alpha}\le\|\widetilde{v}_t-z_t\|_{\mathcal{B}_\alpha}+\|z_t-\varphi\|_{\mathcal{B}_\alpha}\le\|\widetilde{v}_t-z_t\|_{\mathcal{B}_\alpha}+b_1.$$

By construction of the set Γ_b, we can see that $\|\widetilde{x}_s-\varphi\|_{\mathcal{B}_\alpha}\le 1$, for $s\in[0,b]$. It follows that $\|\widetilde{x}_s\|_{\mathcal{B}_\alpha}\le\sigma$ for $s\in[0,b]$. Then

$$\|f(\widetilde{x}_s,\widetilde{x}'_s)\|\le k_0(\sigma)\|\widetilde{x}_s\|_{\mathcal{B}_\alpha}+\|f(0,0)\|\le k_1(\sigma).$$

Let $t\in[0,b]$, by assumption $(\mathbf{A}_1-\mathbf{iii})$ implies

$$\|\widetilde{v}_t-z_t\|_{\mathcal{B}_\alpha}=|\widetilde{v}_t-z_t|_{\mathcal{B}_\alpha}+|\widetilde{v}'_t-z'_t|_{\mathcal{B}_\alpha}\le\Delta_b\sup_{0\le s\le b}(|v(s)-z(s)|_\alpha+|v'(s)-z'(s)|_\alpha)$$

and from equation (3.2), we have

$$\begin{aligned} |v(t)-z(t)|_\alpha &\le \Big\|-(-A)^{\alpha-1}\int_0^t\Big(\int_0^{t-s}AC(\sigma)f(x_s,x'_s)d\sigma\Big)ds\Big\| \\ &\le \|(-A)^{\alpha-1}\|\mu_0k_1(\sigma)b^2 \\ &< \frac{1-b_1}{2\Delta_b}. \end{aligned} \tag{3.3}$$

Using equation (2.1), Proposition 1 and equation (3.2), we have

$$v'(t)=S(t)A\varphi(0)+C(t)\varphi'(0)+\int_0^t C(t-s)f(\widetilde{x}_s,\widetilde{x}'_s)ds \text{ for } t\ge 0$$

and

$$z'(t)=\begin{cases} S(t)A\varphi(0)+C(t)\varphi'(0) \text{ for } t\ge0 \\ \\ \varphi'(t) \qquad \text{for } t\le 0. \end{cases}$$

It follows that

$$\begin{aligned}|v'(t)-z'(t)|_\alpha &\leq \Big\| -(-A)^{\alpha-1}\int_0^t AC(t-s)f(x_s,x_s')ds\Big\| \\ &\leq \|(-A)^{\alpha-1}\|\mu_0 k_1(\sigma)b \\ &< \frac{1-b_1}{2\Delta_b}. \qquad (3.4)\end{aligned}$$

Adding the two previous inequalities (3.3) and (3.4), we have $\|\widetilde{v}_t - z_t\|_{\mathcal{B}_\alpha} < 1-b$. It follows that $\|v_t-\varphi\|_{\mathcal{B}_\alpha} < 1$ for any $t\in[0,b]$, which implies that $v\in\Gamma_b$.

Step 2: $\mathcal{F}$ is a strict contraction in Γ_b

Let $x,w\in\Gamma_b$ and $t\in[0,b]$. Then we have

$$\begin{aligned}|\mathcal{F}(x)(t)-\mathcal{F}(w)(t)|_\alpha &= \Big|\int_0^t S(t-s)(f(x_s,x_s')-f(w_s,w_s'))ds\Big|_\alpha \\ &\leq \Big\| -(-A)^{\alpha-1}\int_0^t\Big(\int_0^{t-s} AC(\sigma)[f(x_s,x_s')-f(w_s,w_s')]d\sigma\Big)ds\Big\| \\ &\leq \|(-A)^{\alpha-1}\|\mu_0 b\int_0^t \|f(x_s,x_s')-f(w_s,w_s')\|ds \\ &\leq \|(-A)^{\alpha-1}\|\mu_0 b^2 k_0(\sigma)\Delta_b\|x-w\|_{\mathcal{B}_\alpha}.\end{aligned}$$

By equation (3.2), it follows that

$$|\mathcal{F}(x)(t)-\mathcal{F}(w)(t)|_\alpha < \frac{1}{2}\|x-w\||_{\mathcal{B}_\alpha}.$$

Using the same reasoning as previously, we have

$$\begin{aligned}|(\mathcal{F}(x))'(t)-(\mathcal{F}(w))'(t)|_\alpha &= \Big|\int_0^t C(t-s)(f(x_s,x_s')-f(w_s,w_s'))ds\Big|_\alpha \\ &< \frac{1}{2}\|x-w\||_{\mathcal{B}_\alpha}.\end{aligned}$$

Adding the two previous inequalities, we have

$$\|(\mathcal{F}(x))-(\mathcal{F}(v))\|_{\mathcal{B}_\alpha} < \|x-v\||_{\mathcal{B}_\alpha}.$$

Consequently $\mathcal{F}$ is a strict contraction in Γ_b and $\mathcal{F}$ has a unique fixed point x in Γ_b.

