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Asymptotics Solutions of a Singularly Perturbed Integro-differential Fractional Order Derivative Equation with Rapidly Oscillating Coefficients

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Abstract

In this paper, the regularization method of S.A. Lomov is generalized to singularly perturbed integro-differential fractional order derivative equation with rapidly oscillating coefficients. The main purpose of the study is to reveal the influence of the integral term and rapidly oscillating coefficients on the asymptotic of the solution of the original problem. To study the influence of rapidly oscillating coefficients on the leading term of the asymptotic of solutions, we consider a simple case, i.e. the case of no resonance (when an entire linear combination of frequencies of a rapidly oscillating cosine does not coincide with the frequency of the spectrum of the limit operator.

Keywords: Singularly perturbed, fractional order derivation, integro-differential equation, rapidly oscillating coefficients, iterative problems.
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1. Introduction

As is known, the study of various issues related to dynamic stability, the properties of media with a periodic structure, and other applied problems is reduced to the study of differential and integro-differential

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equations with rapidly oscillating coefficients. Various methods have been developed for solving such equations, one of which is the splitting method [1, 2, 3, 4] and the regularization method [5, 6]. However, in the splitting method, problems with an integral operator proportional to a small parameter are considered, which significantly narrow the scope of this method. In the well-known works of the regularization method, singularly perturbed differential equations were considered, containing only rapidly oscillating coefficients for unknown functions [7]. A generalization of the idea of the regularization method for integro-differential equations with rapidly oscillating coefficients was studied in [8, 9, 10], for singularly perturbed integral and integro-differential equations with rapidly oscillating inhomogeneities in [11, 12, 13, 14, 15, 16]. With the advent of the concept of consonant fractional derivative [17], singularly perturbed differential equations with fractional derivatives were studied [18]. On the basis of the above results, singularly perturbed integro-differential equations with fractional derivatives began to be systematically studied [19, 20].

In this paper, we generalize this problem to a singularly perturbed integro-differential equation with a fractional derivative and rapidly oscillating coefficients. As in previous works, the main goal of the study is to reveal the influence of the integral term and rapidly oscillating coefficients on the asymptotics of the solution of the original problem. The effect of rapidly oscillating coefficients for equations with fractional derivatives on the asymptotic behavior of solutions is an interesting and nontrivial problem. A simple case is considered, i.e. the case of no resonance (when an entire linear combination of frequencies of a rapidly oscillating cosine does not coincide with the frequency of the spectrum of the limit operator).

An initial problem is considered for a singularly perturbed integro-differential equation:

$$L_\varepsilon z(t,\varepsilon) \equiv \varepsilon z^{(\alpha)} - A(t)z - \varepsilon g(t) cos\frac{\beta(t)}{\varepsilon} z - \int\limits_{t_0}^{t} K(t,s)z(s,\varepsilon)ds = h(t),$$

$$z(t_0,\varepsilon) = z^0,\ \ t\in[t_0,T],\ \ t_0>0 \tag{1.1}$$

for a scalar unknown function $z(t,\varepsilon)$, in which $A(t),\ h(t),\ \beta'(t)>0,\ (\forall t\in[t_0,T]),\ g(t)$ are known functions, $0<\alpha<1$, z^0 – constant number, $\varepsilon>0$ is a small parameter. The problem is posed of constructing a regularized [5,6] asymptotic solution to problem (1.1).

We give the definition of a conformable fractional derivative. Conformable derivative is an extended classical derivative that was proposed in [17]. This derivative has overcome the barriers with other derivatives. It is described as.

Suppose $f:(0,\infty)\to\mathbb{R}$, then conformable derivative of f with order α is given by [17]

$$T_\alpha[f(t)] = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha} - f(t))}{\varepsilon},$$

for $t>0,\ \alpha\in(0,1)$. If $T_\alpha[f(t)]$ exists for$x$ in some interval $(0,\alpha)$ with $\alpha>0$, and $\lim\limits_{t\to 0} T[f(t)]$ also exists, then $T_\alpha[f(0)] = \lim\limits_{x\to 0} T_\alpha[f(t)]$. moreover, if $T_\alpha[f(t)]$ exists on $[0,\infty)$, then f is said to be $\alpha-$differentiable at t. The following properties are associated with the conformable derivative [17]:

- $T_\alpha(af+bg) = aT_\alpha(f) + bT_\alpha(g),\ a,b\in\mathbb{R}$,
- $T_\alpha(t^\mu) = \mu t^{\mu-\alpha},\ \ \mu\in\mathbb{R}$,
- $T_\alpha(fg) = fT_\alpha + gT_\alpha(f)$,
- $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha}{g^2}$,
- If f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dg}$.

According conformable derivative, we rewrite the original fractional order equation (1.1) in the following form:

$$L_\varepsilon z(t,\varepsilon) \equiv \varepsilon t^{(1-\alpha)}\frac{dz}{dt} - A(t)z - \varepsilon g(t) cos\frac{\beta(t)}{\varepsilon} z - \int\limits_{t_0}^{t} K(t,s)z(s,\varepsilon)ds = h(t),$$

$$z(t_0,\varepsilon) = z^0,\ t\in[t_0,T],\ \ t_0>0. \tag{1.2}$$

In problem (1.2), the frequency of the rapidly oscillating cosine is $\beta'(t)$. In what follows, the function $\lambda_1(t) = A(t)$ is called the spectrum of problem (1.2), and functions $\lambda_2(t) = -i\beta'(t),\ \lambda_3(t) = +i\beta'(t)$ spectrum of a rapidly oscillating coefficient.

Let us introduce the following notation:

$\lambda(t) = (\lambda_1(t), ..., \lambda_3(t))$,

$m = (m_1, ..., m_3)-$ multi-index with non-negative components $m_j,\ j = \overline{1,3}$,

$|m| = \sum\limits_{j=1}^{3} m_j-$ multi-index height m,

$(m, \lambda(t)) = \sum\limits_{j=1}^{3} m_j \lambda_j(t)$.

