Lie Symmetry Analysis and Conservation Laws of Non Linear Time Fractional WKI Equation

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Abstract

In this study, we considered with the time fractional WKI equation, Lie group analysis method are applied to frational WKI equation with the Riemann-Liouville derivative. The invariance properties of this equation were found. Besides we present corresponding infinitesimal generators for the WKI equation. And then the symmetry reductions are constructed with the Erdelyi-Kober fractional operator. Furthermore, we calculate Lie point symmetries associated the nonlocal conserved vectors utilizing the new conservation theorem method for two different cases of time fractional.

Keywords—Fractional Conservation laws, Lie group analysis, Time fractional WKI equation, New conservation theorem method.

1 Introduction

Motion of inextebsible plane curves in Euclidean space are given by the modified Korteweg de Vries (mKdV) equation

$$k_t + k_{sss} + \left(\frac{3}{2}\right) k^2 k_s = 0$$

where k is the curvature and s is the arclength of the curve. The plane curve motion flow is governed by [1].

$$\gamma_t = -k_s \mathbf{n} - \frac{1}{2} k^2 \mathbf{t}$$

here γ denotes the curve, **n** and **t** are the normal and tangent vector fields, respectively. Assume that this flow can be shown as the graph (x, u (x,t)) of some function u on the x-axis. Using the fact that the normal speed of the curve γ, $u_t / (1+u_x{}^2)^{1/2}$ is given by $-k_s$, one finds that u satisfies the famous Wadati-Konno-Ichikawa (WKI) equation [2].

$$u_t = \left(\frac{u_{xx}}{(1+u_x{}^2)^{3/2}} \right)_x$$

Wadati et al. found that WKI equation is solvable in the WKI scheme of the inverse scattering method [3,4]. This WKI-scheme for u is connected to the AKNS-scheme for k by a gauge transformation explicitly displayed in [5]. Furthermore Qu et al. have studied the group-invariant solutions of the two-component WKI equation and its similarity reductions to systems of ordinary differential equations were also given [6]. Besides they have constructed conservation laws in another work [7].

In recent years, fractional differential equations (FDEs) has attracted due to an exact description in complex phonemena in various applications such as control theory, signal processing, fluid flow, population dynamics, fractional dynamics. Many powerful and efficient methods have been developed to obtain exact solutions of FDEs [8-13]. One of them is Lie gorup analysis method and it is well known that this method plays an important role to analyze, finding exact solutions and constructing conservation laws. Lie symmetries of integer order differential equations have been studied by many scientists [14-18]. But studies about the invariance properties of FDEs are quite new.

For example J. Hu et al. [19] considered the fractional KdV type equation and obtained a group of dilation. Utilizing the dilation symmetry they have reduced to an fractional ordinary differential equation with Erdelyi Kober operator. In Ref. [20], the authors have made an attempt to extend the Lie group analysis and constructing conservation laws to FDEs.

In this paper, we perform the Lie gorup analysis method and construct the conservation laws for nonlinear time fractional WKI equation,

$$u_t^{\alpha}=\left(\frac{u_{xx}}{(1+u_x^{\ 2})^{3/2}}\right)_x,\qquad 0<\alpha<2, \tag{1}$$

Where $u_t^{\alpha}=D_t^{\alpha}u$ denotes the modified Riemann-Liouville fractional derivative of order α with respect to the variable t and defined by,

$$D_t^{\alpha}u=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\displaystyle\int_0^t\dfrac{u(\tau)-u(0)}{(t-\tau)^{\alpha}}, & 0<\alpha<1\\ [u^{(n)}(t)]^{(\alpha-n)}, & n\le\alpha\le n+1,\quad 1\le n\end{cases}$$

and Euler Gamma function $\Gamma(\sigma)$ is given by the integral,

$$\Gamma(\sigma)=\int_0^{\infty}e^{-t}t^{\sigma-1}dt$$

The outline of this paper is organized as follows: In Section 2, a brief description of invariant group analysis method is given for FPDEs. In Section 3 we employ symmetry analysis and similarity reduction for fractional WKI equation. In Section 4, conservation laws of Eq.(1) were constructed by Ibragimov method. Consequently, main results are summarized.

2 Lie Symmetry Theory

In this section, we give the general procedure to find Lie point symmetries for time FPDEs which include two independent variables and one dependent variable [21].

