Two New Types of Irresolute Functions via e-open Sets

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Abstract
The main purpose of this paper is to introduce two new types of irresolute functions called completely e-irresolute and completely weakly e-irresolute functions via e-open sets introduced by Ekici. We obtain some characterizations of these functions. Also, we investigate some fundamental properties between these new notions and separation and covering.
Keywords — completely e-irresolute, completely weakly e-irresolute, countably e-compact, e-closed compact, e-Lindelöf, strongly e-regular space, e-normal space.

1 Introduction and Preliminaries

Throughout this paper (X,τ) and (Y,σ) (or simply X and Y) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise stated. Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. $\mathcal{U}(x)$ denotes all open neighborhoods of the point $x \in X$. A subset A of a space X called regular open [17] (resp. regular closed [17]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$) . The δ-interior [19] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by $int_\delta(A)$. The subset A is called δ-open [19] if $A = int_\delta(A)$, i.e., a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set $A \subset X$ is called δ-closed [19] if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x \mid U \in \mathcal{U}(x) \Rightarrow int(cl(U)) \cap A \neq \emptyset\}$. The family of all δ-open (resp. δ-closed) sets in X is denoted by $\delta O(X)$ (resp. $\delta C(X)$).

A subset A of a space X called e-open [17] if $A \subset int(cl_\delta(A)) \cup cl(int_\delta(A))$. The complement of an e-open set is said to be e-closed. The e-interior [7] of a subset A of X is the union of all e-open sets of X contained in A and is denoted by $e\text{-}int(A)$. The e-closure [7] of a subset A of X is the intersection of all e-closed sets of X containing A and is denoted by $e\text{-}cl(A)$. The family of all e-open (resp. regular open) sets of X are denoted by $eO(X)$ (resp. $RO(X)$). The family of all e-closed (resp. regular closed) sets of X is denoted by $eC(X)$ (resp. $RC(X)$) and the family of all e-open (resp. regular open) sets of X containing a point $x \in X$ is denoted by $eO(X,x)$ (resp. $RO(X,x)$).

Definition 1. A function $f: X \to Y$ is said to be:
(a) strongly continuous [9] (briefly s.c.) if $f^{-1}[V]$ is both open and closed in X for each subset V of Y;
(b) completely continuous [2] (briefly c.c.) if $f^{-1}[V]$ is regular open in X every open set V of Y;
(c) e-irresolute [6] (briefly e.i.) if $f^{-1}[V]$ is e-closed (resp. e-open) in X for every e-closed (resp. e-open) subset V of Y;
(d) e-continuous [7] (briefly e.c.) if $f^{-1}[V]$ is e-open in X every open set V of Y.

2 Completely e-irresolute Functions

Definition 2. A function $f: X \to Y$ is said to be completely e-irresolute (briefly c.e.i.) if the inverse image of each e-open subset of Y is regular open in X.

Remark 3. It is not difficult to see that every strongly continuous function is completely e-irresolute and every completely e-irresolute function is e-irresolute. But the converse of the implications are not true in general as shown by the following examples.

s.c. → c.e.i. → e.i.

Example 4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\sigma = \{\emptyset, X, \{b, c\}\}$. Then the identity function $f: (X, \tau) \to (X, \sigma)$ is e-irresolute but not completely e-irresolute.

QUESTION. Is there any completely e-irresolute function which is not strongly continuous?

Theorem 5. Let $f: X \to Y$ be a function, then the following statements are equivalent:
(a) f is completely e-irresolute;
(b) $f^{-1}[e - int(B)] \subset int_\delta(f^{-1}[B])$ for every subset B of Y;
(c) $f[cl_\delta(A)] \subset e - cl(f[A])$ for every subset A of X;
(d) $cl_\delta(f^{-1}[B]) \subset f^{-1}[e - cl(B)]$ for every subset B of Y;
(e) $f^{-1}[V]$ is regular closed in X for each e-closed set V in Y;
(f) $f^{-1}[V]$ is regular open in X for each e-open set V in Y.

Proof. $(\boldsymbol{a}) \Rightarrow (\boldsymbol{b})$: Let $B \subset Y$ and $x \in f^{-1}[e - int(B)]$.

$x \in f^{-1}[e - int(B)] \Rightarrow e - int(B) \in eO(Y, f(x))$

$\overset{(a)}{\Rightarrow} (\exists U \in RO(X, x))(f[U] \subset e - int(B) \subset B)$

$\Rightarrow (\exists U \in RO(X, x))(U \subset f^{-1}[B]) \Rightarrow x \in int_\delta(f^{-1}[B])$.

$(\boldsymbol{b}) \Rightarrow (\boldsymbol{c})$: Let $A \subset X$.