Let $\widehat{x}$ be another mild solution of equation (1.1) on Γ_b corresponding to φ. Then we have

$$\begin{aligned}\|x-\widehat{x}\|_{\mathcal{B}_\alpha} &= \|(\mathcal{F}(x))-(\mathcal{F}(\widehat{x}))\|_{\mathcal{B}_\alpha} \\ &< \|x-\widehat{x}\||_{\mathcal{B}_\alpha},\end{aligned}$$

which gives a contradiction. Consequently equation (1.1) has one and only one mild solution which is defined on $]-\infty,b]$ and denoted by $x(.,\varphi)$. By the same reasoning, $x(.,\varphi)$ can be extended to a maximal interval of existence $[0,b_\varphi[$

Step 3: The mild solution $x(.,\varphi)$ is uniformly continuous

Let $t,\ t+h \in [0,b_\varphi[,\ h>0$ and μ be a positive number such that $\|AC(t\| \leq \mu$ for all $t \in [0,b_\varphi[$. Then, we have

$$\begin{aligned}
x(t+h)-x(t) &= C(t+h)\varphi(0)-C(t)\varphi(0)+S(t+h)\varphi'(0)-S(t)\varphi'(0)+\int_0^{t+h} S(t+h-s)f(x_s,x'_s)ds \\
&\quad -\int_0^t S(t-s)f(x_s,x'_s)ds \\
&= C(t+h)\varphi(0)-C(t)\varphi(0)+S(t+h)\varphi'(0)-S(t)\varphi'(0) \\
&\quad +\int_0^t S(s)\Big[f(x_{t+h-s},x'_{t+h-s})-f(x_{t-s},x'_{t-s})\Big]ds \\
&\quad +\int_t^{t+h} S(s)f(x_{t+h-s},x'_{t+h-s})ds.
\end{aligned}$$

Using $(\mathbf{H}_4)$, we have

$$\begin{aligned}
|x(t+h,\varphi)-x(t,\varphi)|_\alpha &\leq |C(t+h)\varphi(0)-C(t)\varphi(0)|_\alpha+|S(t+h)\varphi'(0)-S(t)\varphi'(0)|_\alpha+\|(-A)^{\alpha-1}\|\mu b_\varphi k_1(\sigma)h \\
&\quad +\|(-A)^{\alpha-1}\|\mu\Lambda_{b_\varphi}b_\varphi^2\sup_{-r\leq\theta\leq 0}(|x_h(\theta)-\varphi(\theta)|_\alpha+|x'_h(\theta)-\varphi'(\theta)|_\alpha) \\
&\quad +\|(-A)^{\alpha-1}\|\mu\Delta_{b_\varphi}b_\varphi\int_0^t\sup_{0\leq\sigma\leq s}(|x_{s+h}-x_s|_\alpha+|x'_{s+h}-x'_s|_\alpha)ds,
\end{aligned}$$

where

$$\Delta_{b_\varphi}=\max_{0\leq t\leq b_\varphi}K(t) \ \text{ and } \ \Lambda_{b_\varphi}=\sup_{0\leq t\leq b_\varphi}M(t).$$

Consequently

$$\begin{aligned}
|x(t+h,\varphi)-x(t,\varphi)|_\alpha &\leq \sup_{t \text{ such that } t+h\in[0,b_\varphi[}\Big(|C(t+h)\varphi(0)-C(t)\varphi(0)|_\alpha+|S(t+h)\varphi'(0)-S(t)\varphi'(0)|_\alpha\Big) \\
&\quad +\|(-A)^{\alpha-1}\|\mu b_\varphi k_1(\sigma)h+\|(-A)^{\alpha-1}\|\mu\Lambda_{b_\varphi}b_\varphi^2\|x_h(.,\varphi)-\varphi\|_{\mathcal{B}_\alpha} \\
&\quad +\|(-A)^{\alpha-1}\|\mu\Delta_{b_\varphi}b_\varphi\int_0^t\sup_{0\leq\sigma\leq s}\|x_{\sigma+h}(.,\varphi)-x_\rho(.,\varphi)\|_{\mathcal{B}_\alpha}ds.
\end{aligned}$$

From [14] (see Proposition 2.4), $t\to C(t)\varphi(0)+S(t)\varphi'(0)$ belongs to $C^2([0,b_\varphi];X)$, by a similar reasoning, we also have

$$\begin{aligned}
\|x'_{t+h}(.)-x'_t(.)\|_\alpha &\leq \sup_{t \text{ such that } t+h\in[0,b_\varphi[}\Big(|AS(t+h)\varphi(0)-AS(t)\varphi(0)|_\alpha+|C(t+h)\varphi'(0)-C(t)\varphi'(0)|_\alpha\Big) \\
&\quad +\|(-A)^{\alpha-1}\|\mu k_1(\sigma)h+\|(-A)^{\alpha-1}\|\mu\Lambda_{b_\varphi}b_\varphi\|x_h(.,\varphi)-\varphi\|_{\mathcal{B}_\alpha} \\
&\quad +\|(-A)^{\alpha-1}\|\mu\Delta_{b_\varphi}\int_0^t\sup_{0\leq\sigma\leq s}\|x_{\sigma+h}(.,\varphi)-x_\rho(.,\varphi)\|_{\mathcal{B}_\alpha}ds.
\end{aligned}$$

Adding the previous two inequalities, we have

$$\|x_{t+h}(.)-x_t(.)\|_{\mathcal{B}_\alpha} \quad \leq \quad \gamma(h)+\|(-A)^{\alpha-1}\|\mu c_0(r)\Delta_{b_\varphi}(1+b_\varphi)\int_0^t \|x_{s+h}-x_s\|_{\mathcal{B}_\alpha}ds,$$

with

$$\begin{aligned}\gamma(h) \quad = \quad & \sup_{t \text{ such that } t+h\in[0,b_\varphi[} \Big(|C(t+h)\varphi(0)-C(t)\varphi(0)|_\alpha+|S(t+h)\varphi'(0)-S(t)\varphi'(0)|_\alpha\Big) \\ & + \sup_{t \text{ such that } t+h\in[0,b_\varphi[} \Big(|AS(t+h)\varphi(0)-AS(t)\varphi(0)|_\alpha+|C(t+h)\varphi'(0)-C(t)\varphi'(0)|_\alpha\Big) \\ & +\|(-A)^{\alpha-1}\|\mu k_1(\sigma)(1+b_\varphi)h+\|(-A)^{\alpha-1}\|\mu\Lambda_{b_\varphi}b_\varphi\|x_h(.,\varphi)-\varphi\|_{\mathcal{B}_\alpha}(1+b_\varphi).\end{aligned}$$

