Fundamental Journal of Mathematics and Applications, 6 (1) (2023) 70-77
Research Article

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1235668

An Examination for the Intersection of Two Ruled Surfaces

Emel Karaca

Department of Mathematics, Faculty of Polatlı Science and Arts, Ankara Hacı Bayram Veli University, Ankara, Türkiye

Article Info

Keywords: *Ruled surface, Surface intersection, Tangent bundle*
2010 AMS: *05A15, 11B37, 11B83*
Received: *16 January 2023*
Accepted: *23 March 2023*
Available online: *28 March 2023*

Abstract

In this study, firstly, each natural lift curve of the main curve is corresponded to the ruled surface by exploiting E. Study mapping and the relation among the subset of the tangent bundle of unit 2-sphere, $T\bar{M}$ and ruled surfaces in $\mathbb{R}^3$. Secondly, the intersection of two ruled surfaces, which are obtained by using the relation given above, is examined for the condition of the zero-set of $\lambda(u,v)=0$. Then, all redundant and non-redundant solutions of the zero-set are investigated. Furthermore, the degenerate situations $(u,v)=0$, where the whole plane is degenerated by the zero-set, are denoted. Finally, some examples are given to verify the results.

1. Introduction

In mathematics, the ruled surface, whose parametric expression is given as below, is defined as the set of points drawn by a moving straight line. That is, the parametric representation of the ruled surface ϕ, acquired by the set $\{\vec{k}(u),\vec{q}(u)\}$, is

$$\vec{h}(u,v)=\vec{k}(u)+v\vec{q}(u),\ \ u\in I,\ \ v\in\mathbb{R}$$

for the set $\{\vec{k}(u),\vec{q}(u)\}$. Here, $k=\vec{k}(u)$ is a point and $q=\vec{q}(u)$ is a non-null vector in $\mathbb{R}^3$. Moreover, $\vec{k}(u)$ and $\vec{q}(u)$ are called the base curve and various of the generating lines, respectively, see [1]. From several significant applications of this surface, many mathematicians dealt with this surface in literature. Some of them are as follows: in [2], considering geometric invariants of space curves, a categorization of special developable surfaces under special condition was investigated. In [3], the theory of Mannheim curves was extended to the ruled surfaces. In [4], sectional curvature of ruled surfaces was computed in Minkowski space. In [5], some important properties of special ruled surfaces were investigated according to modified orthogonal frame. In [6], the evolution of several associated type ruled surfaces was defined. Furthermore, the Mannheim offset of developable ruled surface was defined. In [7], a correspondence among unit dual sphere, DS^2, the tangent bundle of the unit 2-sphere, TS^2, and non-null ruled surfaces was mentioned in detail. In [8], taking this correspondence into consideration, the ruled surface was described by using E. Study mapping and the relation between the tangent bundle of unit 2-sphere and non-null ruled surfaces. In [9], the non-null ruled surfaces were introduced by exploiting E. Study mapping and the isomorphism between pseudo-spheres and the tangent bundles of pseudo-spheres in E^3_1. In [10], Frenet vector fields and invariants of timelike ruled surfaces were explored. In [11], the ruled surface according to the Darboux frame was introduced. In [12], in 3-dimensional contact metric manifold, the properties of the ruled surface were defined. In [13], some geometric interpretations for timelike ruled surfaces were examined.
The surface intersection problem has important research fields in differential geometry, geometric modeling, architecture, computer aided design, etc. Several algorithms were considered for the intersection of two surfaces in literature. In [14], the intersection of two ruled surfaces was investigated under some special conditions. In [15], in higher dimension, the problems of curve and surface intersections were formulated. Additionally, the algebraic set to a lower dimensional space was constructed. In [16], an adaptive algorithm was developed for finding the intersection curves. In [17], a boundary method for surface

Email address and ORCID number: **emel.karaca@hbv.edu.tr, 0000-0003-0703-939X(E. Karaca)**
Cite as: E. Karaca, ***An Examination for the Intersection of Two Ruled Surfaces*****, Fundam. J. Math. Appl., 6(1) (2023), 70-77.**

intersection was studied for smooth parametric surfaces defined over rectangular and triangular domains. In [18], a hybrid algorithm for the calculation of the intersection of an algebraic surface and a rational polynomial parametric surface patch was computed.
The theory of curves has substantial field in geometry, engineering, computer modeling mentioned above, etc. The relation among Frenet operators for given two smooth curves opened new research areas for many mathematicians in literature. One of the curves, which is compared with Frenet operators of the main curve, is the natural lift curve. Namely, the natural lift curve, which was firstly encountered in J. A. Thorpe's book in [19], is defined as a smooth curve obtained by the unit tangent vectors of any given smooth curve:
for the curve Γ, $\bar{\Gamma}$ is called the natural lift of Γ on $T\bar{M}$, which provides the following equation:

$$\bar{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}(u)) = (q^{'}(u)|_{\gamma(u)}, \vartheta^{'}(u)|_{\vartheta(u)}).$$