$A \subset X \Rightarrow f[A] \subset Y \Rightarrow Y \setminus f[A] \subset Y \overset{(b)}{\Rightarrow}$

$\overset{(b)}{\Rightarrow} f^{-1}[e - int(Y \setminus f[A])] \subset int_\delta(f^{-1}[Y \setminus f[A]])$

$\Rightarrow X \setminus f^{-1}[e - cl(f[A])] \subset X \setminus cl_\delta(f^{-1}[f[A]])$

$\Rightarrow cl_\delta(A) \subset cl_\delta(f^{-1}[f[A]]) \subset f^{-1}[e - cl(f[A])]$

$\Rightarrow f[cl_\delta(A)] \subset e - cl(f[A])$.

$(\boldsymbol{c}) \Rightarrow (\boldsymbol{d})$: Let $B \subset Y$.

$B \subset Y \Rightarrow f^{-1}[B] \subset X \overset{(c)}{\Rightarrow}$

$\overset{(c)}{\Rightarrow} f[cl_\delta(f^{-1}[B])] \subset e - cl(f[f^{-1}[B]]) \subset e - cl(B)$

$\Rightarrow cl_\delta(f^{-1}[B]) \subset f^{-1}[e - cl(B)]$.

$(\boldsymbol{d}) \Rightarrow (\boldsymbol{e})$: Let $V \in eC(Y)$.

$V \in eC(Y) \Rightarrow V = e - cl(V) \overset{(d)}{\Rightarrow}$

$\overset{(d)}{\Rightarrow} cl_\delta(f^{-1}[V]) \subset f^{-1}[e - cl(V)] = f^{-1}[V]$

$\Rightarrow f^{-1}[V] = cl_\delta(f^{-1}[V]) \Rightarrow f^{-1}[V] \in \delta C(X)$.

$(\boldsymbol{e}) \Rightarrow (\boldsymbol{f})$: Obvious.

$(\boldsymbol{f}) \Rightarrow (\boldsymbol{a})$: Let $V \in eO(Y)$ and $x \in f^{-1}[V]$.

$(V \in eO(Y))(x \in f^{-1}[V]) \Rightarrow V \in eO(Y, f(x)) \overset{(f)}{\Rightarrow}$

$\overset{(f)}{\Rightarrow} (U := f^{-1}[V] \in RO(X, x))(f[U] \subset V)$.

Theorem 6. Let $f: (X, \tau) \to (Y, \sigma)$ be a bijective function. Then the following statements are equivalent:
(a) f is completely e-irresolute;
(b) $e - int(f[A]) \subset f[int_\delta(A)]$ for every subset of X.

Proof. $(\boldsymbol{a}) \Rightarrow (\boldsymbol{b})$: Let $A \subset X$.

$A \subset X \Rightarrow X \setminus A \subset X \overset{(a)}{\Rightarrow}$

$$\left.\begin{array}{r}\overset{(a)}{\Rightarrow} f[X \setminus int_\delta(A)] = f[cl_\delta(X \setminus A)] \subset e - cl(f[X \setminus A]) \\ f \text{ is bijection}\end{array}\right\}$$

$\Rightarrow Y \setminus f[int_\delta(A)] \subset Y \setminus e - int(f[A])$

$\Rightarrow e - int(f[A]) \subset f[int_\delta(A)]$.

$(\boldsymbol{b}) \Rightarrow (\boldsymbol{a})$: Let $A \subset X$.

$A \subset X \Rightarrow X \setminus A \subset X \overset{(b)}{\Rightarrow}$

$$\left.\begin{array}{r}\overset{(b)}{\Rightarrow} e - int(f[X \setminus A]) \subset f[int_\delta(X \setminus A)] \\ f \text{ is bijection}\end{array}\right\} \Rightarrow$$

$\Rightarrow Y \setminus e - cl(f[A]) \subset Y \setminus f[cl_\delta(A)]$

$\Rightarrow f[cl_\delta(A)] \subset e - cl(f[A])$.

Lemma 7. [10] Let Y be an open subset of a topological space X. Then the following hold:
(a) If A is regular open in X, then so is $A \cap Y$ in the subspace (Y, τ_Y).
(b) If B is regular open in (Y, τ_Y), then there exists a regular open set R in X such that $B = R \cap Y$.

Theorem 8. If $f: (X, \tau) \to (Y, \sigma)$ is a completely e-irresolute function and A is any open subset of X, then the restriction $f_A: A \to Y$ is completely e-irresolute.

Proof. Let $F \in eO(Y)$.

$$\left.\begin{array}{r}F \in eO(Y) \xRightarrow{f \text{ is c.e.i.}} f^{-1}[F] \in RO(X) \\ A \in \tau\end{array}\right\} \xRightarrow{Lemma\ 7}$$

$\xRightarrow{Lemma\ 7} (f_A)^{-1}[F] = f^{-1}[F] \cap A \in RO(A)$.

Lemma 9. [3] Let Y be a preopen subset of a topological space X. Then $Y \cap A$ is regular open in Y for each regular open subset A of X.