Problem (1) will be considered under the following conditions:

1) $A(t), \beta(t), g(t), h(t) \in C[t_0, T],$ Re $a(t) < 0\ \ \forall t \in [t_0, T],\ \ K(t,s) \in C^\infty(t_0 \le s \le\ \ \le t \le T)$,

2) relations

$$(m, \lambda(t)) = 0,\ (m, \lambda(t)) = \lambda_j(t), j \in \{1, ..., 3\}$$

for all multi-indices m with $|m| \ge 2$ either are not satisfied for any $t \in [t_0, T]$, or are fulfilled identically on the entire segment $[t_0, T]$. In other words, the resonance multi-indices are exhausted by the following sets:

$$\Gamma_0 = \{m :\ (m, \lambda(t)) \equiv 0,\ |m| \ge 2, \forall t \in [t_0, T]\},$$

$$\Gamma_j = \{m :\ (m, \lambda(t)) \equiv \lambda_j(t),\ |m| \ge 2, \forall t \in [t_0, T]\},\ \ j = \overline{1,3}.$$

Thus, we begin to develop an algorithm for constructing a regularized asymptotic solution to [5, 6] problem (1.2).

2. Regularization of the problem (1.2)

Denote by $\sigma_j = \sigma_j(\varepsilon)$ independent of magnitude $\sigma_1 = e^{-\frac{i}{\varepsilon}\beta(t_0)}$, $\sigma_2 = e^{+\frac{i}{\varepsilon}\beta(t_0)}$, and rewrite equation (1.2) as

$$L_\varepsilon z(t,\sigma,\varepsilon) \equiv \varepsilon t^{(1-\alpha)} \frac{dz}{dt} - A(t)z - \varepsilon \frac{g(t)}{2} \left(e^{-\frac{i}{\varepsilon}\int\limits_{t_0}^{t} \beta'(\theta)d\theta} \sigma_1 + e^{+\frac{i}{\varepsilon}\int\limits_{t_0}^{t} \beta'(\theta)d\theta} \sigma_2 \right) z -$$

$$-\int\limits_{t_0}^{t} K(t,s) z(s,\sigma,\varepsilon) ds = h(t),\quad z(t_0,\sigma,\varepsilon) = z^0,\quad t \in [t_0, T]. \tag{2.1}$$

Introduce the regularized variables:

$$\tau_1 = \frac{1}{\varepsilon}\int_0^t \theta^{(\alpha-1)} \lambda_1(\theta) d\theta \equiv \frac{\psi_1(t)}{\varepsilon},\quad \tau_j = \frac{1}{\varepsilon}\int_0^t \lambda_j(\theta) d\theta \equiv \frac{\psi_j(t)}{\varepsilon},\ \ j = 2,3$$

and instead of problem (2.1), consider the problem

$$\tilde{L}_\varepsilon \tilde{z}(t,\tau,\sigma,\varepsilon) \equiv \varepsilon t^{(1-\alpha)} \frac{\partial \tilde{z}}{\partial t} + \lambda_1(t) \frac{\partial \tilde{z}}{\partial \tau_1} + t^{(1-\alpha)} \sum_{j=2}^{3} \lambda_j(t) \frac{\partial \tilde{z}}{\partial \tau_j} - \lambda_1(t)\tilde{z} -$$

$$-\varepsilon \frac{g(t)}{2} \left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2 \right) \tilde{z} - \int\limits_{t_0}^{t} K(t,s) \tilde{z}(s, \frac{\psi(s)}{\varepsilon}, \sigma, \varepsilon) ds = \tag{2.2}$$

$$= h(t),\quad \tilde{z}(t,\tau,\sigma,\varepsilon)|_{t=t_0,\tau=0} = z^0, t \in [t_0, T]$$

for the function $\tilde{z} = \tilde{z}(t, \tau, \sigma, \varepsilon)$, where is indicated: $\tau = (\tau_1, \tau_2, \tau_3)$, $\psi = (\psi_1, \psi_2, \psi_3)$. It is clear that if $\tilde{z} = \tilde{z}(t, \tau, \sigma, \varepsilon)-$ is a solution of the problem (2.2), then the function is $\tilde{z} = \tilde{z}\left(t, \frac{\psi(t)}{\varepsilon}, \sigma, \varepsilon\right)$ an exact solution to problem (2.1), therefore, problem (2.2) is extended with respect to problem (1.2). However, it cannot be considered fully regularized, since it does not regularize the integral

$$J\tilde{z} \equiv J\left(\tilde{z}(t, \tau, \sigma, \varepsilon)|_{t=s, \tau=\psi(s)/\varepsilon}\right) = \int_{t_0}^{t} K(t,s)\tilde{z}(s, \frac{\psi(s)}{\varepsilon}, \varepsilon)ds.$$

For its regularization, we introduce the class M_ε asymptotically invariant with respect to the operator $J\tilde{z}$ (see [5], p. 62]). Consider first the space U of vector functions $z(t, \tau, \sigma)$, representable by the sums

$$z(t, \tau, \sigma) = z_0(t, \sigma) + \sum_{i=1}^{3} z_i(t, \sigma)e^{\tau_i} + \sum_{2 \le |m| \le N_z}^{*} z^m(t, \sigma)e^{(m, \tau)},$$

$$z_i(t, \sigma), z^m(t, \sigma) \in C^\infty\left([t_0, T], \mathbf{C}\right), i = \overline{0, 3}, \;\; 2 \le |m| \le N_z \tag{2.3}$$

where asterisk $*$ above the sum sign indicates that the summation for $|m| \equiv m_1 + m_2 + m_3 \ge 2$ it occurs only on the non-resonant multi-indexes, i.e. $m \notin \bigcup_{j=0}^{3} \Gamma_j$, $\sigma = (\sigma_1, \sigma_2)$.

Note that here the degree N_z of the polynomial $z(t, \tau, \sigma)$ relative to the exponentials e^{τ_j} depends on the element z. In addition, the elements of space U depend on bounded in $\varepsilon > 0$ terms of constants $\sigma_1 = \sigma_1(\varepsilon)$ and $\sigma_2 = \sigma_2(\varepsilon)$ and which do not affect the development of the algorithm described below, therefore, in the record of element (2.3) of this space U, we omit the dependence on $\sigma = (\sigma_1, \sigma_2)$ for brevity. We show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator J.