There are some studies about the geometric interpretations about the natural lift curve. Some of them are as follows: in [20], some properties of the natural lift curve were investigated in $\mathbb{R}^3$. In [21], the correspondence between the natural lift curve and its involute curve was given. In [22], the Frenet frames of the natural lift curve and its Bertrand mate were examined. In [23], dual spherical curves of the natural lift curve were denoted in terms of Frenet vector fields. In [24], the condition being the natural lifts of the spherical indicatrix of the curve is an integral curve of the geodesic spray was introduced. In [25], the authors proved that if the natural lifts geodesic spray of spherical indicator curvatures of Mannheim partner curve was an integral curve, Mannheim Curve was obtained. In [26], the condition being the natural lifts of the spherical indicatrix of the evolute curve are an integral curve of geodesic spray was expressed.
There is no research about the intersection of two ruled surfaces generated by the natural lift curves in literature. Therefore, in this paper, using the mentioned isomorphism and using some properties about dual numbers given in [27], we obtain two ruled surfaces acquired by the natural lift curves. Furthermore, we analyze the cases for the intersection of two ruled surfaces by examining the zero-set of $\lambda(u,v) = 0$. Then, we categorize all redundant and non-redundant solutions of the zero-set. Moreover, in the subsections, we consider the degenerate situations $(u,v) = 0$, where the whole plane is degenerated by the zero-set.
This paper is organized as follows: in Section 2, some basic definitions and theorems about the dual numbers and the ruled surfaces acquired by the natural lift curve are mentioned. In Section 3, the intersection of two ruled surfaces acquired by the natural lift curves is examined by calculating the zero-set $\lambda(u,v) = 0$. Additionally, all redundant and non-redundant solutions are denoted. Then, some examples are given to verify the results. In Section 4, obtained results are discussed in detail.

2. Preliminaries

The set of dual numbers is

$$\mathbb{D} = \{X = x + \varepsilon x^*; (x, x^*) \in \mathbb{R} \times \mathbb{R}, \ \ \varepsilon^2 = 0\},$$

where $\vec{x}$ and $\vec{x}^*$ are real and dual parts of $\vec{X}$, respectively. If $\vec{x}$ and $\vec{x}^*$ are vectors in $\mathbb{R}^3$, $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ is called a dual vector. For $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ and $\vec{Y} = \vec{y} + \varepsilon\vec{y}^*$, the basic operations are given as follows:
the addition is

$$\vec{X} + \vec{Y} = (\vec{x} + \vec{y}) + \varepsilon(\vec{x}^* + \vec{y}^*),$$

and the inner product is

$$\langle \vec{X}, \vec{Y} \rangle = \langle \vec{x}, \vec{y} \rangle + \varepsilon(\langle \vec{x}^*, \vec{y} \rangle + \langle \vec{x}, \vec{y}^* \rangle).$$

Moreover, the vector product is

$$\vec{X} \times \vec{Y} = \vec{x} \times \vec{y} + \varepsilon(\vec{x} \times \vec{y}^* + \vec{x}^* \times \vec{y}).$$

The norm of $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$ is

$$|\vec{X}| = \sqrt{\langle \vec{x}, \vec{x} \rangle} + \varepsilon \frac{\langle \vec{x}, \vec{x}^* \rangle}{\sqrt{\langle \vec{x}, \vec{x} \rangle}}, \ \ \vec{x} \neq 0.$$

The dual vector is called unit dual vector, if $|\vec{X}| = 1$. The unit dual sphere, which consists of all unit dual vectors, is given as the following set:

$$DS^2 = \{\vec{X} = \vec{x} + \varepsilon\vec{x}^* \in \mathbb{D}^3 : |\vec{X}| = 1\}. \tag{2.1}$$

For more information about dual vectors, see [27].
Let S^2 be a unit 2-sphere in $\mathbb{R}^3$. The tangent bundle of S^2 is given by

$$TS^2 = \{(q, \vartheta) \in \mathbb{R}^3 \times \mathbb{R}^3 : |q| = 1, \ \ \langle q, \vartheta \rangle = 0\},$$

where "$\langle , \rangle$" is the inner product and "$| , |$" is the norm in $\mathbb{R}^3$, respectively, see [8]. Let $T\bar{M}$ also be a subset of TS^2, defined by

$$T\bar{M} = \{(\bar{q}, \bar{\vartheta}) \in \mathbb{R}^3 \times \mathbb{R}^3 : |\bar{q}| = 1, \ \langle \bar{q}, \bar{\vartheta} \rangle = 0\}. \tag{2.2}$$