Theorem 10. If $f: (X, \tau) \to (Y, \sigma)$ is a completely e-irresolute function and A is preopen subset of X, then $f_A: A \to Y$ is completely e-irresolute.

Proof. It is clear from Lemma 9.

Theorem 11. Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ be two functions. Then the following hold:
(a) If f is completely e-irresolute and g is e-irresolute, then $g \circ f$ is completely e-irresolute;

(b) If f is completely continuous and g is completely e-irresolute, then $g \circ f$ is completely e-irresolute;
(c) If f is completely e-irresolute and g is e-continuous, then $g \circ f$ is completely continuous.

Proof. Straightforward.

Definition 12. A space X is said to be almost connected [5] (resp. e-connected [8]) if there does not exist disjoint regular open (resp. e-open) sets A and B such that $A \cup B = X$.

Theorem 13. If $f: X \to Y$ is completely e-irresolute surjection and X is almost connected, then Y is e-connected.

Proof. Suppose that Y is not e-connected.

$$Y \text{ is not } e-\text{connected} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists A, B \in eO(Y) \setminus \{\emptyset\})(A \cap B = \emptyset)(A \cup B = Y)\\ f \text{ is completely } e-\text{irresolute surjection}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (f^{-1}[A], f^{-1}[B] \in RO(X) \setminus \{\emptyset\})$$
$$(f^{-1}[A \cap B] = f^{-1}[\emptyset])(f^{-1}[A \cup B] = f^{-1}[Y])$$
$$\Rightarrow (f^{-1}[A], f^{-1}[B] \in RO(X) \setminus \{\emptyset\})$$
$$(f^{-1}[A] \cap f^{-1}[B] = \emptyset)(f^{-1}[A] \cup f^{-1}[B] = X)$$

This means that X is not almost connected.

Definition 14. A topological space X is said to be:
(a) nearly compact [14] if every regular open cover of X has a finite subcover;
(b) nearly countably compact [4] if every countable cover by regular open sets has a finite subcover;
(c) nearly Lindelöf [5] if every cover of X by regular open sets has a countable subcover;
(d) e-compact [8] if every e-open cover of X has a finite subcover;
(e) countably e-compact if every e-open countable cover of X has a finite subcover;
(f) e-Lindelöf if every cover of X by e-open sets has a countable subcover.

Theorem 15. Let $f: (X, \tau) \to (Y, \sigma)$ be a completely e-irresolute surjection. Then the following statements hold:
(a) If X is nearly compact, then Y is e-compact;
(b) If X is nearly Lindelöf, then Y is e-Lindelöf;
(c) If X is nearly countably compact, then Y is countably e-compact.

Proof. (a) Let X be nearly compact and $\mathcal{A}$ be an e-open cover of Y.

$$\big(\mathcal{A} \subset eO(Y)\big)(Y = \cup\, \mathcal{A}) \xRightarrow{f \text{ is c.e.i.}}$$
$$\left.\begin{array}{r}\xRightarrow{f \text{ is c.e.i.}} \big(\mathcal{B} := \{f^{-1}[A] | A \in \mathcal{A}\} \subset RO(X)\big)(X = \cup\, \mathcal{B})\\ X \text{ is nearly compact}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(X = \cup\, \mathcal{B}^*)$$
$$\xRightarrow{f \text{ is surjective}} (f[\mathcal{B}^*] \subset f[\mathcal{B}] = \mathcal{A})(|f[\mathcal{B}^*]| < \aleph_0)$$
$$(Y = f[X] = f[\cup\, \mathcal{B}^*] = \cup_{B \in \mathcal{B}^*} f[B]).$$

(b) Let X be nearly Lindelöf and $\mathcal{A}$ be an e-open cover of Y.

$$\big(\mathcal{A} \subset eO(Y)\big)(|\mathcal{A}| \leq \aleph_0)(Y = \cup\, \mathcal{A}) \xRightarrow{f \text{ is c.e.i.}}$$
$$\left.\begin{array}{r}\xRightarrow{f \text{ is c.e.i.}} \big(\mathcal{B} := \{f^{-1}[A] | A \in \mathcal{A}\} \subset RO(X)\big)(X = \cup\, \mathcal{B})\\ X \text{ is nearly countably compact}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(X = \cup\, \mathcal{B}^*)\\ f \text{ is surjective}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (f[\mathcal{B}^*] \subset f[\mathcal{B}] = \mathcal{A})(|f[\mathcal{B}^*]| < \aleph_0)$$
$$(Y = f[X] = f[\cup\, \mathcal{B}^*] = \cup_{B \in \mathcal{B}^*} f[B]).$$

(c) Let X be nearly countably compact and $\mathcal{A}$ be an e-open countable cover of Y.