Before describing the space U, we introduce the sets of resonant multi-indices. We introduce the notations:

$$\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t)),$$

$$(m, \lambda(t)) = \sum_{j=1}^{3} m_j \lambda_j(t), \quad |m| = \sum_{j=1}^{3} m_j,$$

$$\Gamma_0 = \{m : (m, \lambda(t)) \equiv 0, \, \forall |m| \ge 2\},$$

$$\Gamma_j = \{m : (m, \lambda(t)) \equiv \lambda_j(t), \, \forall |m| \ge 2\}, \, j = 1, 2, 3$$

(the set Γ_0 corresponds to a point of the spectrum $\lambda_0(t) \equiv 0$, generated by the integral operator (see [9]).

For the space U we take the space of functions $z(t, \tau, \sigma)$, represented by sums

$$J\tilde{z}(t, \tau, \varepsilon) \equiv \int_{t_0}^{t} K(t,s)z_0(s)ds + \int_{t_0}^{t} K(t,s)z_1(s)e^{\frac{1}{\varepsilon}\int_0^s \theta^{(\alpha-1)}\lambda_1(\theta)d\theta}ds +$$

$$+ \sum_{i=2}^{3} \int_{t_0}^{t} K(t,s)z_i(s)e^{\frac{1}{\varepsilon}\int_0^s \lambda_i(\theta)d\theta}ds + \sum_{2 \le |m| \le N_z}^{*} \int_{t_0}^{t} K(t,s)z^m(s)e^{\frac{1}{\varepsilon}\int_0^s (m, \lambda(\theta))d\theta}ds.$$

Integrating by parts, we write the image of the operator J on the element (2.3) of the space U as a series

$$J\tilde{z}(t, \tau, \varepsilon) = \int_{t_0}^{t} K(t,s)z_0(s)ds +$$

$$+\sum_{i=1}^{3}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[(I_i^{\nu}\left(K(t,s)z_i(s)\right))_{s=t}e^{\tau_i}-(I_i^{\nu}\left(K(t,s)z_i(s(s))\right))_{s=t_0}\right]+$$

$$+\sum_{2\leq\{m\}\leq N_z}^{*}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[(I_m^{\nu}\left(K(t,s)z^m(s)\right))_{s=t}e^{(m,\tau)}-\right.$$

$$\left.-(I_m^{\nu}\left(K(t,s)z^m(s)\right))_{s=t_0}\right]$$

where are indicated:

$$I_1^0=\frac{1}{s^{(\alpha-1)}\lambda_1(s)}\cdot, I_1^{\nu}=\frac{1}{s^{(\alpha-1)}\lambda_i(s)}\frac{\partial}{\partial s}I_1^{\nu-1},$$

$$I_i^0=\frac{1}{\lambda_i(s)}\cdot, I_i^{\nu}=\frac{1}{\lambda_i(s)}\frac{\partial}{\partial s}I_i^{\nu-1},\ \ i=2,3,$$

$$I_m^0=\frac{1}{(m,\lambda(s))}\cdot, I_m^{\nu}=\frac{1}{(m,\lambda(s))}\frac{\partial}{\partial s}I_m^{\nu-1}(\nu\geq 1,|m|\geq 2).$$

It is easy to show (see, for example, [21], pp. 291-294) that this series converges asymptotically for $\varepsilon\to+0$ (uniformly in $t\in[t_0,T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon\to+0$) with respect to the operator J.

We introduce operators $R_\nu: U\to U$, acting on each element $z(t,\tau)\in U$ of the form (5) according to the law:

$$R_0z(t,\tau)=\int_{t_0}^{t}K(t,s)z_0(s)ds, \tag{2.4$_0$}$$

$$R_1z(t,\tau)=\sum_{i=1}^{3}\left[\left(I_i^0\left(K(t,s)z_i(s)\right)\right)_{s=t}e^{\tau_i}-\left(I_i^0\left(K(t,s)z_i(s)\right)\right)_{s=t_0}\right]+$$

$$+\sum_{|m|=2}^{N_z}\left[\left(I_m^0\left(K(t,s)z^m(s)\right)\right)_{s=t}e^{(m,\tau)}-\left(I_m^0\left(K(t,s)z^m(s)\right)\right)_{s=t_0}\right], \tag{2.4$_1$}$$

$$R_{\nu+1}z\left(t,\tau\right)=\sum_{i=1}^{3}(-1)^{\nu}\left[(I_i^{\nu}\left(K(t,s)z_i(s)\right))_{s=t}e^{\tau_i}-(I_i^{\nu}\left(K(t,s)z_i(s)\right))_{s=t_0}\right]+$$

$$+\sum_{2\leq|m|\leq N_z}^{*}(-1)^{\nu}\left[(I_m^{\nu}\left(K(t,s)z^m(s)\right))_{s=t}e^{(m,\tau)}-\right. \tag{2.4$_{\nu+1}$}$$

$$\left.-(I_m^{\nu}\left(K(t,s)z^m(s)\right))_{s=t_0}\right],\nu\geq 1.$$

Now let $\tilde{z}(t,\tau,\varepsilon)$ be an arbitrary continuous function on $(t,\tau)\in G=[t_0,T]\times\{\tau: Re\tau_1<0,\ Re\tau_j\leq 0,\ j=2,3\}$, with asymptotic expansion

$$\tilde{z}(t,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k z_k(t,\tau), y_k(t,\tau)\in U \tag{2.5}$$

converging as $\varepsilon\to+0$ (uniformly in $(t,\tau)\in G$). Then the image $J\tilde{z}\left(t,\tau,\varepsilon\right)$ of this function is decomposed into an asymptotic series

$$J\tilde{z}(t,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k Jz_k(t,\tau)=\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}z_s(t,\tau)|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator J on series of the form (2.5):

$$\tilde{J}\tilde{z} \equiv \tilde{J}\left(\sum_{k=0}^{\infty}\varepsilon^k z_k(t,\tau)\right) = \sum_{r=0}^{\infty}\varepsilon^r\left(\sum_{k=0}^{r}R_{r-k}z_k(t,\tau)\right).$$