Here, $\bar{q}$ and $\bar{\vartheta}$ are the derivatives of q and ϑ, respectively, see [23]. From Eqs. (2.1) and (2.2), the correspondence between the unit dual sphere and the subset of the tangent bundle of unit 2-sphere is given by

$$\begin{array}{ccc} T\bar{M} & \longrightarrow & DS^2, \\ \bar{\Gamma} = (\bar{q}, \bar{\vartheta}) & \longmapsto & \vec{\bar{\Gamma}} = \vec{\bar{q}} + \varepsilon \vec{\bar{\vartheta}}. \end{array}$$

Theorem 2.1 (E. Study mapping)**.** *There is one-to-one relation between the oriented lines in $\mathbb{R}^3$ and the points of DS^2.*

Theorem 2.2. *Let $\bar{\Gamma}(u) = (\vec{\bar{q}}(u), \bar{\vartheta}(u)) \in T\bar{M}$. In $\mathbb{R}^3$, the ruled surface acquired by the natural lift curve $\bar{\Gamma}(u)$ can be expressed by*

$$\bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u),$$

where

$$\beta(u) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u)$$

is the base curve of $\bar{\phi}$.

As a result, the isomorphism among $T\bar{M}$, DS^2 and $\mathbb{R}^3$ can be written by

$$\begin{array}{ccc} T\bar{M} & \longrightarrow & DS^2 \longrightarrow \mathbb{R}^3, \\ \bar{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}(u)) & \longmapsto & \vec{\bar{\Gamma}}(u) = \vec{\bar{q}}(u) + \varepsilon \vec{\bar{\vartheta}}(u) \longmapsto \bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u). \end{array}$$

Here $\bar{\phi}(u,v)$ is the ruled surface in $\mathbb{R}^3$ related to the dual curve $\bar{\Gamma}(u) = \vec{\bar{q}}(u) + \varepsilon \vec{\bar{\vartheta}}(u) \in DS^2$ (or to the natural lift curve $\bar{\Gamma}(u) \in T\bar{M}$), see [23].

3. Some characterizations for the intersection of two ruled surfaces

Let $\bar{\phi}_1(u,s)$ and $\bar{\phi}_2(v,t)$ be the ruled surfaces acquired by the natural lift curves $\bar{\alpha}(u) = (\alpha_1(u), \alpha_1^*(u))$ and $\tilde{\alpha}(v) = (\alpha_2(v), \alpha_2^*(v))$, where $\bar{\alpha}(u)$ and $\tilde{\alpha}(v)$ on $T\bar{M}$, respectively. Considering the isomorphism mentioned above, the ruled surface generated by $\bar{\alpha}(u) = (\alpha_1(u), \alpha_1^*(u))$ is

$$\bar{\phi}_1(u,s) = \alpha_1(u) \times \alpha_1^*(u) + s\alpha_1^*(u),$$

where the base curve is

$$C(u) = \alpha_1(u) \times \alpha_1^*(u).$$

Moreover, the ruled surface generated by $\tilde{\alpha}(v) = (\alpha_2(v), \alpha_2^*(v))$ is

$$\bar{\phi}_2(v,t) = \alpha_2(v) \times \alpha_2^*(v) + t\alpha_2^*(v),$$

where the base curve is

$$D(v) = \alpha_2(v) \times \alpha_2^*(v).$$

Let L_1 and L_2 denote the rullings of $\bar{\phi}_1$ and $\bar{\phi}_2$, respectively. As $\bar{\phi}_1$ and $\bar{\phi}_2$ intersect, we write

$$\bar{\phi}_1(u,s) = \bar{\phi}_2(v,t).$$

That is,

$$C(u) - D(v) = -s\alpha_1^*(u) + t\alpha_2^*(v). \tag{3.1}$$

It is obvious that $C(u) - D(v)$ is the linear combination of $\alpha_1^*(u)$ and $\alpha_2^*(v)$. In this equation, $C(u) - D(v)$, $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are linearly dependent. Hence, the following condition is satisfied:

$$\lambda(u,v) = det(\alpha_1^*(u), \alpha_2^*(v), C(u) - D(v)) = 0.$$

Now, we will investigate the solutions of the determinant given as above in detail.