$$\big(\mathcal{A} \subset eO(Y)\big)(|\mathcal{A}| \leq \aleph_0)(Y = \cup\, \mathcal{A}) \xRightarrow{f \text{ is c.e.i.}}$$
$$\left.\begin{array}{r}\xRightarrow{f \text{ is c.e.i.}} \big(\mathcal{B} := \{f^{-1}[A] | A \in \mathcal{A}\} \subset RO(X)\big)(X = \cup\, \mathcal{B})\\ X \text{ is nearly countably compact}\end{array}\right\} \Rightarrow$$
$$\left.\begin{array}{r}\Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(X = \cup\, \mathcal{B}^*)\\ f \text{ is surjective}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (f[\mathcal{B}^*] \subset f[\mathcal{B}] = \mathcal{A})(|f[\mathcal{B}^*]| < \aleph_0)$$
$$(Y = f[X] = f[\cup\, \mathcal{B}^*] = \cup_{B \in \mathcal{B}^*} f[B]).$$

Definition 16. A topological space X is said to be:
(a) S-closed [18] (resp. e-closed compact) if every regular closed (resp. e-closed) cover of X has a finite subcover;
(b) Countable S-closed compact [1] (resp. countable e-closed compact) if every countable cover of X by regular closed (resp. e-closed) sets has a finite subcover;
(c) S-Lindelöf [11] (resp. e-closed Lindelöf) if every cover of X by regular closed (resp. e-closed) sets has a countable subcover.

Theorem 17. Let $f: (X, \tau) \to (Y, \sigma)$ be a completely e-irresolute surjection. Then the following statements hold:
(a) If X is S-closed, then Y is e-closed compact;
(b) If X is S-Lindelöf, then Y is e-closed Lindelöf;
(c) If X is countable S-closed compact, then Y is countable e-closed compact.

Proof. $(\boldsymbol{a})$ Let X be S-closed and $\mathcal{A}$ be an e-closed cover of Y.

$$\big(\mathcal{A} \subset eC(Y)\big)(Y = \cup\, \mathcal{A}) \xRightarrow{f \text{ is c.e.i.}}$$
$$\left.\begin{array}{r}\xRightarrow{f \text{ is c.e.i.}} \big(\mathcal{B} := \{f^{-1}[A] | A \in \mathcal{A}\} \subset RC(X)\big)(X = \cup\, \mathcal{B})\\ X \text{ is } S-\text{closed}\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(X = \cup\, \mathcal{B}^*)$$

$\xrightarrow{f\ is\ surjective} (f[\mathcal{B}^*] \subset f[\mathcal{B}] = \mathcal{A})(|f[\mathcal{B}^*]| < \aleph_0)$
$(Y = f[X] = f[\cup \mathcal{B}^*] = \cup_{B\in\mathcal{B}^*} f[B])$.

(***b***) Let X be S-Lindelöf and $\mathcal{A}$ be an e-closed countable cover of Y.

$$\big(\mathcal{A} \subset eC(Y)\big)(Y = \cup \mathcal{A}) \xrightarrow{f\ is\ c.e.i.}$$
$$\left.\begin{array}{r}\xrightarrow{f\ is\ c.e.i.} \big(\mathcal{B} := \{f^{-1}[A] | A \in \mathcal{A}\} \subset RC(X)\big)(X = \cup \mathcal{B}) \\ X\ is\ S - Lindelöf\ closed\end{array}\right\} \Rightarrow$$
$$\Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(|\mathcal{B}^*| \leq \aleph_0)(X = \cup \mathcal{B}^*)$$

$\xrightarrow{f\ is\ surjective} (f[\mathcal{B}^*] \subset f[\mathcal{B}] = \mathcal{A})(|f[\mathcal{B}^*]| \leq \aleph_0)$
$(Y = f[X] = f[\cup \mathcal{B}^*] = \cup_{B\in\mathcal{B}^*} f[B])$.

(***c***) Let X be countable S-closed compact and $\mathcal{A}$ be an e-closed countable cover of Y.

$$\big(\mathcal{A} \subset eC(Y)\big)(|\mathcal{A}| \leq \aleph_0)(Y = \cup \mathcal{A}) \xrightarrow{f\ is\ c.e.i.}$$
$$\left.\begin{array}{r}\big(\mathcal{B} := \{f^{-1}[A] | A \in \mathcal{A}\} \subset RC(X)\big)(|\mathcal{B}| \leq \aleph_0)(X = \cup \mathcal{B}) \\ X\ is\ countable\ S - closed\ compact\end{array}\right\}$$
$$\left.\begin{array}{r}\Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(X = \cup \mathcal{B}^*) \\ f\ is\ surjective\end{array}\right\} \Rightarrow$$
$\Rightarrow (f[\mathcal{B}^*] \subset f[\mathcal{B}] = \mathcal{A})(|f[\mathcal{B}^*]| < \aleph_0)$
$(Y = f[X] = f[\cup \mathcal{B}^*] = \cup_{B\in\mathcal{B}^*} f[B])$.