Although the operator $\tilde{J}$ is formally defined, its utility is obvious, since in practice it is usual to construct the N-th approximation of the asymptotic solution of the problem (2.1), in which impose only N-th partial sums of the series (2.5), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2.1):

$$L_\varepsilon\tilde{z}(t,\tau,\sigma,\varepsilon) \equiv \varepsilon t^{(1-\alpha)}\frac{\partial\tilde{z}}{\partial t} + \lambda_1(t)\frac{\partial\tilde{z}}{\partial\tau_1} + t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial\tilde{z}}{\partial\tau_j} - \lambda_1(t)\tilde{z} -$$

$$-\tilde{J}\tilde{z} - \varepsilon\frac{g(t)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2)\tilde{z} = h(t), \tilde{z}(t_0,0,\sigma,\varepsilon) = z^0, t\in[t_0,T]. \quad (2.6)$$

3. Iterative problems and their solvability in the space U

Substituting the series (2.5) into (2.6) and equating the coefficients of the same powers of ε, we obtain the following iterative problems:

$$L\,z_0(t,\tau,\sigma) \equiv \lambda_1(t)\frac{\partial z_0}{\partial\tau_1} + t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial z_0}{\partial\tau_j} - \lambda_1(t)z_0 - R_0z_0 = h(t),$$

$$z_0(t_0,0) = z^0; \quad (3.1_0)$$

$$L\,z_1(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_0}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)z_0 + R_1z_0,$$

$$z_1(t_0,0) = 0; \quad (3.1_1)$$

$$L\,z_2(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_1}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)z_1 + R_1z_1 + R_2z_0,$$

$$z_0(t_0,0) = 0; \quad (3.1_2)$$

....

$$L\,z_k(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_{k-1}}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)z_{k-1} + R_kz_0 + \ldots +$$

$$+ \ldots + R_1z_{k-1},\ z_k(t_0,0) = 0, \quad k\geq 1. \quad (3.1_k)$$

Each iterative problem (3.1_k) has the form

$$L\,z(t,\tau,\sigma) \equiv \lambda_1(t)\frac{\partial z}{\partial\tau_1} + t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial z}{\partial\tau_j} - \lambda_1(t)z - R_0z = H(t,\tau,\sigma),$$

$$z(t_0,0) = z^* \quad (3.2)$$

where $H(t,\tau,\sigma) = H_0(t,\sigma) + \sum\limits_{i=1}^{3}H_i(t,\sigma)e^{\tau_i} + \sum\limits_{2\leq|m|\leq N_z}^{*}H^m(t,\sigma)e^{(m,\tau)}$ is the known function of space U, z^*- is the known function of the complex space $\mathbf{C}$, and the operator R_0 has the form (see (2.4_0))

$$R_0z \equiv R_0\left(z_0(t) + \sum_{j=1}^{3}z_j(t)e^{\tau_j} + \sum_{2\leq|m|\leq N_z}^{*}z^m(t)e^{(m,\tau)}\right) \equiv \int_{t_0}^{t}K(t,s)z_0(s)ds.$$

We introduce scalar (for each $x \in [t_0, T]$) product in space U:

$$< u, w > \equiv < u_0(t) + \sum_{j=1}^{3} u_j(t) e^{\tau_j} + \sum_{2 \le |m| \le N_u}^{*} u^m(t) e^{(m,\tau)}, w_0(t) + \sum_{j=1}^{3} w_j(t) e^{\tau_j} +$$

$$+ \sum_{2 \le |m| \le N_w}^{*} w^m(t) e^{(m,\tau)} > \equiv \sum_{j=0}^{3} (u_j(t), w_j(t)) +$$

$$+ \sum_{2 \le |m| \le min(N_u, N_w)}^{*} (u^m(t), w^m(t)),$$

where we denote by $(*, *)$ the usual scalar product in the complex space $\mathbf{C}:\ (u, v) = u \cdot \bar{v}$. Let us prove the following statement.

Theorem 3.1. *Let conditions 1), 2) be fulfilled and the right-hand side* $H(t, \tau, \sigma) = H_0(t, \sigma) + \sum_{j=1}^{3} H_j(t, \sigma) e^{\tau_j} + \sum_{2 \le |m| \le N_H}^{*} H^m(t, \sigma) e^{(m,\tau)}$ *of equation (3.2) belongs to the space* U. *Then the equation (3.2) is solvable in* U, *if and only if*

$$< H(t, \tau), e^{\tau_1} > \equiv 0, \forall t \in [t_0, T]. \tag{3.3}$$

Proof. We will determine the solution of equation (3.2) as an element (2.5) of the space U :

$$z(t, \tau, \sigma) = z_0(t, \sigma) + \sum_{j=1}^{3} z_j(t, \sigma) e^{\tau_j} + \sum_{2 \le |m| \le N_H}^{*} z^m(t, \sigma) e^{(m,\tau)}. \tag{3.4}$$

Substituting (3.4) into equation (3.2), and equating here the free terms and coefficients separately for identical exponents, we obtain the following equations of equations:

$$\lambda_1(t) z_0(t, \sigma) - \int_{t_0}^{t} K(t, s) z_0(s, \sigma) ds = H_0(t, \sigma), \tag{3.5}$$

$$0 \cdot z_1(t, \sigma) = H_1(t, \sigma), \tag{3.5_1}$$

$$\left[t^{(1-\alpha)} \lambda_j(t) - \lambda_1(t)\right] z_j(t, \sigma) = H_j(t, \sigma),\ j = \overline{2, 3}, \tag{3.5_j}$$

$$\left[t^{(1-\alpha)} (m, \lambda(t)) - \lambda_1(t)\right] z^m(t, \sigma) = H^m(t, \sigma), 2 \le |m| \le N_H. \tag{3.5_m}$$