3.1. Redundant and non-redundant solutions

The condition of being $\lambda(u,v)=0$ is a desired condition. However, it is not sufficient condition for examining the intersection of two rulling lines L_1 and L_2. Some redundant points, which do not related to real intersection points of these ruled surfaces, could be contained by the solution set of $\lambda(u,v)=0$. Hence, we categorize all probabilities for redundant solutions as below: the solutions of $\lambda(u,v)=0$ denote the linear dependency of $C(u)-D(v), \alpha_1^*(u)$ and $\alpha_2^*(v)$:

$$c_1\alpha_1^*(u)+c_2\alpha_2^*(v)+c_3(C(u)-D(v))=0 \tag{3.2}$$

for some non-zero constants c_1,c_2,c_3. There are two conditions for the solution of Eq. (3.2):
(i) if $c_3\neq 0$, then Eq. (3.1) is obtained. We say that two rulling lines intersect. Furthermore, there has not been any redundant solution of $\lambda(u,v)=0$ under this condition.
(ii) If $c_3=0$, we get

$$\alpha_1^*(u)=-\frac{c_1}{c_2}\alpha_2^*(v)$$

for $c_1\neq 0$ and $c_2\neq 0$. It is deduced that $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are parallel or opposite.
Additionally, (u,v) provides $\lambda(u,v)=0$ without checking the intersection of L_1 and L_2. In this way, (u,v) is called redundant if $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are parallel or opposite. However, the related rullings do not coincide. The condition of being parallel or opposite for $\alpha_1^*(u)$ and $\alpha_2^*(v)$ can be expressed by employing the zero-set of another function as below:

$$\Delta(u,v)=\|\alpha_1^*(u)\times\alpha_2^*(v)\|^2=\|\alpha_1^*(u)\|^2\|\alpha_2^*(v)\|^2-\langle\alpha_1^*(u),\alpha_2^*(v)\rangle^2=0.$$

The zero set of $\Delta(u,v)=0$ is a subset of the zero set of $\lambda(u,v)=0$. Furthermore, L_1 and L_2 overlap each other iff $\Delta(u,v)=0$ and there exist the following equations:

$$\begin{aligned}
\delta_1(u,v) &= \|\alpha_1^*(u)\times(C(u)-D(v))\|^2\\
&= \|\alpha_1^*(u)\|^2\|C(u)-D(v)\|^2-\langle\alpha_1^*(u),C(u)-D(v)\rangle^2=0,\\
\delta_2(u,v) &= \|\alpha_2^*(v)\times(C(u)-D(v))\|^2\\
&= \|\alpha_2^*(v)\|^2\|C(u)-D(v)\|^2-\langle\alpha_2^*(v),C(u)-D(v)\rangle^2=0.
\end{aligned}$$

$\Delta(u,v)=\delta_1(u,v)=\delta_2(u,v)=0$ iff two rulling lines L_1 and L_2 overlap each other.
As a result, $\Delta(u,v)+\delta_1(u,v)+\delta_2(u,v)=0$, since $\Delta(u,v),\delta_1(u,v),\delta_2(u,v)\geq 0$. Consequently, the solution of $\lambda(u,v)=0$ is redundant iff $\Delta(u,v)=0$ and $\Delta(u,v)+\delta_1(u,v)+\delta_2(u,v)\neq 0$.

3.2. Birational correspondence

As L_1 and L_2 intersect, s and t are expressed as rational bivariate functions u and v. Calculating the inner product of Eq. (3.1) with $\alpha_1^*(u)$ and $\alpha_2^*(v)$, we have

$$\begin{pmatrix} \|\alpha_1^*(u)\|^2 & -\langle\alpha_1^*(u),\alpha_2^*(v)\rangle\\ -\langle\alpha_1^*(u),\alpha_2^*(v)\rangle & \|\alpha_2^*(v)\|^2\end{pmatrix}\begin{pmatrix} s\\ t\end{pmatrix}=\begin{pmatrix}\langle\alpha_1^*(u),D(v)-C(u)\rangle\\ \langle\alpha_2^*(v),C(u)-D(v)\rangle\end{pmatrix}.$$

As $\Delta(u,v)$ is not equal to 0, this matrix equation becomes non-singular. Furthermore, there exists unique rational solutions of $s(u,v)$ and $t(u,v)$. So, we find

$$\begin{aligned}
s(u,v) &= \frac{\|\alpha_2^*(v)\|^2\langle\alpha_1^*(u),D(v)-C(u)\rangle+\langle\alpha_1^*(u),\alpha_2^*(v)\rangle\langle\alpha_2^*(v),C(u)-D(v)\rangle}{\|\alpha_1^*(u)\|^2\|\alpha_2^*(v)\|^2-\langle\alpha_1^*(u),\alpha_2^*(v)\rangle^2},\\
t(u,v) &= \frac{\|\alpha_1^*(u)\|^2\langle\alpha_2^*(v),C(u)-D(v)\rangle+\langle\alpha_1^*(u),\alpha_2^*(v)\rangle\langle\alpha_1^*(u),D(v)-C(u)\rangle}{\|\alpha_1^*(u)\|^2\|\alpha_2^*(v)\|^2-\langle\alpha_1^*(u),\alpha_2^*(v)\rangle^2}.
\end{aligned}$$