Definition 18. A topological space X is said to be almost regular [13] (resp. strongly e-regular) if for any regular closed (resp. e-closed) set $F \subset X$ and any point $x \in X \setminus F$, there exists disjoint open (resp. e-open) sets U and V such that $x \in U$ and $F \subset V$.

Theorem 19. If f is completely e-irresolute e-open bijection from an almost regular space X onto a space Y, then Y is strongly e-regular.

Proof. Let $F \in eC(Y)$ and $f(x) = y \notin F$.

$$\left.\begin{array}{r} f(x) = y \notin F \in eC(Y) \xrightarrow{f\ is\ c.e.i.} x \notin f^{-1}[F] \in RC(X) \\ X \text{ is almost regular}\end{array}\right\} \Rightarrow$$
$$\Rightarrow \big(\exists U, V \in eO(X)\big)(x \in U)(f^{-1}[F] \subset V)(U \cap V = \emptyset)$$
$\xrightarrow{f \text{ is } e-\text{open bijection}} \big(f[U], f[V] \in eO(Y)\big)(y = f(x) \in f[U])$
$(F \subset f[V])(f[U] \cap f[V] = \emptyset)$.

Definition 20. A topological space X is said to be:
(a) almost normal [15] if for each closed set A and each regular closed set B such that $A \cap B = \emptyset$, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.
(b) strongly e-normal if for every pair of disjoint e-closed subsets A and B of X, there exist disjoint e-open sets U and V such that $A \subset V$ and $B \subset V$.

Theorem 21. If $f: (X, \tau) \to (Y, \sigma)$ is completely e-irresolute e-open bijection from an almost normal space X into a space Y, then Y is strongly e-normal.

Proof. Let $A, B \in eC(Y)$ and $A \cap B = \emptyset$.

$$\left.\begin{array}{r}\big(A, B \in eC(Y)\big)(A \cap B = \emptyset) \\ f\ is\ c.e.i.\end{array}\right\} \Rightarrow$$
$$\Rightarrow \big(f^{-1}[A], f^{-1}[B] \in RC(X)\big)(f^{-1}[A \cap B] = f^{-1}[\emptyset])$$
$$\left.\begin{array}{r}\Rightarrow \big(f^{-1}[A], f^{-1}[B] \in RC(X)\big)(f^{-1}[A] \cap f^{-1}[B] = \emptyset) \\ RC(X) \subset C(X)\end{array}\right\} \Rightarrow$$
$\Rightarrow \big(f^{-1}[A] \in C(X)\big)\big(f^{-1}[B] \in RC(X)\big)$
$(f^{-1}[A] \cap f^{-1}[B] = \emptyset)$

$$\xrightarrow{X \text{ is almost normal}}$$
$$\left.\begin{array}{r}(\exists U, V \in \tau)(f^{-1}[A] \subset U)(f^{-1}[B] \subset V)(U \cap V = \emptyset) \\ f \text{ is } e - \text{open bijection}\end{array}\right\} \Rightarrow$$
$\Rightarrow \big(f[U], f[V] \in eO(Y)\big)(A \subset f[U])(B \subset f[V])$
$(f[U] \cap f[V] = \emptyset)$.

Definition 22. A topological space (X, τ) is said to be e-T_1 [6] (resp. r-T_1 [5]) if for each pair of distinct points x and y of X, there exist e-open (resp. regular open) sets U_1 and U_2 such that $x \in U_1$ and $y \in U_2$, $x \notin U_2$ and $y \notin U_1$.

Theorem 23. If $f: X \to Y$ is completely e-irresolute injection and Y is e-T_1, then X is r-T_1.

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x, y \in X)(x \neq y) \xrightarrow{f\ is\ injective} f(x) \neq f(y) \\ Y\ is\ e - T_1\end{array}\right\} \Rightarrow$$
$\Rightarrow \Big(\exists F_1 \in eO\big(Y, f(x)\big)\Big)\Big(\exists F_2 \in eO\big(Y, f(y)\big)\Big)(f(x) \notin F_2)$
$(f(y) \notin F_1)$

$\xrightarrow{f\ is\ c.e.i.} \big(f^{-1}[F_1] \in RO(X, x)\big)\big(f^{-1}[F_2] \in RO(X, y)\big)$
$(x \notin f^{-1}[F_2])(y \notin f^{-1}[F_1])$.

Definition 24. A topological space X is said to be e-T_2 [8] (resp. r-T_2 [16]) for each pair of distinct points x and y in X, there exist disjoint e-open (resp. regular open) sets A and B in X such that $x \in A$ and $y \in B$.

Theorem 25. If $f: X \to Y$ is completely e-irresolute injection and Y is e-T_2, then X is r-T_2.