Since the $\lambda_1(t) \ne 0$, the equation (3.5) can be written as

$$z_0(t, \sigma) = \int_{t_0}^{t} \left(-\lambda_1^{-1}(t) K(t, s)\right) z_0(s, \sigma) ds - \lambda_1^{-1}(t) H_0(t, \sigma). \tag{3.5_0}$$

Due to the smoothness of the kernel $\left(-\lambda_1^{-1}(t) K(t, s)\right)$ and heterogeneity $-\lambda_1^{-1}(t) H_0(t, \sigma)$, this Volterra integral equation has a unique solution $z_0(t, \sigma) \in C^\infty([t_0, T], \mathbf{C})$. The equations (3.5_2) and (3.5_3) also have unique solutions

$$z_j(t, \sigma) = \left[t^{(1-\alpha)} \lambda_j(t) - \lambda_1(t)\right]^{-1} H_j(t, \sigma) \in C^\infty([t_0, T], \mathbf{C}), j = 2, 3 \tag{3.6}$$

since $\lambda_2(t), \lambda_3(t)$ not equal to $\lambda_1(t)$. The equation (3.5_1) is solvable in space $C^\infty([t_0,T],\mathbf{C})$ if and only $(H_1(t,\tau), e^{\tau_1}) \equiv 0\,\forall t \in [t_0,T]$ hold. It is not difficult to see that these identities coincide with identities (3.2).

Further, since $t^{(1-\alpha)}(m,\lambda(t)) \neq \lambda_1(t)$, $\forall m \notin \bigcup\limits_{j=1}^{3}\Gamma_j, j=\overline{1,3}$, $|m|\geq 2$, (see (2.3)), the equations (3.5_m) has a unique solution

$$z^m(t) = \left[t^{(1-\alpha)}(m,\lambda(t)) - \lambda_1(t)\right]^{-1} H^m(t) \in C^\infty([t_0,T],\mathbf{C}),$$

$$\forall\,|m|\geq 2,\ m\notin \bigcup_{j=0}^{3}\Gamma_j.$$

Thus, condition (3.3) is necessary and sufficient for the solvability of equations (3.2) in the space U. □

Remark 3.2. *If identity (3.3) holds, then under conditions 1), 2), equation (3.2) has the following solution in the space U :*

$$z(t,\tau,\sigma) = z_0(t,\sigma) + \alpha_1(t,\sigma)e^{\tau_1} + \sum_{j=2}^{3}\left[t^{(1-\alpha)}\lambda_j(t) - \lambda_1(t)\right]^{-1} H_j(t,\sigma)e^{\tau_j} +$$

$$+\sum_{2\leq|m|\leq N_H}^{*}\left[t^{(1-\alpha)}(m,\lambda(t)) - \lambda_1(t)\right]^{-1} H^m(t,\sigma) \tag{3.7}$$

where $\alpha_1(t,\sigma)\in C^\infty([t_0,T],\mathbf{C})$ are arbitrary function, $z_0(t,\sigma)$ is the solution of an integral equation (3.5_0).

4. The unique solvability of the general iterative problem in the space U. Residual term theorem

Let us proceed to the description of the conditions for the unique solvability of equation (3.2) in space U. Along with problem (3.2), we consider the equation

$$Lz(t,\tau) = -t^{(1-\alpha)}\frac{\partial z}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)z + R_1 z + Q(t,\tau) \tag{4.1}$$

where $z = z(t,\tau)$ is the solution (3.7) of the equation (3.2), $Q(z,\tau)\in U$ is the well-known function of the space U. The right part of this equation:

$$G(t,\tau) \equiv -t^{(1-\alpha)}\frac{\partial z}{\partial t} + \frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)z + R_1 z + Q(t,\tau) =$$

$$= -t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j} + \sum_{2\leq|m|\leq N_H}^{*} z^m(t)e^{(m,\tau)}\right] +$$

$$+\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j} + \sum_{2\leq|m|\leq N_H}^{*} z^m(t)e^{(m,\tau)}\right] +$$

$$+R_1\left[z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j} + \sum_{2\leq|m|\leq N_H}^{*} z^m(t)e^{(m,\tau)}\right] + Q(t,\tau)$$

may not belong to space U, if $z = z(t,\tau) \in U$. Indeed, taking into account the form (4.1) of the function $z = z(t,\tau) \in U$, we consider in $G(t,\tau)$, for example, the terms

$$Z(t,\tau) \equiv \frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} z^m(t)e^{(m,\tau)}\right] =$$

$$= \frac{g(t)}{2} z_0(t)\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right) + \sum_{j=1}^{3}\frac{g(t)}{2} z_j(t)\left(e^{\tau_j+\tau_2}\sigma_1 + e^{\tau_j+\tau_3}\sigma_2\right) +$$

$$+\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\sum_{2\le|m|\le N_H}^{*} z^m(t)e^{(m,\tau)}.$$

Here, for example, terms with exponents

$$\begin{gathered} e^{\tau_2+\tau_3} = e^{(m,\tau)}|_{m=(0,1,1)},\ e^{\tau_2+(m,\tau)}\left(if\ m_1 = 0, m_2+1 = m_3\right), \\ e^{\tau_3+(m,\tau)}\left(if\ m_1 = 0,\ m_3+1 = m_2\right), e^{\tau_2+(m,\tau)}\left(if\ m_1 = 0, m_2 = m_3\right), \\ e^{\tau_3+(m,\tau)}\left(if\ m_1 = 0,\ m_2 = m_3\right),\ e^{\tau_2+(m,\tau)}\left(if\ m_1 = 1,\ m_2 = m_3\right), \\ e^{\tau_3+(m,\tau)}\left(if\ m_1 = 1,\ m_2 = m_3\right) \end{gathered} \tag{$*$}$$

do not belong to space U, since multi-indexes

$$(0,n,n)\ \in\Gamma_0,\ (0,n+1,n)\in\Gamma_1,\ (0,n,n+1)\in\Gamma_2,\ \forall n\in N$$

are resonant. Then, according to the well-known theory (see, [5], p. 234), we embed these terms in the space U according to the following rule (see $(*)$):