Since $t\to C(t)\varphi(0)+S(t)\varphi'(0)$ belongs to $C^2([0,b_\varphi];X)$, then

$$\sup_{t \text{ such that } t+h\in[0,b_\varphi[} \Big(|C(t+h)\varphi(0)-C(t)\varphi(0)|_\alpha+|S(t+h)\varphi'(0)-S(t)\varphi'(0)|_\alpha\Big)\to 0 \text{ as } h\to 0$$

$$\sup_{t \text{ such that } t+h\in[0,b_\varphi[} \Big(|AS(t+h)\varphi(0)-AS(t)\varphi(0)|_\alpha+|C(t+h)\varphi'(0)-C(t)\varphi'(0)|_\alpha\Big)\to 0 \text{ as } h\to 0.$$

From Gronwall's lemma, it follows that

$$\|x_{t+h}(.)-x_t(.)\|_{\mathcal{B}_\alpha}\to 0 \text{ as } h\to 0$$

uniformly for $t\geq 0$ such that $t+h\in[0,b_\varphi[$. Using the same reasoning, one can show a similar result for $h<0$. Consequently $x(.,\varphi)$ is uniformly continuous.

Step 4: $b_\varphi=+\infty$ or $\overline{\lim}_{t\to b_\varphi^-}(|x(t,\varphi)|_\alpha+|x'(t,\varphi)|_\alpha)=+\infty$

Suppose that $b_\varphi<+\infty$ and $\overline{\lim}_{t\to b_\varphi^-}(|x(t,\varphi)|_\alpha+|x'(t,\varphi)|_\alpha)<+\infty$, then there exists a constant $\sigma>0$ such that $(|x(t,\varphi)|_\alpha+|x'(t,\varphi)|_\alpha)\leq\sigma$ for all $t\in[0,b_\varphi[$. Since $x(.,\varphi)$ is uniformly continuous, then

$$\lim_{t\to b_\varphi^-}(x(t,\varphi)+x'(t,\varphi)) \text{ exists,}$$

which contradicts the maximality of $[0,b_\varphi[$ and it is extended to $[0,b_\varphi+\eta]$ for some $\eta>0$, which contradicts the maximality of $[0,b_\varphi[$.□

Theorem 3.6. *Assume that* $(\boldsymbol{H_0})$, $(\boldsymbol{H_2})$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H_4})$ *and* $(\boldsymbol{H_5})$ *hold. Let* $\varphi\in\mathcal{B}_\alpha$ *such that* $\varphi(0)\in D(A)$ *and* $\varphi'(0)\in E$. *Then, the mild solution* $x(.,\varphi)$ *continuously depends on initial data in the sense that if* $\varphi\in\mathcal{B}_\alpha$ *and* $t\in[0,b_\varphi[$, *then there exist positive constants* τ *and* ε *such that, for* $\varphi,\xi\in\mathcal{B}_\alpha$ *and* $\|\varphi-\xi\|_{\mathcal{B}_\alpha}<\varepsilon$, *we have*

$t\in[0,b_\xi[$ *and* $|x(s,\varphi)-x(s,\xi)|_\alpha+|x'(s,\varphi)-x'(s,\xi)|_\alpha\leq\tau\|\varphi-\xi\|_{\mathcal{B}_\alpha}$ *for all* $s\in]-\infty,t]$.

Proof. Let $\varphi\in\mathcal{B}_\alpha$ and $t\in[0,b_\varphi[$ be fixed. Set

$$\sigma=1+\sup_{-r\leq s\leq t}\|x_s(.,\varphi)\|_{\mathcal{B}_\alpha}$$

and

$$\lambda(t)=\Big(\Lambda_{b_0}+2M_1e^{\omega t}+\mu t\Big)\exp\Big(\Delta_{b_0}\|(-A)^{\alpha-1}\|\mu k_0(\sigma)t^2(1+t)\Big).$$

Let $\varepsilon\in]0,1[$ and $\xi\in\mathcal{B}_\alpha$ such that $\|\varphi-\xi\|_{\mathcal{B}_\alpha}<\varepsilon$. Then

$$\|\xi\|_{\mathcal{B}_\alpha}\leq\|\varphi\|_{\mathcal{B}_\alpha}+\varepsilon<\sigma.$$

Set

$$b_0 := \sup\{s > 0 : \|x_\rho(.,\xi)\|_{\mathcal{B}_\alpha} \leq \sigma \text{ for } \ \rho \in [0,s]\}.$$