Proof. Let $x, y \in X$ and $x \neq y$.

$$\left.\begin{array}{r}(x, y \in X)(x \neq y) \xrightarrow{f\ is\ injective} f(x) \neq f(y) \\ Y\ is\ e - T_2\end{array}\right\} \Rightarrow$$
$\Rightarrow \Big(\exists A \in eO\big(Y, f(x)\big)\Big)\Big(\exists B \in eO\big(Y, f(y)\big)\Big)(A \cap B = \emptyset)$
$\xrightarrow{f\ is\ c.e.i.} \big(f^{-1}[A] \in RO(X, x)\big)\big(f^{-1}[B] \in RO(X, y)\big)$

$(f^{-1}[A] \cap f^{-1}[B] = \emptyset)$.

Theorem 26. Let Y be an e-T_2 space. If $f: X \to Y$ and $g: X \to Y$ are completely e-irresolute, then the set $A = \{x | f(x) = g(x)\} \in \delta C(X)$.

Proof. Let $x \notin A$.

$$\left.\begin{array}{r} x \notin A \Rightarrow f(x) \neq g(x) \\ Y \text{ is } e - T_2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left(\exists V_1 \in eO\big(Y, f(x)\big)\right)\left(\exists V_2 \in eO\big(Y, g(x)\big)\right)$$
$$(V_1 \cap V_2 = \emptyset)$$
$$\xRightarrow{f \text{ and } g \text{ are c.e.i.}}$$

$$\Rightarrow \big(f^{-1}[V_1] \in RO(X,x)\big)\big(g^{-1}[V_2] \in RO(X,x)\big)$$
$$(f^{-1}[V_1 \cap V_2] = \emptyset)(g^{-1}[V_1 \cap V_2] = \emptyset)$$

$$\Rightarrow \big(U := f^{-1}[V_1] \cap g^{-1}[V_2] \in RO(X,x)\big)(U \cap A = \emptyset)$$

$$\Rightarrow x \notin cl_\delta(A).$$

Then A is δ-closed in X.

Theorem 27. Let Y be an e-T_2 space. If $f: X \to Y$ is completely e-irresolute, then the set $B = \{(x,y) | f(x) = f(y)\} \in \delta C(X \times X)$.

Proof. Let $(x,y) \notin B$.

$$\left.\begin{array}{r} (x,y) \notin B \Rightarrow f(x) \neq f(y) \\ Y \text{ is } e - T_2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left(\exists V_1 \in eO\big(Y, f(x)\big)\right)\left(\exists V_2 \in eO\big(Y, f(y)\big)\right)$$
$$(V_1 \cap V_2 = \emptyset)$$

$$\xRightarrow{f \text{ is c.e.i.}} \big(f^{-1}[V_1] \in RO(X,x)\big)\big(f^{-1}[V_2] \in RO(X,y)\big)$$
$$(f^{-1}[V_1] \cap f^{-1}[V_2] = \emptyset)$$

$$\Rightarrow \left(U := f^{-1}[V_1] \times f^{-1}[V_2] \in RO\big(X \times X, (x,y)\big)\right)$$
$$(U \cap B = \emptyset)$$

$$\Rightarrow (x,y) \notin cl_\delta(B).$$

Then B is δ-closed in $X \times X$.

3 Completely Weakly e-irresolute Functions

Definition 28. A function $f: X \to Y$ is said to be completely weakly e-irresolute (briefly c.w.e.i.) if for each $x \in X$ and for any e-open set V containing $f(x)$, there exists an open set U containing x such that $f[U] \subset V$.

Remark 29. We have the following diagram from Definition 1 and Definition 2 and Definition 28. The converses of these implications are not true in general as shown by the following examples.

c.e.i. → c.w.e.i. → e.i.

Example 30. Let $X = \{a,b,c\}$, $\tau = \big\{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\big\}$ and $\sigma = \big\{\emptyset, X, \{a\}, \{a,b\}\big\}$. Then the identity function $f: (X,\tau) \to (X,\sigma)$ is e-irresolute but not completely weakly e-irresolute.

QUESTION. Is there any completely weakly e-irresolute function which is not completely e-irresolute?

Theorem 31. Let $f: X \to Y$ be a function, then the following statements are equivalent:
(a) f is completely weakly e-irresolute;
(b) $f^{-1}[e - int(B)] \subset int(f^{-1}[B])$ for every subset B of Y;
(c) $f[cl(A)] \subset e - cl(f[A])$ for every subset A of X;
(d) $cl(f^{-1}[B]) \subset f^{-1}[e - cl(B)]$ for every subset B of Y;
(e) $f^{-1}[V]$ is closed in X for each e-closed set V in Y;
(f) $f^{-1}[V]$ is open in X for each e-open set V in Y.