$$\begin{gathered} \widehat{e^{\tau_2+\tau_3}} = e^0 = 1,\ \widehat{e^{\tau_2+(m,\tau)}} = e^0 = 1\left(if\ m_1 = 0, m_2+1 = m_3\right), \\ \widehat{e^{\tau_3+(m,\tau)}} = e^0 = 1\left(if\ m_1 = 0,\ m_3+1 = m_2\right), \\ \widehat{e^{\tau_2+(m,\tau)}} = e^{\tau_2}\left(if\ m_1 = 0, m_2 = m_3\right), \widehat{e^{\tau_3+(m,\tau)}} = e^{\tau_3}\left(if\ m_1 = 0,\ m_2 = m_3\right), \\ \widehat{e^{\tau_2+(m,\tau)}}\left(if\ m_1 = 1,\ m_2 = m_3\right) = e^{\tau_1},\ \widehat{e^{\tau_3+(m,\tau)}} = e^{\tau_1}\left(if\ m_1 = 1,\ m_2 = m_3\right). \end{gathered}$$

In other words, terms with resonant exponentials $e^{(m,\tau)}$ replaced by members with exponents $e^0, e^{\tau_1}, e^{\tau_2}, e^{\tau_3}$ according to the following rule:

$$\widehat{e^{(m,\tau)}}|_{m\in\Gamma_0} = e^0 = 1,\ \widehat{e^{(m,\tau)}}|_{m\in\Gamma_1} = e^{\tau_1},\ \widehat{e^{(m,\tau)}}|_{m\in\Gamma_2} = e^{\tau_2},\ \widehat{e^{(m,\tau)}}|_{m\in\Gamma_3} = e^{\tau_3}.$$

After embedding, the right-hand side of equation (4.1) will look like

$$\hat{G}(t,\tau) = -t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} z^m(t)e^{(m,\tau)}\right] + Q(t,\tau).$$

As indicated in [5], the embedding $G(t,\tau)\to\widehat{G}(t,\tau)$ will not affect the accuracy of the construction of asymptotic solutions of problem (1.2), since $G(t,\tau)$ at $\tau = \frac{\psi(t)}{\varepsilon}$ coincides with $\widehat{G}(t,\tau)$.

Theorem 4.1. *Let conditions 1), 2) be fulfilled and the right-hand side* $H(t,\tau) = H_0(t) + \sum_{j=1}^{3} H_j(t)e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} H^m(t)e^{(m,\tau)} \in U$ *of equation (3.2) satisfy condition (3.3). Then problem (3.2) under additional conditions*

$$< \widehat{G}(t,\tau), e^{\tau_1} >\equiv 0 \ \forall t \in [t_0, T] \tag{4.2}$$

where $Q(t,\tau) = Q_0(t) + \sum_{k=1}^{3} Q_k(t)e^{\tau_i} + \sum_{2\le|m|\le N_z}^{*} Q^m(t)e^{(m,\tau)}$ *is the known function of space* U, *is uniquely solvable in* U.

Proof. Since the right-hand side of equation (3.2) satisfies condition (3.3), this equation has a solution in space U in the form (3.7), where $\alpha_1(t) \in C^\infty([t_0,T], \mathbf{C})$ is arbitrary function. Submit (4.1) to the initial condition $y(t_0, 0) = y^*$. We get $\alpha_1(t_0, t) = y_*$, where denoted

$$z_* = z^* + \lambda_1^{-1}(t_0)H_0(t_0) - \frac{H_2(t_0)}{t_0{}^{(1-\alpha)}\lambda_2(t_0) - \lambda_1(t_0)} - \frac{H_3(t_0)}{t_0{}^{(1-\alpha)}\lambda_3(t_0) - \lambda_1(t_0)} -$$

$$- \sum_{2\le|m|\le N_H}^{*} \frac{H^m(t_0)}{\left[t_0{}^{(1-\alpha)}\left(m, \lambda(t_0)\right) - \lambda_1(t_0)\right]}.$$

Now we subordinate the solution (3.7) to the orthogonality condition (4.2). We write $G(t,\tau)$ in more detail the right side of equation (3.2):

$$G(t,\tau) \equiv -t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t) + \alpha_1(t)e^{\tau_1} + h_{21}(t)e^{\tau_2} + \right.$$

$$\left. + h_{31}(t)e^{\tau_3} \sum_{2\le|m|\le N_H}^{*} P^m(t)e^{(m,\tau)}\right] +$$

$$+\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\frac{\partial}{\partial t}\left[z_0(t) + \alpha_1(t)e^{\tau_1} + h_{21}(t)e^{\tau_2} + h_{31}(t)e^{\tau_3} + \right.$$

$$\left. + \sum_{2\le|m|\le N_H}^{*} P^m(t)e^{(m,\tau)}\right] + Q(t,\tau) +$$

$$+R_1\left[z_0(t) + \alpha_1(t)e^{\tau_1} + h_{21}(t)e^{\tau_2} + h_{31}(t)e^{\tau_3} + \sum_{|m|=2}^{N_z} P^m(t)e^{(m,\tau)}\right].$$

Embedding this function into space U we will have

$$\hat{G}(t,\tau) \equiv -t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t) + \alpha_1(t)e^{\tau_1} + h_{21}(t)e^{\tau_2} + h_{31}(t)e^{\tau_3} + \right.$$

$$\left. + \sum_{2\le|m|\le N_H}^{*} P^m(t)e^{(m,\tau)}\right] +$$

$$+\left\{\frac{g(t)}{2}z_0(t)e^{\tau_2}\sigma_1 + \frac{g(t)}{2}z_0(t)e^{\tau_3}\sigma_2 + \sum_{j=1}^{3}\frac{g(t)}{2}z_j(t)e^{\tau_j+\tau_2}\sigma_1 + \right.$$

$$+ \sum_{j=1}^{3}\frac{g(x)}{2}z_j(t)e^{\tau_j+\tau_3}\sigma_2 +$$

$$+\sum_{2\le|m|\le N_H}^{*}\frac{g(t)}{2}z^m(t)e^{(m,\tau)+\tau_2}\sigma_1+\sum_{2\le|m|\le N_H}^{*}\frac{g(t)}{2}z^m(t)e^{(m,\tau)+\tau_3}\sigma_2\Bigg\}^{\wedge}+$$