Let us consider that a nonlinear FPDE including two independent variables x and t is given by,

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}=F(x,t,u,u_x,u_{xx},u_{xxx},...) \tag{2}$$

We suppose that equation (2) is invariant under the continuous transformations,

$$\begin{aligned} t^*&=t+\varepsilon\tau(x,t,u)+O(\varepsilon^2)\\ x^*&=x+\varepsilon\xi(x,t,u)+O(\varepsilon^2)\\ u^*&=u+\varepsilon\eta(x,t,u)+O(\varepsilon^2)\\ \partial^{\alpha}u^*/\partial t^{*\alpha}&=\partial^{\alpha}u/\partial t^{\alpha}+\varepsilon\eta^{\alpha t}(x,t,u)+O(\varepsilon^2)\\ \partial^{j}u^*/\partial x^{*m}&=\partial^{m}u/\partial x^{m}+\varepsilon\eta^{mx}(x,t,u)+O(\varepsilon^2)\end{aligned} \tag{3}$$

where ε is the group parameter and ξ,τ,η are infinitesimals and the extended infinitesimals of order three are given by the prolongation formula (see [22])

$$\begin{aligned}\eta^{x}&=D_x(\eta)-u_xD_x(\xi)-u_tD_x(\tau),\\ \eta^{xx}&=D_x(\eta^{x})-u_{xt}D_x(\tau)-u_{xx}D_x(\xi),\\ \eta^{xxx}&=D_x(\eta^{xx})-u_{xxt}D_x(\tau)-u_{xxx}D_x(\xi),\end{aligned} \tag{4}$$

Here D_x denotes the total derivative operator and is defined by

$$D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+...$$

with the associated vector field of the form

$$X=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u}, \tag{5}$$

Now for the invariance of Eq.(2) under Eq.(3), we must have

$$Pr^{(\alpha,3)}X(\Delta)|_{\Delta=0}=0,$$

where

$$\Delta=\tfrac{\partial^{\alpha}u}{\partial t^{\alpha}}-F(x,t,u,u_x,u_{xx},u_{xxx},...)$$

The α th extended infinitesimal related to Riemann-

Liouville fractional time derivative, which reads (see [22]):

$$\eta_\alpha^0 = D_t^\alpha(\eta) + \xi D_t^\alpha(\mathrm{u}_x) - D_t^\alpha(\xi\,\mathrm{u}_x) + D_t^\alpha(\mathrm{D}_t(\tau)\,\mathrm{u})$$
$$-D_t^{\alpha+1}(\tau\,\mathrm{u}) + \tau D_t^{\alpha+1}(\mathrm{u}).$$

where the symbol D_t^α shows the total fractional derivative operator.

By utulizing the generalized Leibnitz rule in the fractional sense

$$D_t^\alpha[\mathrm{u(t)\,v(t)}] = \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}\,\mathrm{u(t)\,v(t)}, \qquad \alpha>0, \tag{6}$$

in which

$$\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(\mathrm{n}-\alpha)}{\Gamma(1-\alpha)\Gamma(\mathrm{n}+1)}$$

Thus from Eq.(6), we expressed η_α^0 as follows:

$$\eta_\alpha^0 = D_t^\alpha(\eta) - \alpha D_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi)\, D_t^{\alpha-n}(u_x)$$
$$-\sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{n-\alpha}(u) \tag{7}$$

Generalization of the chain rule for a composite function of the form

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-\mathrm{g(t)}]^r \frac{d^m}{dt^m}[\mathrm{g(t)}]^{k-r}\frac{d^k f(g)}{dg^k} \tag{8}$$

Further, employing the chain rule(8) and the generalized Leibnitz rule (6) with $f(\mathrm{t}) = 1$, α-th prolongation formula becomes

$$D_t^\alpha(\eta) = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_u \frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} D_t^{\alpha-n}(\mathrm{u})\right] + \mu$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}$$
$$[-u]^r \frac{\partial^m}{\partial t^m}\left[u^{k-r}\right]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

Therefore we obtain the explicit form of η_α^0 from the Eq.(7),

$$\eta_\alpha^0 = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu$$
$$+\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi)\, D_t^{\alpha-n}(u_x)$$

Definition: In Ref.[13], $u = \theta(\mathrm{x,t})$ is an invariant solution of Eq.(2) related to the infinitesimal operator (5) if and only if

1) $u = \theta(\mathrm{x,t})$ satisfies Eq.(2).