Proof. $(\boldsymbol{a}) \Rightarrow (\boldsymbol{b})$: Let $B \subset Y$ and $x \in f^{-1}[e - int(B)]$.

$$x \in f^{-1}[e - int(B)] \Rightarrow e - int(B) \in eO\big(Y, f(x)\big)$$
$$\overset{(a)}{\Rightarrow} \big(\exists U \in \mathcal{U}(x)\big)(f[U] \subset e - int(B) \subset B)$$
$$\Rightarrow \big(\exists U \in \mathcal{U}(x)\big)(U \subset f^{-1}[B]) \Rightarrow x \in int(f^{-1}[B]).$$

$(\boldsymbol{b}) \Rightarrow (\boldsymbol{c})$: Let $A \subset X$.

$$A \subset X \Rightarrow f[A] \subset Y \Rightarrow Y \setminus f[A] \subset Y \overset{(b)}{\Rightarrow}$$
$$\overset{(b)}{\Rightarrow} f^{-1}[e - int(Y \setminus f[A])] \subset int\big(f^{-1}\big[Y \setminus f[A]\big]\big)$$
$$\Rightarrow X \setminus f^{-1}[e - cl(f[A])] \subset X \setminus cl\big(f^{-1}\big[f[A]\big]\big)$$
$$\Rightarrow cl(A) \subset cl\big(f^{-1}\big[f[A]\big]\big) \subset f^{-1}[e - cl(f[A])]$$
$$\Rightarrow f[cl(A)] \subset e - cl(f[A]).$$

$(\boldsymbol{c}) \Rightarrow (\boldsymbol{d})$: Let $B \subset Y$.

$$B \subset Y \Rightarrow f^{-1}[B] \subset X \overset{(c)}{\Rightarrow}$$
$$\overset{(c)}{\Rightarrow} f[cl(f^{-1}[B])] \subset e - cl\big(f\big[f^{-1}[B]\big]\big) \subset e - cl(B)$$
$$\Rightarrow cl(f^{-1}[B]) \subset f^{-1}[e - cl(B)].$$

$(\boldsymbol{d}) \Rightarrow (\boldsymbol{e})$: Let $V \in eC(Y)$.

$$V \in eC(Y) \Rightarrow V = e - cl(V) \overset{(d)}{\Rightarrow}$$
$$\overset{(d)}{\Rightarrow} cl(f^{-1}[V]) \subset f^{-1}[e - cl(V)] = f^{-1}[V]$$
$$\Rightarrow f^{-1}[V] = cl(f^{-1}[V]) \Rightarrow f^{-1}[V] \in C(X).$$

$(\boldsymbol{e}) \Rightarrow (\boldsymbol{f})$: Obvious.

$(\boldsymbol{f}) \Rightarrow (\boldsymbol{a})$: Let $V \in eO(Y)$ and $x \in f^{-1}[V]$.

$$\big(V \in eO(Y)\big)(x \in f^{-1}[V]) \Rightarrow V \in eO\big(Y, f(x)\big) \overset{(f)}{\Rightarrow}$$
$$\overset{(f)}{\Rightarrow} \big(U := f^{-1}[V] \in \mathcal{U}(x)\big)(f[U] \subset V).$$

Theorem 32. Let $f: (X,\tau) \to (Y,\sigma)$ be a bijective function. Then the following statements are equivalent:
(a) f is completely weakly e-irresolute;
(b) $e - int(f[A]) \subset f[int(A)]$ for every subset of X.

Proof. $(\boldsymbol{a}) \Rightarrow (\boldsymbol{b})$: Let $A \subset X$.

$$A \subset X \Rightarrow X \setminus A \subset X \overset{(a)}{\Rightarrow}$$
$$\left.\begin{array}{r}\overset{(a)}{\Rightarrow} f[X \setminus int(A)] = f[cl(X \setminus A)] \subset e - cl(f[X \setminus A]) \\ f \text{ is bijection}\end{array}\right\} \Rightarrow$$
$$\Rightarrow Y \setminus f[int(A)] \subset Y \setminus e - int(f[A])$$
$$\Rightarrow e - int(f[A]) \subset f[int(A)].$$

$(\boldsymbol{b}) \Rightarrow (\boldsymbol{a})$: Let $A \subset X$.

$$\left.\begin{array}{r}A \subset X \Rightarrow X \setminus A \subset X \overset{(b)}{\Rightarrow} e - int(f[X \setminus A]) \subset f[int(X \setminus A)] \\ f \text{ is bijection}\end{array}\right\}$$
$$\Rightarrow Y \setminus e - cl(f[A]) \subset Y \setminus f[cl(A)]$$
$$\Rightarrow f[cl(A)] \subset e - cl(f[A]).$$

Theorem 33. Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ be any two functions. Then the following statements hold:
(a) If f is c.w.e.i. and g is e-irresolute, then $g \circ f: X \to Z$ is c.w.e.i.
(b) If f is completely continuous and g is c.w.e.i., then $g \circ f$ is c.e.i.
(c) If f is strongly continuous and g is c.e.i., then $g \circ f$ is c.e.i.
(d) If f and g are c.e.i., then $g \circ f$ is c.e.i.
(e) If f is c.e.i. and g is c.w.e.i., then $g \circ f$ is c.e.i.
(f) If f is c.w.e.i. and g is e-continuous, then $g \circ f$ is continuous.
(g) If f is e-continuous and g is c.w.e.i., then $g \circ f$ is e-irresolute.
(h) If f is continuous and g is c.w.e.i., then $g \circ f$ is c.w.e.i.