$$+R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+$$

$$+Q(t,\tau)=-t^{(1-\alpha)}\frac{\partial}{\partial t}\Bigg[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+$$

$$=\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\Bigg]+Q(t,\tau)+$$

$$+\frac{g(t)}{2}\{z_0(t)e^{\tau_2}\sigma_1+z_0(t)e^{\tau_3}\sigma_2+\underline{\alpha_1(t)e^{\tau_1+\tau_2}\sigma_1}+h_{21}(t)e^{2\tau_2}\sigma_1+$$

$$+h_{31}(t)e^{\tau_3+\tau_2}\sigma_1+\underline{\alpha_1(t)e^{\tau_1+\tau_3}\sigma_2}+h_{21}(t)e^{\tau_2+\tau_3}\sigma_2+h_{31}(t)e^{2\tau_3}\sigma_2+$$

$$+\underline{\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)+\tau_2}\sigma_1}+\underline{\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)+\tau_3}\sigma_2}\Bigg\}^{\wedge}+$$

$$+R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right].$$

The embedding operation acts only on resonant exponentials, leaving the coefficients unchanged at these exponents. Given that the expression

$$R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]$$

linearly depends on $\alpha_1(t)$ (see formula (2.4_1)), we also conclude that after the embedding operation the function $\hat{G}(t,\tau)$ will linearly depend on the scalar function $\alpha_1(t)$. Given that in condition (4.2) scalar multiplication by functions e^{τ_1}, containing only the exponent e^{τ_1}, in the expression for $\widehat{G}(t,\tau)$ it is necessary to keep only the term with the exponent e^{τ_1}. Then condition (4.2) takes the form

$$<-t^{(1-\alpha)}\frac{\partial}{\partial t}\left(\alpha_1(t)e^{\tau_1}\right)+\left(\sum_{|m^1|=2:m^1\in\Gamma_1}^{N}w^{m^1}\left(\alpha_1(t),t\right)\right)e^{\tau_1}+$$

$$+Q_1(t)e^{\tau_1},e^{\tau_1}>=0\ \forall t\in[t_0,T]$$

where $w^{m^1}(\alpha_1(t),t)$ are some functions linearly dependent on $\alpha_1(t)$. Performing scalar multiplication here, we obtain a linear ordinary differential equation (relative t) for a function $\alpha_1(t)$. Given the initial condition $\alpha_1(t_0)=y_*$, found above, we find uniquely the function $\alpha_1(t)\in C^\infty[t_0,T]$ and therefore, we will uniquely construct a solution to equation (3.2) in the space U. □

As mentioned above, the right-hand sides of iterative problems (3.1_k) (if solved sequentially) may not belong to space U. Then, according to [5] (p. 234), the right-hand sides of these problems must be embedded into U, according to the above rule. As a result, we obtain the following problems:

$$L\,z_0(t,\tau,\sigma)\equiv\lambda_1(t)\frac{\partial z_0}{\partial\tau_1}+t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial z_0}{\partial\tau_j}-\lambda_1(t)z_0-R_0z_0=h(t),$$

$$z_0(t_0,0)=z^0; \tag{$\overline{3.1}_0$}$$

$$Lz_1(t,\tau)=-t^{(1-\alpha)}\frac{\partial z_0}{\partial t}+\left[\frac{g(t)}{2}(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2)z_0\right]^{\wedge}+R_1z_0,$$

$$z_1(t_0,0)=0; \tag{$\overline{3.1}_1$}$$

$$Lz_2(t,\tau)=-t^{(1-\alpha)}\frac{\partial z_1}{\partial t}+\left[\frac{g(t)}{2}(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2)z_1\right]^{\wedge}+R_1z_1+R_2z_0,$$

$$z_2(t_0,0)=0; \tag{$\overline{3.1}_2$}$$

....

$$Lz_k(t,\tau)=-t^{(1-\alpha)}\frac{\partial z_{k-1}}{\partial t}+\left[\frac{g(t)}{2}(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2)z_{k-1}\right]^{\wedge}+R_kz_0+\ldots+$$

$$+\ldots+R_1z_{k-1},\; z_k(t_0,0)=0,\; k\geq 1. \tag{$\overline{3.1}_k$}$$

(images of linear operators $\frac{\partial}{\partial t}$ and R_ν do not need to be embedding in space U, since these operators operate from U to U). Such a change will not affect the construction of the asymptotic solution of the original problem (1.1) (or the equivalent problem (1.2)), so on the restriction $\tau=\frac{\psi(t)}{\varepsilon}$ series of problems ($\overline{3.1}_k$) will coincide with a series of problems (3.1_k) (see [5], pp. 234-235).

Applying Theorems 1 and 2 to iterative problems (3.1_k) (in this case, the right-hand sides $H^{(k)}(t,\tau)$ of these problems are embedded in the space U, i.e. $H^{(k)}(t,\tau)$ we replace with $\hat{H}^{(k)}(t,\tau)\in U$), we find uniquely their solutions in space U and construct series (2.5). Just as in [21], we prove the following statement

Theorem 4.2. *Suppose that conditions 1), 2) are satisfied for equation (1.2). Then, when $\varepsilon\in(0,\varepsilon_0](\varepsilon_0>0$ is sufficiently small), equation (1.2) has a unique solution $z(t,\varepsilon)\in C^1([t_0,T],\mathbf{C})$, in this case, the estimate*

$$||z(t,\varepsilon)-z_{\varepsilon N}(t)||_{C[t_0,T]}\leq c_N\varepsilon^{N+1},\;\; N=0,1,2,\ldots$$

holds true, where $z_{\varepsilon N}(t)$ is the restriction (for $\tau=\frac{\psi(t)}{\varepsilon}$) of the N - partial sum of series (2.5) (with coefficients $z_k(t,\tau)\in U$, satisfying the iteration problems (8_k)), and the constant $c_N>0$ does not depend on $\varepsilon\in(0,\varepsilon_0]$.