We can see that $b_0 \geq t$. In fact if we suppose that $b_0 < t$, then for $s \in [0, b_0]$, we have

$$\begin{aligned}
|x_s(.,\varphi) - x_s(.,\xi)|_{\mathcal{B}_\alpha} &\leq \Lambda_{b_0}|\varphi - \xi|_{\mathcal{B}_\alpha} + \Delta_{b_0} \sup_{0\leq\gamma\leq s} |x(\gamma,\varphi) - x(\gamma,\xi)|_\alpha \\
&\leq \Lambda_{b_0}|\varphi - \xi|_{\mathcal{B}_\alpha} + \Delta_{b_0} \sup_{0\leq\gamma\leq s} \Big\{ M_1 e^{\omega\gamma}\Big(|\varphi(0) - \xi(0)|_\alpha + |\varphi'(0) - \xi'(0)|_\alpha\Big) \\
&\quad + \|(-A)^{\alpha-1}\|\mu\gamma k_0(\sigma) \int_0^\gamma \|x_\rho(.,\varphi) - x_\rho(.,\xi)\|_{\mathcal{B}_\alpha} d\rho\Big\}.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
|x'_s(.,\varphi) - x'_s(.,\xi)|_{\mathcal{B}_\alpha} &\leq \Lambda_{b_0}|\varphi' - \xi'|_{\mathcal{B}_\alpha} + \Delta_{b_0} \sup_{0\leq\gamma\leq s} \Big\{|AS(\xi)[\varphi(0) - \xi(0)|_\alpha + |C(\xi)[\varphi'(0) - \xi'(0)|_\alpha \\
&\quad + \|(-A)^{\alpha-1}\|\mu k_0(\sigma) \int_0^\gamma \|x_\rho(.,\varphi) - x_\rho(.,\xi)\|_{\mathcal{B}_\alpha} d\rho\Big\} \\
&\leq \Lambda_{b_0}|\varphi' - \xi'|_{\mathcal{B}_\alpha} + \Delta_{b_0} \sup_{0\leq\gamma\leq s} \Big\{\Big| \int_0^\gamma AC(\rho)[\varphi(0) - \xi(0)] d\rho\Big|_\alpha + |C(\gamma)[\varphi'(0) - \xi'(0)|_\alpha \\
&\quad + \|(-A)^{\alpha-1}\|\mu k_0(\sigma) \int_0^\gamma \|x_\rho(.,\varphi) - x_\rho(.,\xi)\|_{\mathcal{B}_\alpha} d\rho\Big\} \\
&\leq \Lambda_{b_0}|\varphi' - \xi'|_{\mathcal{B}_\alpha} + \Delta_{b_0} \sup_{0\leq\gamma\leq s} \Big\{\mu\gamma|\varphi(0) - \xi(0)]|_\alpha + M_1 e^{\omega\gamma}|\varphi'(0) - \xi'(0)|_\alpha \\
&\quad + \|(-A)^{\alpha-1}\|\mu k_0(\sigma) \int_0^\gamma \|x_\rho(.,\varphi) - x_\rho(.,\xi)\|_{\mathcal{B}_\alpha} d\rho\Big\}.
\end{aligned}$$

By adding the previous inequality, we have

$$\begin{aligned}
\|x_s(.,\varphi) - x_s(.,\xi)\|_{\mathcal{B}_\alpha} &\leq \Lambda_{b_0}\|\varphi - \xi\|_{\mathcal{B}_\alpha} + \Delta_{b_0} \sup_{0\leq\gamma\leq s} \Big\{(2M_1 e^{\omega\gamma} + \mu\xi)\|\varphi - \xi\|_{\mathcal{B}_\alpha} \\
&\quad + \|(-A)^{\alpha-1}\|\mu\xi k_0(\sigma)(1+\gamma) \int_0^\gamma \|x_\rho(.,\varphi) - x_\rho(.,\xi)\|_{\mathcal{B}_\alpha} d\rho\Big\} \\
&\leq (\Lambda_{b_0} + 2M_1 e^{\omega t} + \mu t)\|\varphi - \xi\|_{\mathcal{B}_\alpha} \\
&\quad + \Delta_{b_0}\|(-A)^{\alpha-1}\|\mu k_0(\sigma) t(1+t) \int_0^s \|x_\rho(.,\varphi) - x_\rho(.,\xi)\|_{\mathcal{B}_\alpha} d\rho.
\end{aligned}$$

By Gronwall's lemma, we deduce that

$$\|x_s(.,\varphi) - x_s(.,\xi)\|_{\mathcal{B}_\alpha} \leq \lambda(t)\|\varphi - \xi\|_{\mathcal{B}_\alpha}. \tag{3.5}$$

This implies that

$$\|x_s(.,\xi)\|_{\mathcal{B}_\alpha} \leq \lambda(t)\varepsilon + \sigma - 1 < \sigma \text{ for all } s \in [0, b_0].$$

Consequently b_0 cannot be a number $s > 0$ so large that $\|x_s(.,\xi)\|_{\mathcal{B}_\alpha} < \sigma$, for $\rho \in [0,s]$; it follows that $b_0 \geq t$ and $t < b_\gamma$. Since, $\|x_s(.,\varphi)\|_{\mathcal{B}_\alpha} < \sigma$ for $s \in [0,t]$, then from inequality (3.5), we have the desired result.□

Proposition 4. Assume that $(\mathbf{H_0})$, $(\mathbf{H_2})$, $(\mathbf{H}_3)$, $(\mathbf{H_4})$ and $(\mathbf{H_5})$ hold. Let $\varphi \in \mathcal{B}_\alpha$ such that $\varphi(0) \in D(A)$ and $\varphi'(0) \in E$. Let k_1 be a continuous function on $\mathbb{R}^+$ and $k_2 \in L^1_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ be such that $\|f(\varphi,\varphi')\| \leq k_1(t)\|\varphi\|_{\mathcal{B}_\alpha} + k_2(t)$ for $t \geq 0$ and $\varphi, \varphi' \in \mathcal{B}_\alpha$. Then equation (1.1) has a unique mild solution which is defined for all $t \geq 0$.