Proof. Straightforward.

Definition 34. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be almost open [12] if $f[U]$ is open in Y for every regular open set U of X.

Theorem 35. If $f: (X, \tau) \to (Y, \sigma)$ is almost open surjection and $g: (Y, \sigma) \to (Z, \eta)$ is any function such that $g \circ f: (X, \tau) \to (Z, \eta)$ is completely e-irresolute, then g is completely weakly e-irresolute.

Proof. Let $V \in eO(Z)$.

$$\left.\begin{array}{r}V \in eO(Z) \xRightarrow{g \circ f \text{ is c.e.i.}} (g \circ f)^{-1}[V] = f^{-1}\left[g^{-1}[V]\right] \in RO(X) \\ f \text{ is almost open surjection}\end{array}\right\}$$
$$\Rightarrow f\left[f^{-1}\left[g^{-1}[V]\right]\right] = g^{-1}[V] \in \sigma.$$

Theorem 36. If $f: (X, \tau) \to (Y, \sigma)$ is open surjection and $g: (Y, \sigma) \to (Z, \eta)$ is any function such that $g \circ f: (X, \tau) \to (Z, \eta)$ is completely weakly e-irresolute, then g is completely weakly e-irresolute.

Proof. Let $V \in eO(Z)$.

$$\left.\begin{array}{r}V \in eO(Z) \xRightarrow{g \circ f \text{ is c.w.e.i.}} (g \circ f)^{-1}[V] = f^{-1}\left[g^{-1}[V]\right] \in \tau \\ f \text{ is open surjection}\end{array}\right\}$$
$$\Rightarrow f\left[f^{-1}\left[g^{-1}[V]\right]\right] = g^{-1}[V] \in \sigma.$$

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4 References

[1] Abd El-Monsef, M.E.; El-Deep, S.N.; Mohmoud, R.A. β-open sets and β-continuous mappings. Bulletin of the Faculty of Science Assiut University. 1983; 12, 77-90.
[2] Arya, S.P.; Gupta, R. On strongly continuous maps. Kyungpook Mathematical Journal. 1974; 14, 131-143.

[3] Allam, A.A.; Zaharan, A.M.; Hasanein, I.A. On almost continuous, δ-continuous and set connected mappings. Indian Journal of Pure and Applied Mathematics. 1987; 18(11), 991-996.
[4] Ergun, N. On nearly paracompact spaces. Istanbul Universitesi Fen Fakültesi Mecmuası A Serisi. 1980; 45, 65-87.

[5] Ekici, E. Generalization of perfectly continuous, regular set-connected and clopen functions. Acta Mathematica Hungarica. 2005; 107(3), 193-206.

[6] Ekici, E. New forms of contra continuity. Carpathian Journal of Mathematics. 2008; 24(1), 37-45.

[7] Ekici, E. On e-open sets, $\mathcal{DP}^*$-sets and $\mathcal{DPE}^*$-sets and decompositions of continuity. Arabian Journal for Science and Engineering. 2008; 33(2A), 269-281.

[8] Ekici, E. Some weak forms of δ-continuity and e^*-first-countable spaces (submitted)

[9] Levine, N. Strong continuity in topological spaces. The American Mathematical Monthly. 1960; 67, 269.

[10] Long, P.E.; Herrington, L.L. Basic properties of regular-closed functions. Rendiconti del Circolo Matematico di Palermo. 1978; 27, 20-28.

[11] Maio, G.D. S-closed spaces, S-sets and S-continuous functions. Accademia delle Scienze di Torino. 1984; 118, 125-134.

[12] Singal, M.K.; Singal, A. Almost continuous mappings. Yokohama Mathematical Journal. 1968; 16, 63-73.

[13] Singal, M.K.; Arya, S.P. On almost regular spaces, Glasnik Matematicki. 1969; 4(24), 89-99.

[14] Singal, M.K.; Singal, A.R.; Mathur, A. On nearly-compact spaces. Bollettino dell'Unione Matematica Italiana. 1969; 4(2), 702-710.

[15] Singal, M.K.; Arya, S.P. Almost normal and almost completely regular spaces. Glasnik Matematicki. 1970; 5(25), 141-152.

[16] Staum, R. The algebra of bounded continuous functions into a nonarchimedean field. Pacific Journal of Mathematics. 1974; 50, 169-185.

[17] Stone, M.H. Applications of the theory of Boolean rings to general topology. Transaction of the American Mathematical Society. 1937; 41, 373-381.

[18] Thomson, T. S-closed spaces. Proceeding of the American Mathematical Society. 1976; 60, 335-338.

[19] Velicko, N.V. H-closed topological spaces. American Mathematical Society Translations Series 2. 1968; 78(2), 103-118.