In this situation, as $\Delta(u,v)\approx 0$, the calculation of $s(u,v)$ and $t(u,v)$ is unstable numerically. In this situation, the squared distance $\delta(u,v)$ is measured by using two parallel ruling lines and distinguish the lines if their squared distance is bigger than a particular contribution: $\delta(u,v)\geq\varepsilon^2$, where $\delta(u,v)$ is expressed as $\frac{\delta_1(u,v)}{\|\alpha_1^*(u)\|^2}$, the squared distance between $D(v)$ and L_1 or $\frac{\delta_2(u,v)}{\|\alpha_2^*(v)\|^2}$, the squared distance between $C(u)$ and L_2.
Assume that $\tilde{C}$, which has the projection C onto the uv-plane, is a section of the intersection curve of $\bar{\phi}_1$ and $\bar{\phi}_2$, respectively. In this situation, these ruled surfaces do not coincide. If $\tilde{C}$ is a connected curve section, C is accepted as a connected section of $\lambda(u,v)=0$. Generally, the opposite condition is not true. There is not any unique solution for $s(u,v)$ and $t(u,v)$ as a connected curve section C of $\lambda(u,v)=0$ consists of a point (u,v) of $\Delta(u,v)=0$. Moreover, in some degenerate cases, the intersection curve could be empty or a single point, while the zero-set of $\lambda(u,v)\equiv 0$ is the all plane. In these conditions, there is not any relation between an intersection curve $\tilde{C}$ and C of $\lambda(u,v)=0$. Genarally, with the exception of the following conditions, there exists birational correspondence between $\tilde{C}$ and C on $\lambda(u,v)=0$:

(i) parallel ruling lines,
(ii) degenerate cases
(iii) peaks and self-intersections.

Assume that $\bar{\phi}_1$ has an peak P such that $\bar{\phi}_1(u,s) = P$, for $u_0 \leq u \leq u_1$, and P lies on the other ruled surface $\bar{\phi}_2(v_0,t_0)$. Moreover, the zero-set of $\lambda(u,v) = 0$ includes a line section: $\{(u,v_0) : u_0 \leq u \leq u_1\}$. The whole line section is related to a single point P in the intersection of $\bar{\phi}_1$ and $\bar{\phi}_2$. Q lies on the intersection curve: $Q = \bar{\phi}_1(u_1,s_1) = \bar{\phi}_1(u_2,s_2) = \bar{\phi}_2(v_1,t_1)$. The same intersection point Q relates to the two different solutions (u_1,v_1) and (u_2,v_1). Therefore, there is not birational relation between C and $\tilde{C}$ in these situations.
All singular points of $\bar{\phi}_1(u,s)$ are on the striction curve:

$$\bar{C}(u) = (\alpha_1(u) \times \alpha_1^*(u)) - \frac{\langle(\alpha_1(u) \times \alpha_1^*(u))', (\alpha_1^*)'(u)\rangle}{\langle(\alpha_1^*)'(u), (\alpha_1^*)'(u)\rangle}.$$

If the curve $\bar{C}$ degenerates into a point, this point is the peak of a conical surface $\bar{\phi}_1(u,s)$. Let $\bar{\phi}_1(u,s)$ be noncylindrical ruled surface. Then, all singular points of $\bar{\phi}_1(u,s)$ could be distinguished along the striction curve $\bar{C}$ by controlling the condition as follows:

$$\langle \bar{C}'(u) \times \alpha_1(u), (\alpha_1)'(u)\rangle = 0.$$

Self-intersection points of $\bar{\phi}_1$ could be distinguished by the intersection of $\bar{\phi}_1(u,s)$ with $\bar{\phi}_1(v,t)$. $u - v = 0$, which is described as the diogonal line, is included in the zero-sets of all variate functions. By eliminating the diagonal line from these zero-sets, the self-intersection of $\bar{\phi}_1(u,s)$ could be characterized.

Example 3.1. *Let us consider* $\alpha_1(u) = (0,0,1)$ *and the vector* $\alpha_1^*(u) = (\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}, 0)$ *in* $\mathbb{R}^3$. *Since* $\|\alpha_1(u)\| = 1$ *and* $\langle\alpha_1(u), \alpha_1^*(u)\rangle = 0$, *the natural lift curve* $\alpha(u) = (\alpha_1(u), \alpha_1^*(u)) \in T\bar{M}$. *Then, the ruled surface corresponding to the natural lift curve* $\alpha(u) = (\alpha_1(u), \alpha_1^*(u))$ *is given as*

$$\bar{\phi}_1(u,s) = \left(\frac{-2u}{1+u^2}, \frac{u^2-1}{1+u^2}, 0) + s(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}, 0\right)$$