Proof. Let $]-\infty, b_\varphi[$ denote the maximal interval of existence of the mild solution $x(t,\varphi)$ of equation (1.1). Then

$$b_\varphi = +\infty \text{ or } \overline{\lim}_{t\to b_\varphi^-}(|x(t,\varphi)|_\alpha + |x'(t,\varphi)|_\alpha) = +\infty.$$

If $b_\varphi < +\infty$, then $\overline{\lim}_{t\to b_\varphi^-}(|x(t,\varphi)|_\alpha + |x'(t,\varphi)|_\alpha) = +\infty$. Consequently, assumption $(\mathbf{A}_1 - \mathbf{iii})$ implies that

$$\begin{aligned}
|x(t,\varphi)|_\alpha &\leq |C(t)\varphi(0)|_\alpha + |S(t)\varphi'(0)|_\alpha + \Big|\int_0^t S(t-s)f(x_s,x'_s)ds\Big|_\alpha \\
&\leq M_1e^{\omega b_\varphi}(|\varphi(0)|_\alpha + |\varphi'(0)|_\alpha) + \Big\|(-A)^{\alpha-1}\int_0^s \Big(\int_0^{t-s} AC(\xi)f(x_s(.,\varphi),x'_s(.,\varphi))d\xi\Big)ds\Big\| \\
&\leq M_1e^{\omega b_\varphi}(|\varphi(0)|_\alpha + |\varphi'(0)|_\alpha) + \|(-A)^{\alpha-1}\|\mu b_\varphi\int_0^t (k_1(s)\|x_s\|_{\mathcal{B}_\alpha} + k_2(s))ds \\
&\leq k_0 + \|(-A)^{\alpha-1}\|\mu b_\varphi\int_0^t k_1(s)\sup_{0\leq\xi\leq s}(|x_s(\xi,\varphi)|_\alpha + |x'_s(\xi,\varphi)|_\alpha)ds \text{ for } t\in[0,b_\varphi[,
\end{aligned}$$

where

$$\begin{aligned}
k_0 &= (2M_1e^{\omega b_\varphi} + \mu b_\varphi)(|\varphi(0)|_\alpha + |\varphi'(0)|_\alpha) + \Lambda_{b_\varphi}|\varphi|_{\mathcal{B}_\alpha}b_\varphi(1+b_\varphi)\sup_{0\leq s\leq b_\varphi}k_1(s) \\
&\quad + \|(-A)^{\alpha-1}\|\mu(b_\varphi+1)\int_0^{b_\varphi}k_2(s)ds.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|x'(t,\varphi)|_\alpha &\leq \mu b_\varphi|\varphi'(0)|_\alpha + M_1e^{\omega b_\varphi}|\varphi'(0)|_\alpha + \|(-A)^{\alpha-1}\|\mu\int_0^t \|f(x_s,x'_s)\|ds \\
&\leq k_0 + \|(-A)^{\alpha-1}\|\mu\int_0^t k_1(s)\sup_{0\leq\xi\leq s}(|x_s(\xi,\varphi)|_\alpha + |x'_s(\xi,\varphi)|_\alpha)ds \text{ for } t\in[0,b_\varphi[.
\end{aligned}$$

By Gronwall's lemma, we deduce that

$$\|x_t(\varphi)\|_{\mathcal{B}_\alpha} \leq 2k_0\exp\Big(\|(-A)^{\alpha-1}\mu(b_\varphi+1)\int_0^t k_1(s)\Big)ds < +\infty \text{ for } t\in[0,b_\varphi[,$$

and

$$\overline{\lim}_{t\to b_\varphi^-}(|x(t,\varphi)|_\alpha + |x'(t,\varphi)|_\alpha) < +\infty,$$

which gives a contradiction.□

As an immediat consequence, we get the following result.

Proposition 5. Assume that $(\mathbf{H_0})$, $(\mathbf{H_2})$, $(\mathbf{H}_3)$ and $(\mathbf{H_5})$ hold and there exists a positive constant L such that for $\varphi_1, \varphi_2 \in \mathcal{B}_\alpha$

$$\|f(\varphi_1, \varphi_1') - f(\varphi_2, \varphi_2')\| \leq L\|\varphi_1 - \varphi_2\|_{\mathcal{B}_\alpha} \text{ for } t \geq 0.$$

Let $\varphi \in \mathcal{B}_\alpha$ such that $\varphi(0) \in D(A)$ and $\varphi'(0) \in E$. Then equation (1.1) has a unique mild solution which is defined for all $t \geq 0$.

4. **Existence of strict solutions**

Theorem 4.1. *Assume that $(\boldsymbol{H_0})$, $(\boldsymbol{H_2})$, $(\boldsymbol{H}_3)$ and $(\boldsymbol{H_4})$ hold and f is continuously differentiable and f' is locally lipschitz continuous. Let φ be in $C^3(]-\infty, 0], D((-A)^\alpha)$ such that $\varphi(0), \varphi''(0) \in D(A)$, $\varphi'(0), \varphi^{(3)}(0) \in E$, $\varphi''(0) = A\varphi(0)+ f(\varphi, \varphi')$ and $\varphi^{(3)}(0) = A\varphi'(0)$. Then the corresponding mild solution x becomes a strict solution of equation* (1.1)*.*

Proof. Let φ be in $C^3(]-\infty, 0], X)$ such that $\varphi(0), \varphi''(0) \in D(A)$, $\varphi'(0), \varphi^{(3)}(0) \in E$, $\varphi''(0) = A\varphi(0) + f(\varphi, \varphi')$ and $\varphi^{(3)}(0) = A\varphi'(0)$. Let x be the corresponding mild solution of equation (1.1) on $[0, b_\varphi[$ and let $a < b_\varphi$. By a same reasoning as previously (Theorem 3.6), there exists a unique continuous function η such that

$$\eta(t) = \begin{cases} C(t)(A\varphi(0) + f(\varphi, \varphi')) + S(t)A\varphi'(0) + \displaystyle\int_0^t C(t-s)f'(x_s, x_s')v_s ds \\ \\ v_0 = \varphi''. \end{cases}$$