2) $u = \theta(\mathrm{x,t})$ is an invariant surface of (2), in other words,

$$X\theta = \left(\xi(\mathrm{x,t,u})\frac{\partial}{\partial x} + \tau(\mathrm{x,t,u})\frac{\partial}{\partial t} + \eta(\mathrm{x,t,u})\frac{\partial}{\partial u}\right)\theta(\mathrm{x,t}) = 0.$$

3 Lie Symmetry Analysis and Reduction of The Time Fractional WKI Equation

In the previous section, we have given some preliminaries about the Lie symmetry method on the FPDEs. In this subsection, using the above discussion and the Lie theory, we employee the time fractional WKI equation

$$u_t^\alpha - \frac{1}{2}\frac{u_{xxx}}{(1+u_x{}^2)^3} + \frac{3u_{xx}{}^2 u_x}{(1+u_x{}^2)^4} = 0 \tag{9}$$

According to the Lie point theory, applying the prolongation $Pr^{(\alpha,3)}X$ to Eq.(9), we can arrive at the following invariance criterion,

$$\eta^{\alpha t} + \left(\frac{3u_x u_{xxx}}{(1+u_x{}^2)^4} - \frac{24u_{xx}{}^2 u_x{}^2}{(1+u_x{}^2)^5} + \frac{3u_{xx}{}^2}{(1+u_x{}^2)^4}\right)\eta^x$$
$$+\frac{6u_x u_{xx}}{(1+u_x{}^2)^4}\eta^{xx} - \frac{1}{2(1+u_x{}^2)^3}\eta^{xxx} = 0. \tag{10}$$

Substituting (4) and (7) into (10) and we get a polynomial in terms of some derivatives $u_x, u_t, u_{xx}, \ldots$ and dependent variable u. Then let the coe¢ cients of $u_x, u_t, u_{xx}, \ldots$ equal to zero. So we get some of determining equations:

$$\eta_{uu} = \tau_u = \tau_x = \xi_u = \xi_t = \xi_{xx} = 0,$$

$$2\xi_u - \eta_x = 0,$$

$$-\frac{1}{2}\alpha\tau_t + \frac{3}{2}\xi_x = 0,$$

$$2\eta_u - 5\xi + \alpha\tau_t = 0,$$

$$\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}\tau = 0, \qquad \text{for } n=1,2,...$$

The special solution of determining system helps us to obtain the coefficient functions. Solving these equations consistently with the help of package program, we obtain the following forms of the coefficient functions,

$$\xi = c_1 + c_3\alpha x, \quad \tau = 3c_3 t, \quad \eta = c_2 f(t) + c_3\alpha u.$$

where c_1 and c_2 are arbitrary constants. Hence the infinitesimal operator becomes

$$X = (\alpha x c_3 + c_1)\frac{\partial}{\partial x} + 3tc_3\frac{\partial}{\partial t} + (c_2 f(t) + \alpha u c_3)\frac{\partial}{\partial u}.$$

Thus, infinitesimal generators of every one parameter Lie group of point symmetries of the Eq.(1) are,

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = f(t)\frac{\partial}{\partial t}, \qquad X_3 = 3t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u}.$$

For the symmetry generator X_3 corresponding characteristic equation is given by

$$\frac{dx}{\alpha x} = \frac{dt}{3t} = \frac{du}{\alpha u},$$

which solving them yields the corresponding invariants

$$\Omega = xt^{\frac{-\alpha}{3}}, \qquad u = t^{\frac{\alpha}{3}}\phi(\Omega). \tag{11}$$

Using the above invariants we yield a special a nonlinear ODE of fractional order. We will prove the following theorem corresponding this case.