5. Construction of the solution of the first iteration problem

Using Theorem 1, we will try to find a solution to the first iteration problem ($\overline{3.1}_0$). Since the right side $h(t)$ of the equation ($\overline{3.1}_0$) satisfies condition (3.3), this equation has (according to (3.7)) a solution in the space U in the form

$$z_0(t,\tau)=z_0^{(0)}(t)+\alpha_1^{(0)}(t)e^{\tau_1} \tag{5.1}$$

where $\alpha_1^{(0)}(t)\in C^\infty([t_0,T],\mathbf{C})$ are arbitrary function, $y_0^{(0)}(t)$ is the solution of the integral equation

$$z_0^{(0)}(t)=\int_{t_0}^{t}\left(-\lambda_1^{-1}(t)K(t,s)\right)z_0^{(0)}(s)ds-\lambda_1^{-1}(t)h(t). \tag{5.2}$$

Subordinating (5.1) to the initial condition $z_0(t_0,0)=z^0$, we have

$$z_0^{(0)}(t_0)+\alpha_1^{(0)}(t_0)=z^0\;\;\Leftrightarrow\;\;\alpha_1^{(0)}(t_0)=z^0-z_0^{(0)}(t_0)\Leftrightarrow$$

$$\Leftrightarrow\alpha_1^{(0)}(t_0)=z^0+\lambda_1^{-1}(t_0)h(t_0).$$

To fully compute the function $\alpha_1^{(0)}(t)$, we proceed to the next iteration problem $(\overline{3.1}_1)$. Substituting into it the solution (5.1) of the equation $(\overline{3.1}_0)$, we arrive at the following equation:

$$Lz_1(t,\tau) = -t^{(1-\alpha)}\frac{\partial}{\partial t}z_0^{(0)}(t) - t^{(1-\alpha)}\frac{\partial}{\partial t}\left(\alpha_1^{(0)}(t)\right)e^{\tau_1} +$$

$$+\left[\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left(z_0^{(0)}(t) + \alpha_1^{(0)}(t)e^{\tau_1}\right)\right]^{\wedge} +$$

$$+\frac{K(t,t)\alpha_1^{(0)}(t)}{t^{(1-\alpha)}\lambda_1(t)}e^{\tau_1} - \frac{K(t,t_0)\alpha_1^{(0)}(t_0)}{t_0{}^{(1-\alpha)}\lambda_1(t_0)} +$$

$$+\sum_{j=2}^{3}\left[\frac{K(t,t)z_j^{(0)}(t)}{\lambda_j(t)}e^{\tau_j} - \frac{K(t,t_0)z_j^{(0)}(t_0)}{\lambda_j(t_0)}\right]$$

(here we used the expression (2.4_1) for $R_1 z(t,\tau)$ and took into account that when $z(t,\tau) = z_0(t,\tau)$ the sum (2.4_1) contains only terms with e^{τ_1}).

Let us calculate

$$M = \left[\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left(z_0^{(0)}(t) + \alpha_1^{(0)}(t)e^{\tau_1}\right)\right]^{\wedge} =$$

$$= \frac{g(t)}{2}\left\{\sigma_1 z_0^{(0)}(t)e^{\tau_2} + \sigma_2 z_0^{(0)}(t)e^{\tau_3} + \sigma_1\alpha_1^{(0)}(t)e^{\tau_2+\tau_1} + \sigma_2\alpha_1^{(0)}(t)e^{\tau_3+\tau_1}\right\}^{\wedge}.$$

Let us analyze the exponents of the second dimension included here for their resonance:

$$e^{\tau_2+\tau_1}|_{\tau=\psi(t)/\varepsilon} = e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(-i\beta'(\theta)+A(\theta))d\theta}, \quad e^{\tau_2+\tau_1}|_{\tau=\psi(t)/\varepsilon} = e^{\frac{1}{\varepsilon}\int_{t_0}^{t}(-i\beta'(\theta)+A(\theta))d\theta},$$

$$-i\beta' + A = \left[\begin{array}{c} 0, \\ A, \\ -i\beta', \\ +i\beta', \end{array}\right. \Leftrightarrow \emptyset; \qquad -i\beta' + A = \left[\begin{array}{c} 0, \\ A, \\ -i\beta', \\ +i\beta', \end{array}\right. \Leftrightarrow \emptyset.$$

Thus, exponents $e^{\tau_2+\tau_1}$ and $e^{\tau_3+\tau_1}$ are not resonant. Then, for solvability, equation (5.1) it is necessary and sufficient that the condition

$$-t^{(1-\alpha)}\frac{\partial}{\partial t}\left(\alpha_1^{(0)}(t)\right) + \frac{K(t,t)\alpha_1^{(0)}(t)}{t^{(1-\alpha)}\lambda_1(t)} = 0$$

is satisfied. Attaching the initial initial condition

$$\alpha_1^{(0)}(t_0) = z^0 + \lambda_1^{-1}(t_0)h(t_0)$$

to this equation, we find

$$\alpha_1^{(0)}(t) = \alpha_1^{(0)}(t_0)e^{\int_{t_0}^{t}\left(\frac{K(\theta,\theta)}{\theta^{2(1-\alpha)}\lambda_1(\theta)}\right)d\theta}$$

and therefore, we uniquely calculate the solution (5.1) of the problem $(\overline{3.1}_0)$ in the space U. Moreover, the main term of the asymptotic of the solution to problem (1.2) has the form

$$z_{\varepsilon 0}(t) = z_0^{(0)}(t) + \left[z^0 + \lambda_1^{-1}(t_0)h(t_0)\right]e^{\int_{t_0}^{t}\left(\frac{K(\theta,\theta)}{\theta^{2(1-\alpha)}\lambda_1(\theta)}\right)d\theta + \frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_1(\theta)d\theta} \tag{5.3}$$

where $z_0^{(0)}(t)$ is the solution of the integrated equation (5.2).

6. Conclusions

From expression (5.2) for $z_{\varepsilon 0}(t)$ it's clear that $z_{\varepsilon 0}(t)$ is independent of rapidly oscillating terms. However, already in the next approximation, their influence on the asymptotic solution of problem (1.2) is revealed.

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