where the base curve is

$$C(u) = \left(\frac{-2u}{1+u^2}, \frac{u^2-1}{1+u^2}, 0\right).$$

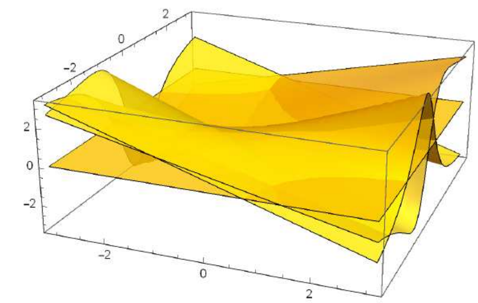

Figure 3.1: The ruled surface $\bar{\phi}_1(u,s)$ acquired by $\bar{\alpha}(u)$

Let us consider another vector couple as $\alpha_2(v) = (\frac{-2v}{1+v^2}, \frac{1-v^2}{1+v^2}, 0)$ *and the vector* $\alpha_2^*(v) = (\frac{1-v^2}{1+v^2}, \frac{2v}{1+v^2}, 0)$ *in* $\mathbb{R}^3$. *Since* $\|\alpha_2(v)\| = 1$ *and* $\langle\alpha_2(v), \alpha_2^*(v)\rangle = 0$, *the natural lift curve* $\tilde{\alpha}(v) = (\alpha_2(v), \alpha_2^*(v)) \in T\bar{M}$. *Then, the ruled surface corresponding to the natural lift curve* $\tilde{\alpha}(v) = (\alpha_2(v), \alpha_2^*(v))$ *is given as*

$$\bar{\phi}_2(v,t) = \left(0, 0, \frac{-v^4-2v^2-1}{v^4+2v^2+1}) + t(\frac{1-v^2}{1+v^2}, \frac{2v}{1+v^2}, 0\right)$$

where the base curve is

$$D(v) = \left(0, 0, \frac{-v^4-2v^2-1}{v^4+2v^2+1}\right).$$

If these ruled surfaces intersect, we get

$$\bar{\phi}_1(u,s) = \bar{\phi}_2(v,t).$$

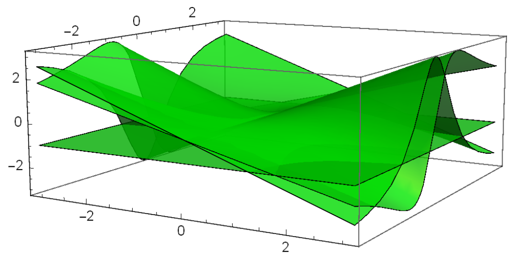

Figure 3.2: The ruled surface $\bar{\phi}_2(v,t)$ generated by $\bar{\alpha}(v)$

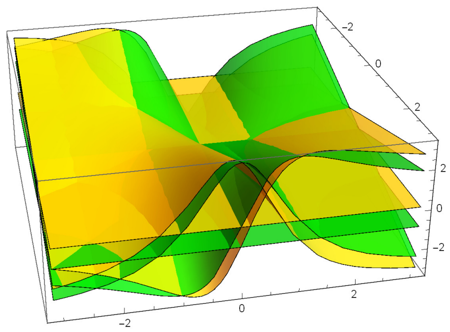

Figure 3.3: The intersection of $\bar{\phi}_1(u,s)$ and $\bar{\phi}_2(v,t)$

Considering these ruled surfaces, we calculate

$$
\begin{aligned}
\lambda(u,v) &= \frac{2(v-u)(1-uv)}{(1+u^2)(1+v^2)},\\
\Delta(u,v) &= \left\| \left(0,0,\frac{2(v-u)(1-uv)}{(1+u^2)(1+v^2)}\right) \right\|^2,\\
\delta_1(u,v) &= \left\| \left(\frac{2u}{1+u^2},\frac{1-u^2}{u^2+1},\frac{u^4+4u^2-1}{u^4+2u^2+1}\right) \right\|^2,\\
\delta_2(u,v) &= \left\| \left(\frac{2v}{1+v^2},\frac{v^2-1}{v^2+1},\frac{(1-v^2)(u^2-1)+4uv}{(u^2+1)(v^2+1)}\right) \right\|^2.
\end{aligned}
$$