Theorem: The similarity transformation $u = t^{\frac{\alpha}{3}}\phi(\Omega)$ along with the similarity variable $\Omega = xt^{\frac{-\alpha}{3}}$ reduces the time fractional WKI equation to the nonlinear ODE of fractional order,:

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},\alpha}\phi\right)(\Omega) - \frac{1}{2}\frac{\phi'''}{(1+(\phi')^2)^3} + \frac{3(\phi'')^2\phi'}{(1+(\phi')^2)^4} = 0. \tag{12}$$

with the Erdelyi-Kober fractional differential operator [22].

$$\left(P_\beta^{\tau,\alpha}\phi\right) := \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta}\Omega\frac{d}{d\Omega}\right)\left(K_\beta^{\tau+\alpha,n-\alpha}\phi\right)(\Omega),$$

$$n = \begin{cases}[\alpha]+1, & \alpha \notin N,\\ \alpha, & \alpha \in N,\end{cases}$$

where

$$\left(K_\beta^{\tau,\alpha}\phi\right)(\Omega) := \begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1}u^{-(\tau+\alpha)}\phi(\Omega u^{\frac{1}{\beta}})du, & \alpha > 0,\\ \phi(\Omega), & \alpha = 0,\end{cases}$$

is the Erdelyi-Kober fractional integral operator.

Proof: In ordet to obtain more general applicability, we get $n-1<\alpha<n$; $n=1,2,3,...$ Then the Riemann-Liouville fractional derivative for the similarity transformation $\Omega = xt^{\frac{-\alpha}{3}}$, $u = t^{\frac{\alpha}{3}}\phi(\Omega)$ one can get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{\frac{\alpha}{3}}\phi(xs^{\frac{-\alpha}{3}})ds\right]. \tag{14}$$

Let $v = t/s$ one can have $ds = -(t/v^2)dv$. Then the Eq.(14) can be expressed as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{2\alpha}{3}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1}v^{-(n+1-\frac{2\alpha}{3})}\phi(\Omega v^{\frac{\alpha}{3}})dv\right]. \tag{15}$$

If one uses the definition of Erdelyi-Kober fractional integral operator (13), then Eq.(15) becomes,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{2\alpha}{3}}\left(K_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3},n-\alpha}\phi\right)(\Omega)\right]. \tag{16}$$

In order to simplify the above equation we consider the relation,

$$\Omega = xt^{\frac{-\alpha}{3}}, \phi \in C^1(0,\infty) \;\Rightarrow$$

$$t\frac{\partial}{\partial t}\phi(\Omega) = tx\left(-\frac{\alpha}{3}\right)t^{-\frac{\alpha}{3}-1}\phi'(\Omega) = -\frac{\alpha}{3}\Omega\frac{\partial}{\partial\Omega}\phi(\Omega). \tag{17}$$

So, we obtain

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{2\alpha}{3}}\left(K_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3},n-\alpha}\phi\right)(\Omega)\right] = \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{2\alpha}{3}}\left(K_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3},n-\alpha}\phi\right)(\Omega)\right)\right]$$

$$=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{2\alpha}{3}-1}\left(n-\frac{2\alpha}{3}-\frac{\alpha}{3}\Omega\frac{\partial}{\partial\Omega}\right)\left(K_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3},n-\alpha}\phi\right)(\Omega)\right].$$

Repeating the similar procedure for n-1 times, we have

$$\frac{\partial^{n}}{\partial t^{n}}\left[t^{n-\frac{2\alpha}{3}}\left(K_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3},n-\alpha}\phi\right)(\Omega)\right]=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{2\alpha}{3}}\left(K_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3},n-\alpha}\phi\right)(\Omega)\right)\right]$$

$$=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{2\alpha}{3}-1}\left(n-\frac{2\alpha}{3}-\frac{\alpha}{3}\Omega\frac{\partial}{\partial\Omega}\right)\left(K_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3},n-\alpha}\phi\right)(\Omega)\right]$$

$$=\ldots=t^{-\frac{2\alpha}{3}}\prod_{j=0}^{n-1}\left(1-\frac{2\alpha}{3}+j-\frac{\alpha}{3}\Omega\frac{d}{d\Omega}\right)\left(K_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3},n-\alpha}\phi\right)(\Omega).$$

Now using the definition of the Erdélyi-Kober fractional differential operator given in (12), the above equation can be written as

$$\frac{\partial^{n}}{\partial t^{n}}\left[t^{n-\frac{2\alpha}{3}}\left(K_{\frac{3}{\alpha}}^{1+\frac{\alpha}{3},n-\alpha}\phi\right)(\Omega)\right]=t^{-\frac{2\alpha}{3}}\left(P_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},\alpha}\phi\right)(\Omega).$$

Substituting the expression (18) into (16), we obtain an expression for the time fractional derivative

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=t^{-\frac{2\alpha}{3}}\left(P_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},\alpha}\phi\right)(\Omega).$$

Thus we find that time fractional WKI equation reduces into an fractional order ODE

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},\alpha}\phi\right)(\Omega)-\frac{1}{2}\frac{\phi'''}{(1+(\phi')^{2})^{3}}+\frac{3(\phi'')^{2}\phi'}{(1+(\phi')^{2})^{4}}=0.$$

The proof is completed.