Let χ a function be defined by

$$\begin{cases} \chi(t) = \varphi'(0) + \displaystyle\int_0^t \eta(s)ds \text{ if } t \geq 0 \\ \\ \chi(t) = \varphi'(t) \text{ if } -r \leq t \leq 0 \\ \\ \chi'(t) = \varphi''(t) \text{ if } -r \leq t \leq 0. \end{cases}$$

We want to prove that $\chi = x'$. We can observe that

$$\chi_t = \varphi' + \int_0^t \eta_s ds \ \text{ for } t \in [0, a].$$

Hence the $t \to \chi_t$ and $t \to \displaystyle\int_0^t C(t-s)f(x_s, \chi_s)ds$ are continuously differentiable and we obtain

$$\begin{aligned} \frac{d}{dt}\int_0^t C(t-s)f(x_s, \chi_s)ds &= \frac{d}{dt}\int_0^t C(s)f(x_{t-s}, \chi_{t-s})ds \\ &= C(t)f(\varphi, \varphi') + \int_0^t C(t-s)f'(x_s, \chi_s)\eta_s ds, \end{aligned}$$

which implies

$$\int_0^t C(s)f(\varphi, \varphi')ds = \int_0^t C(t-s)f(x_s, \chi_s)ds - \int_0^t \int_0^s C(s-\tau)f'(x_\tau, \chi_\tau)\eta_\tau d\tau ds.$$

Consequently we have

$$\begin{aligned}\chi(t) &= \varphi'(0)+\int_0^t C(s)A\varphi(0)\,ds+\int_0^t C(t-s)f(x_s,\chi_s)ds+\int_0^t S(s)A\varphi'(0)\,ds\\ &\quad -\int_0^t\int_0^s C(s-\tau)f'(x_\tau,\chi_\tau)\eta_\tau d\tau ds+\int_0^t\int_0^s C(s-\tau)f'(x_\tau,x'_\tau)\eta_\tau d\tau ds.\end{aligned}$$

By equation (2.1) and Proposition 1, we have

$$\begin{aligned}\int_0^t C(s)A\varphi(0)ds &= S(t)A\varphi(0)\\ \int_0^t S(s)A\varphi'(0)\,ds &= C(t)\varphi'(0)-\varphi'(0),\end{aligned}$$

it follows that

$$\chi(t) = S(t)A\varphi(0)+\int_0^t C(t-s)f(x_s,\chi_s)ds+C(t)\varphi'(0)+\int_0^t\int_0^s C(s-\tau)\Big[f'(x_\tau,x_\tau)\eta_\tau-f'(x_\tau,\chi_\tau)\eta_\tau\Big]d\tau ds.$$

Since for $t\geq 0$, we have

$$x'(t)=AS(t)\varphi(0)+C(t)\varphi'(0)+\int_0^t C(t-s)f(x_s,x'_s)ds,$$

then we have

$$\begin{aligned}|x'(t)-\chi(t)|_\alpha &\leq \Big|\int_0^t S(t-s)[f(x_s,x'_s)-f(x_s,\chi_s)]ds\Big|_\alpha\\ &\quad+\Big|\int_0^t\int_0^s C(s-\tau)[f'(x_\tau,x_\tau)\eta_\tau-f'(x_\tau,\chi_\tau)\eta_\tau]d\tau ds\Big|_\alpha,\end{aligned}$$

consequently
(4.1)

$$|x'(t)-\chi(t)|_\alpha\leq\|A^{\alpha-1}\|\mu\Big(b_1\int_0^t\|f(x_s,x'_s)-f(x_s,\chi_s)\|ds+\int_0^t\int_0^s\|f'(x_\tau,x'_\tau)\eta_\tau-f'(x_\tau,\chi_\tau)\eta_\tau\|d\tau ds\Big).$$

Let

$$\sigma=\max(\sup_{0\leq s\leq b_1}\|x'_s\|_{\mathcal{B}_\alpha},\sup_{0\leq s\leq b_1}\|v_s\|_{\mathcal{B}_\alpha},\sup_{0\leq s\leq b_1}\|\chi_s\|_{\mathcal{B}_\alpha}).$$

There exists $k_0(\sigma)$ such that if $\varphi_1,\varphi_2,\psi_1,\psi_2\in\mathcal{B}_\alpha$ $|\varphi_1|_{\mathcal{B}_\alpha},|\varphi_2|_{\mathcal{B}_\alpha},|\psi_1|_{\mathcal{B}_\alpha},|\psi_2|\leq\sigma$, then

$$\begin{aligned}\|f(\varphi_1,\varphi_2)-f(\psi_1,\psi_2)\| &\leq k_0(\sigma)(|\varphi_1-\psi_1|_{\mathcal{B}_\alpha}+|\varphi_2-\psi_2|_{\mathcal{B}_\alpha}),\\ \|f'(\varphi_1,\varphi_2)-f'(\psi_1,\psi_2)\| &\leq k_0(\sigma)(|\varphi_1-\psi_1|_{\mathcal{B}_\alpha}+|\varphi_2-\psi_2|_{\mathcal{B}_\alpha}).\end{aligned}$$

It follows that for some positive constant $\widetilde{c}(b_1)$, we have

$$\sup_{0\leq s\leq t}|x'(s)-w(s)|_\alpha \leq \widetilde{c}(b_1)\int_0^t|x'_s-\chi_s|_{\mathcal{B}_\alpha}ds.$$