The real solutions of $\lambda(u,v)=0$ and $\Delta(u,v)=0$ represents a planar curve $(v-u)(1-uv)=0$. Hence, the zero-set of $\Delta(u,v)=0$ is the subset of $\lambda(u,v)=0$. It is simply to control that $\Delta(u,v)+\delta_1(u,v)+\delta_2(u,v)>0$ for all (u,v). Thus, all solutions of $\Delta(u,v)=0$ are redundant solutions of $\lambda(u,v)=0$.
The non-redundant solution of $C=\{(u,v):\lambda(u,v)=0,\Delta(u,v)\neq 0\}$ is comprised of four components in the $uv-$ plane, given as follows:

$$
\begin{aligned}
C_1 &= \{(u,v): uv=1, u<-1\},\\
C_2 &= \{(u,v): uv=1, -1<u<0\},\\
C_3 &= \{(u,v): uv=1, 0<u<1\},\\
C_4 &= \{(u,v): uv=1, u>1\}.
\end{aligned}
$$

Hence, the intersection curve contains four connected components. Moreover, the limit points of C are $(1,1)$ and $(-1,-1)$. However, they are not included in the solution set C. In a small neighborhood of these limit points, the parameter values of $s(u,v)$ and $t(u,v)$ diverge to $\mp\infty$.

3.3. Degenerate cases

In some situations, the whole plane is degenerated by the solution set. These situations cover all probabilities for degenerate cases of $\lambda(u,v)=0$. The exceptions occurs when two ruled surfaces coincide. We denote that the two ruled surfaces, which overlap each other, are planes or rational bilinear surfaces as below. Therefore, the determination of all degenerate cases may be decreased for categorizing the special types of input surfaces: whether the surface is a plane, cylinder, cone, quadric, etc. As $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are parallel or opposite for all pairs of (u,v), $\bar{\phi}_1(u,s)$ and $\bar{\phi}_2(v,t)$ are cylindrical surfaces that are parallel to each other. In the contrary case, $\alpha_1^*(u)$ and $\alpha_2^*(v)$ are parallel or opposite for the couple of (u,v) providing the condition of $\Delta(u,v) = 0$. Generally, $\Delta(u,v) = 0$ could not be space-filling curve. Thus, there exists an area $[u_a,u_b] \times [v_a,v_b]$, where $\Delta(u,v) \neq 0$. For $\lambda(u,v) = 0, u_a \leq u \leq u_b, v_a \leq v \leq v_b$, we write

$$C(u) - D(v) = -s(u,v)\alpha_1^*(u) + t(u,v)\alpha_2^*(v).$$

From this equation, we conclude that each ruling line L_1 of $\bar{\phi}_1$ intersects with all other rulling lines L_2 of $\bar{\phi}_2$ and the converse is true. There are three different cases:
first of all is that there is a couple of lines L_1 and L_2 which intersect at P. Each ruling line L_2 of $\bar{\phi}_2$ coincides with both L_1 and L_2. There are two subcases to examine:
if there are infinitely many L_2 running through the point P, $\bar{\phi}_2$ must be a conical surface at P.
Otherwise, infinitely many lines L_2 must be contained in the plane identified by L_1 and L_2. Then, a plane is degenerated by the whole surface $\bar{\phi}_2$.
Similarly, the surface type of $\bar{\phi}_1$ has also been identified. If $\bar{\phi}_2$ is a non-planar conical surface, all ruling lines L_1 of $\bar{\phi}_1$ run through the apex P. Hence, $\bar{\phi}_1$ becomes a conical surface.
Otherwise, $\bar{\phi}_2$ is a plane. All ruling lines L_1 of $\bar{\phi}_1$ are the subset of the plane for $\bar{\phi}_2$. Hence, $\bar{\phi}_1$ and $\bar{\phi}_2$ degenerate into the same plane.
Second of all is that there is a pair of parallel lines L_1 and L_2.
Then, there has been a unique plane identified by these two parallel lines. All ruling lines L_2 of $\bar{\phi}_2$ are subsets of the plane. Therefore, the whole surface $\bar{\phi}_2$ degenerates into the plane. Likewise, $\bar{\phi}_1$ has also been contained in the same plane.
Third of all is that any two different lines L_1 and L_2 are skew. Furthermore, any two different lines L_1 and L_2 are also skew. (Otherwise, we will result in first or second cases given above examined before.) Assume that T is the intersection point of L_1 and L_2. Let us consider as $\bar{\phi}_1 = T$ for all $(u,v) \in [u_a,u_b] \times [v_a,v_b]$. Then $\bar{\phi}_1$ and $\bar{\phi}_2$ coincide. Therefore, $\bar{\phi}_1$ and $\bar{\phi}_2$ indicate the same surface. $\bar{\phi}_1$ generates a rational bilinear surface with special conditions of u and v. Additionally, this surface are considered as a quadric surface.