4 Conservation Laws

Now we construct a conservation law for Eq.(1) in the same way as it defines for the integer order differential equations. Namely a vector $T=(T^{x},T^{t})$ satisfying

$$D_{x}(T^{x})+D_{t}(T^{t})=0$$

for all solutions of Eq.(1) is known as the conserved vector of Eq.(1). Note that Eq.(<ref>1.1</ref>) with the Riemann-Liouville fractional derivative can be rewritten in the conserved form with,

$$T^{t}=D_{t}^{n-1}({}_{0}I_{t}^{n-\alpha}u),\qquad T^{x}=\frac{u_{xx}}{(1+u_{x}^{2})^{3/2}}.$$

Eq.(1) does not have a Lagrangian in classical sense so it means that Eq.(1) can not be constructed variational principle of least action with a Lagrangian depending on the variables x,t,u. To exceed this restriction Ibragimov introduced formal Lagrangian structure and gave the new conservation theorem. In accordance to this method [23], the formal Lagrangian of the fractional WKI equation is given by

$$L=w(x,t)\left[u_{t}^{\alpha}-\frac{1}{2}\frac{u_{xxx}}{(1+u_{x}^{2})^{3}}+\frac{3u_{xx}^{2}u_{x}}{(1+u_{x}^{2})^{4}}\right],$$

Here, w(x,t) is a adjoint variable. Considering this formal Lagrangian, an action integral is

$$\int_{0}^{T}\int_{\Psi}L(t,x,u,w,D_{t}^{\alpha},u_{x},\ldots)dxdt.$$

Agrawal developed the fractional variational approach and one can find the Euler--Lagrange operator with respect to u has the form [24],

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_{x}\frac{\partial}{\partial u_{x}}+(D_{t}^{\alpha})^{*}\frac{\partial}{\partial D_{t}^{\alpha}u}+D_{x}^{2}\frac{\partial}{\partial u_{xx}}-D_{x}^{3}\frac{\partial}{\partial u_{xxxx}},$$

where $(D_{t}^{\alpha})^{*}$ is the adjoint operator of (D_{t}^{α}). For the Riemann-Liouville fractional differential operators,

$$(D_{t}^{\alpha})^{*}=(-1)^{n}I_{T}^{n-\alpha}(D_{t}^{n})={}_{t}^{C}D_{T}^{\alpha}$$

where

$$I_{t}^{n-\alpha}u(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_{t}^{T}\frac{u(\theta,x)}{(\theta-t)^{1-n+\alpha}}d\theta,\qquad n=[\alpha]+1,$$

is the right sided time-fractional integral of order $n-\alpha$ and ${}_{t}^{C}D_{T}^{\alpha}$ is the right sided Caputo operator of fractional differentiation of order α.

As in stated [24], the adjoint equation is similarly to the case of integer order nonlinear differential equations, so we have the adjoint equation to the time fractional WKI equation as Euler--Lagrange equation

$$\frac{\delta L}{\delta u}=F^{*}=(D_{t}^{\alpha})^{*}w+\frac{1}{2}\frac{w_{xxx}u_{x}^{2}-6w_{xx}u_{xx}u_{x}+w_{xxx}}{(1+u_{x}^{2})^{4}}$$