It follows that

$$|x'_t-\chi_t|_{\mathcal{B}_\alpha} \leq \widetilde{c}(b_1)\int_0^t|x'_s-\chi_s|_{\mathcal{B}_\alpha}ds\ \text{ for }\ t\in[0,b_1].$$

Choose b_1 such that

$$\widetilde{c}(b_1)\int_0^{b_1}|x'_s-\chi_s|_{\mathcal{B}_\alpha}ds<1,$$

then we deduce that, $x'_t=\chi_t$ for any $t\in[0,b_1]$. By assumption $(\mathbf{A}_1-\mathbf{ii})$, we conclude that $x(t)=w(t)$ for any $t\in]-\infty,b_1]$. Consequently, the mild solution x is twice continuously differentiable, thus x is a strict solution of equation (1.1).□

5. Application

For illustration, we consider for the following system

(5.1)
$$\begin{cases}\dfrac{\partial^2 z(t,\xi)}{\partial t^2}=\dfrac{\partial^2 z(t,x)}{\partial x^2}+g\Big(\dfrac{\partial}{\partial x}[z(t+\theta,x)],\dfrac{\partial}{\partial x}[z'(t+\theta,x)]\Big) \text{ for } t\geq 0 \text{ and } x\in[0,\pi]\\[2ex] z(t,0)=z(t,\pi)=0 \text{ for } t\geq 0\\[2ex] z(\theta,x)=\varphi_0(\theta,x) \text{ for } \theta\in]-\infty,0] \text{ and } x\in[0,\pi],\end{cases}$$

where $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a positive constant L such that for $x,y,x_1,y_1\in\mathbb{R}$

$$|g(x,y)-g(x_1,y_1)|\leq L(|x-x_1|+|y-y_1|).$$

For example, we can take $g(x,y)=\Big[\sin\Big(\dfrac{x}{2}\Big)+\sin\Big(\dfrac{y}{2}\Big)\Big]$ for $(x,y)\in\mathbb{R}\times\mathbb{R}$. We can see that $|g(x_1,y_1)-g(x_2,y_2)|\leq\dfrac{1}{2}(|x_1-x_2|+|y_1-y_2|)$. The function $\varphi_0:]-\infty,0]\times[0,\pi]\to\mathbb{R}$ can be defined by $\varphi_0(\theta,\xi)=e^{-\theta}\sin\xi$. To rewrite equation (5.1) in the abstract form, we introduce the space $X=L^2([0,\pi];\mathbb{R})$, the set of functions vanishing at 0 and π, equipped with the L^2 norm that is to say for all $u\in X$,

$$u(0)=u(\pi)=0 \text{ and } |u|_{L^2}=\Big(\int_0^\pi|u(s)|^2ds\Big)^{\frac{1}{2}}.$$

For example, let $u:[0,\pi]\to\mathbb{R}$ be defined by $u(t)=\sin t$. We have $u(0)=u(\pi)=0$ and

$$\Big(\int_0^\pi\sin^2(t)ds\Big)^{\frac{1}{2}}=\sqrt{\frac{\pi}{2}},$$

which implies $u\in X$, consequently $X\neq\emptyset$.

Let $e_n(x)=\sqrt{\dfrac{2}{\pi}}\sin(nx),\ x\in[0,\pi],\ n\in\mathbb{N}^*$, then $(e_n)_{n\in\mathbb{N}^*}$ is an orthonormal base for X. Let $A:X\to X$ be defined by

$$\begin{cases}Ay=y''\\ D(A)=\Big\{y\in X:\ \text{y, y' are absolutely continuous},\ y''\in X,\ \ y(0)=y(\pi)=0\Big\},\end{cases}$$

then

$$Ay=\sum_{n=1}^{+\infty}-n^2(y,e_n)e_n,\ y\in D(A),$$

where

$$(h,j)=\int_0^\pi h(s)j(s)ds, \text{ for } h,j\in X.$$

It is well known that A is the infinitesimal generator of strongly continuous cosine family $C(t),\ t\in\mathbb{R}$ in X given by

$$C(t)y=\sum_{n=1}^{+\infty}\cos nt(y,e_n)e_n,\ y\in X,$$

and the associated sine family is given by

$$S(t)y=\sum_{n=1}^{+\infty}\frac{1}{n}\sin(nt)(y,e_n)e_n,\ y\in X.$$

If we choose $\alpha=\dfrac{1}{2}$, then $(\mathbf{H_0})$ is satisfied since

$$(-A)^{\frac{1}{2}}y=\sum_{n=1}^{+\infty}n(y,e_n)e_n,\ y\in D\Big((-A)^{\frac{1}{2}}\Big)$$

and

$$(-A)^{\frac{1}{2}}y=\sum_{n=1}^{+\infty}\frac{1}{n}(y,e_n)e_n,\ y\in X.$$

From [13], the compactness A^{-1} follows from Lemma 2.4, and the fact that the eigenvalues of $(-A)^{\frac{1}{2}}$ are $\lambda_n=\dfrac{1}{n},\ \ n=1,2,...$, then $(\mathbf{H_2})$ is satisfied.