Fractional WKI equation involves only fractional derivative with respect to t, thus x-component conserved vector can be detemined by the formula for the integer order PDEs. The operator T_{i}^{x} is given by

$$T_i^x = W_i \frac{\delta L}{\delta u_x} + D_x(W_i)\frac{\delta L}{\delta u_{xx}} + D_x^2(W_i)\frac{\delta L}{\delta u_{xxx}} \quad (20)$$

and W, Lie characteristic function is

$$W = \eta - u_x \xi - u_t \tau$$

Riemann--Liouville time-fractional derivative is used in Eq.(1) so the operator T^t takes the form

$$T_i^t = \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W_i) D_t^k \frac{\partial L}{\partial(D_t^\alpha u)} - (-1)^n J\left(W_i, D_t^n \frac{\partial L}{\partial(D_t^\alpha u)}\right) \quad (21)$$

where J is the integral

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{f(\tau,x) g(\mu,x)}{(t-\theta)^{1-n+\alpha}} d\mu d\tau.$$

As in determined previous section fractional WKI equation admits three infinitesimal generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = f(t)\frac{\partial}{\partial t}, \qquad X_3 = 3t\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} + \alpha u \frac{\partial}{\partial u}.$$

and the corresponding Lie characteristic functions are found as follows:

$$W_1 = -u_x, \quad W_2 = f(t), \qquad W_3 = \alpha u - \alpha x u_x - 3t u_t.$$

When $\alpha \in (0,1)$ effecting these values into the vector components (20,21) we can get the components of conserved vectors of Eq.(1)

$$T^x = \frac{1}{2(1+u_x^2)^4}[u_x^3 w_{xxx} + u_{xxx} w(1+u_x^2) - w_x u_{xx}(1+u_x^2)$$
$$-6wu_x u_{xx}^2 + u_x w_{xx}],$$

$$T_1^t = -I_t^{1-\alpha}(u_x)w - J(u_x, w_t),$$

$$T_2^x = \frac{-f(t)w_{xx}}{2(1+u_x^2)^3},$$

$$T_2^t = I_t^{1-\alpha}(f(t))w - J(f(t), w_t),$$

$$T_3^x = (u - xu_x - 3tu_t)\left[w\left(\frac{3u_x u_{xxx}}{(1+u_x^2)^4} - \frac{24u_x^2 u_{xx}^2}{(1+u_x^2)^5} + \frac{3u_{xx}^2}{(1+u_x^2)^4}\right)\right.$$
$$\left. - \frac{3wu_x u_{xxx}}{(1+u_x^2)^4} + \frac{24wu_x^2 u_{xx}^2}{(1+u_x^2)^5} - \frac{3wu_{xx}^2}{(1+u_x^2)^4} - \frac{w_{xx}}{2(1+u_x^2)^3}\right]$$
$$+(-xu_{xx} - 3tu_{tx})\left[\frac{3wu_x u_{xx}}{(1+u_x^2)^4} + \frac{w_x}{2(1+u_x^2)^3}\right] - \frac{1}{2}\frac{w(-u_{xx} - xu_{xxx} - 3tu_{xxt})}{(1+u_x^2)^3}$$

$$T_3^t = I_t^{1-\alpha}(\alpha u - \alpha x u_x - 3tu_t)w - J(\alpha u - \alpha x u_x - 3tu_t, w_t).$$

When $\alpha \in (1,2)$ effecting these values into the vector components (20,21) we can get the components of conserved vectors of Eq.(1).

$$T_1^t = -D_t^{\alpha-1}(u_x)w - I_t^{2-\alpha}(u_x)w_t + J(u_x, w_{tt}),$$

$$T_2^t = D_t^{\alpha-1}(f(t))w + I_t^{2-\alpha}(f(t))w_t - J(f(t), w_{tt}),$$

$$T_3^t = D_t^{\alpha-1}(\alpha u - \alpha x u_x - 3tu_t)w + I_t^{2-\alpha}(\alpha u - \alpha x u_x - 3tu_t)w_t$$
$$-J(\alpha u - \alpha x u_x - 3tu_t, w_{tt}).$$

5 Conclusion

In the present study, we illustrate the application of Lie group approach to investigate the time fractional WKI equation. It is obtained that Eq.(1) is spanned by three vector fields and we obtained symmetry properties, similarity reduction forms. Based on the symmetry generators, we have shown that this equation can be reduce to a nonlinear ordinary differential equation of fractional order with the Erdelyi-Kober fractional operator.

Besides we considered nonlocal conservation theorem method for constructing conserved vectors. This method ensures that conservation laws are obtained if the generators of the equation are known. In this way we obtained six conserved vectors for values of α; using each symmetry generators .

6 References

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