From [13], for all $y\in X_{\frac{1}{2}}$, y is absolutely continuous and $\|y\|_{\frac{1}{2}}=\|y'\|=\|A^{\frac{1}{2}}y\|$. We define the phase space

$$\mathcal{B}=BUC^1(]-\infty,0],X)$$

where $BUC^1(]-\infty,0],X)$ the space of bounded uniformly continuous functions defined from $]-\infty,0]$ to X, with the norm

$$|\varphi|_{\mathcal{B}}=\sup_{\theta\le 0}\|\varphi(\theta)\|,\ \text{for } \varphi\in\mathcal{B}$$

This space satisfies axioms $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and $(\mathbf{B})$. The norm in $\mathcal{B}_{\frac{1}{2}}$ is given by

$$|\varphi|_{\mathcal{B}_{\frac{1}{2}}}=\sup_{\theta\in[-r,0]}\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}+\sup_{\theta\in[-r,0]}\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}.$$

Let $f:\mathcal{B}_{\frac{1}{2}}\times\mathcal{B}_{\frac{1}{2}}\to X$ be defined by

$$f(\varphi,\varphi')(x)=g\Big(\frac{\partial}{\partial x}[\varphi(\theta)(x)],\frac{\partial}{\partial x}[\varphi'(\theta)(x)]\Big)\ \text{ for } x\in[0,\pi],\ \varphi\in\mathcal{B}_{\frac{1}{2}}\ \text{ and } \ t\ge 0,$$

where $\varphi\in\mathcal{C}^1$ is defined by

$$\varphi(\theta)(x)=\varphi_0(\theta,x)\ \text{ for } \ \theta\le 0\ \text{ and } \ x\in[0,\pi].$$

Let us pose $v(t) = z(t, x)$. Then equation (5.1) takes the following abstract form

$$
\left\{
\begin{array}{l}
v''(t) = Av(t) + f(v_t, v'_t) \text{ for } t \geq 0 \\[2ex]
v_0 = \varphi \\[2ex]
v'_0 = \varphi'.
\end{array}
\right.
\tag{5.2}
$$

Let $\varphi, \psi \in \mathcal{B}_{\frac{1}{2}}$, since $|g(x_1, y_1) - g(x_2, y_2)| \leq \dfrac{1}{2}(|x_1 - x_2| + |y_1 - y_2|)$, then we have

$$
\begin{aligned}
|f(\varphi, \varphi') - f(\psi, \psi')|_{L^2} &= \Big(\int_0^\pi \Big(g\Big(\frac{\partial}{\partial x}[\varphi(\theta)(x)], \frac{\partial}{\partial x}[\varphi'(\theta)(x)] \Big) - g\Big(\frac{\partial}{\partial x}[\psi(\theta)(x)], \frac{\partial}{\partial x}[\psi'(\theta)(x)] \Big) \Big)^2 dx \Big)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \Big(\int_0^\pi \Big(\Big| \frac{\partial}{\partial x}[\varphi(\theta)(x)] - \frac{\partial}{\partial x}[\psi(\theta)(x)] \Big| + \Big| \frac{\partial}{\partial x}[\varphi'(\theta)(x)] - \frac{\partial}{\partial x}[\psi'(\theta)(x)] \Big| \Big)^2 dx \Big)^{\frac{1}{2}}
\end{aligned}
$$

By using the Minkowski inequality, we have

$$
\begin{aligned}
|f(\varphi, \varphi') - f(\psi, \psi')|_{L^2} &\leq \frac{1}{2} \Big(\int_0^\pi \Big| \frac{\partial}{\partial x}[\varphi(\theta)(x)] - \frac{\partial}{\partial x}[\psi(\theta)(x)] \Big|^2 dx \Big)^{\frac{1}{2}} \\
&\quad + \frac{1}{2} \Big(\int_0^\pi \Big| \frac{\partial}{\partial x}[\varphi'(\theta)(x)] - \frac{\partial}{\partial x}[\psi'(\theta)(x)] \Big|^2 dx \Big)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \sup_{\theta \in]-\infty, 0]} \Big(\int_0^\pi \Big| \frac{\partial}{\partial x}[\varphi(\theta)(x)] - \frac{\partial}{\partial x}[\psi(\theta)(x)] \Big|^2 dx \Big)^{\frac{1}{2}} \\
&\quad + \frac{1}{2} \sup_{\theta \in]-\infty, 0]} \Big(\int_0^\pi \Big| \frac{\partial}{\partial x}[\varphi'(\theta)(x)] - \frac{\partial}{\partial x}[\psi'(\theta)(x)] \Big|^2 dx \Big)^{\frac{1}{2}},
\end{aligned}
$$

which implies that

$$
|f(\varphi, \varphi') - f(\psi, \psi')|_{L^2} \leq \frac{1}{2} \|\varphi - \psi\|_{\mathcal{B}_{\frac{1}{2}}}
$$

Consequently the function f satisfies $(\mathbf{H}_4)$. Then equation (5.2) has a unique mild solution which is defined for $t \geq 0$. For the regularity, we make the following assumptions.

$(\mathbf{H}_6)$ $g \in C^1(\mathbb{R}^- \times \mathbb{R}; \mathbb{R})$, such that $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ are locally lipschitz continuous.

$(\mathbf{H}_7)$

$$
\left\{
\begin{array}{l}
\varphi \in C^3(]-\infty, 0] \times [0, \pi]) \text{ such that } \varphi(0), \varphi''(0) \in D(A),\ \varphi'(0), \varphi^{(3)}(0) \in E \\[2ex]
\dfrac{\partial^2}{\partial \theta^2} \varphi(0, x) = \dfrac{\partial^2}{\partial x^2} \varphi(0, x) + \displaystyle\int_{-\infty}^0 g(\theta, \varphi(\theta, x)) d\theta \text{ for } x \in [0, \pi], \\[2ex]
\dfrac{\partial^3}{\partial \theta^3} \varphi(0)(x) = \dfrac{\partial^2}{\partial x^2} \varphi'(0, x) \text{ for } x \in [0, \pi].
\end{array}
\right.
$$

Conequently, we get the following result

Proposition 6. Equation (5.1) has one and only one strict solution u defined for $t \geq 0$ and $x \in [0, \pi]$.

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