4. Conclusion

In this paper, different from literature, the ruled surfaces acquired by the natural lift curves are defined by using E. Study mapping and the isomorphism between the subset of the unit tangent bundle of unit 2-sphere, $T\bar{M}$, and unit dual sphere, DS^2. Taking the the intersection of ruled surfaces obtained in this way into consideration, the cases for the intersection are investigated by exploiting $\lambda(u,v) = 0$. Therefore, redundant and non-redundant solutions are scrunutized under some conditions. Moreover, being to be degenerate conditions are denoted in detail. Then, obtained results are illustrated by some significiant examples.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: The article has a single author. The author has seen the final version of the article and approved its publication.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] M. Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1976.
[2] S. Izumiya, N. Takeuchi, *New special curves and developable surfaces*, Turk J Math, **28** (2004), 531-537.

[3] K. Orbay, E. Kasap, İ. Aydemir, *Mannheim offsets of ruled surfaces*, Math. Prob. Eng., (2009), doi: 10.155/2009/160917.
[4] S. Ersoy, M. Tosun, *Sectional curvature of principal ruled surfaces in Minkowski space*, Erzincan University Journal of Science and Technology, **13** (Special Issue-I) (2020), 83-91.
[5] K. Eren, H.H. Kosal, *Evolution of space curves and the special ruled surfaces with modified orthogonal frame*, AIMS Mathematics, **5**(3) (2020), 2027–2039.
[6] A. Kelleci, K. Eren, *On evolution of some associated type ruled surfaces*, Math. Sci. Appl. E-Notes, **8**(2) (2020), 178-186.
[7] B. Karakaş, H. Gündoğan, *A relation among* DS^2, TS^2 *and non-cylindirical ruled surfaces*, Math. Commun., **8** (2003), 9-14.
[8] F. Hathout, M. Bekar, Y. Yaylı, *Ruled surfaces and tangent bundle of unit 2-sphere*, Int. J. Geom. Methods Mod. Phys., **2** (2017).
[9] M. Bekar, F. Hathout, Y. Yaylı, *Tangent bundle of pseudo-sphere and ruled surfaces in Minkowski 3-space*, Gen. Lett. Math., **5** (2018), 58-70.
[10] M. Önder, H. Hüseyin Uğurlu, *Frenet frames and invariants of timelike ruled surfaces*, Ain Shams Eng J., **4** (2013), 507-513.
[11] G. Y. Şentürk, S. Yüce, *Characteristic properties of the ruled surface with Darboux frame in* $\mathbb{R}^3$, Kuwait J. Sci., **2** (2015), 14-33.
[12] M. K. Karacan, N. Yüksel, H. Ikiz, *On ruled surface in 3-dimensional almost contact metric manifold*, Int. J. Geom. Methods Mod. Phys., **5** (2017).
[13] M. Önder, Z. Ekinci, *On the kinematic interpretion of timelike ruled surfaces*, Int. J. Geom. Methods Mod. Phys., **12** (2015).
[14] H. S. Heo, M. S. Kim, G. Elber, *The intersection of two ruled surfaces*, Comput Aided Des, **31** (1999), 33-50.
[15] D. Manocha, J. F. Canny, *A new approach for surface intersection*, Int. J. Comput. Geom. and Appl., **1**(4) (1991), 491-516.
[16] M.J. Pratt, *Surface/surface intersection problems*, In J.A. Gregory, editor, The Mathematics of Surfaces II, Oxford, 1986.
[17] R.E. Barnhill, S.N. Kersey, *A marching method for parametric surface/surface intersection*, Comput Aided Geom Des, **7** (1990), 257-280.
[18] G.A. Kriezis, P.V. Prakash, N.M. Patrikalakis, *A method for intersecting algebraic surfaces with rational polynomial patches*, Comput Aided Des, **22**(10) (1990), 645-654.
[19] J.A. Thorpe, *Elementary Topics in Differential Geometry*, Springer Verlag, New York, Heidelberg-Berlin, 1979.
[20] E. Ergün, M. Çalışkan, *On natural lift of a curve*, Pure Math. Sci., **2** (2012), 81-85.
[21] E. Ergün, M. Çalışkan, *On the natural lift curve and the involute curve*, J of Sci. and Arts, **11** (2018), 869-890.
[22] E. Ergün, M. Çalışkan, *On the natural lift curve and the Bertrand mate*, J of Sci. and Arts, **4** (2018), 75-94.
[23] M. Çalışkan, E. Karaca, *Dual spherical curves of natural lift curve and tangent bundle of unit 2-sphere*, J of Sci. and Arts, **3** (2019), 561-574.
[24] S. Şenyurt, *Geodesic curvatures and natural lifts of spherical indicators of timelike Mannheim curves*, Hadronic J., **39**(1) (2016), 41–66.
[25] S. Şenyurt, S. Demet, *Timelike-spacelike Mannheim pair curves spherical indicators geodesic curvatures and natural lifts*, International J. Math. Combin., **2** (2015), 32–54.
[26] S. Şenyurt, S. Demet, *Natural lifts and curvatures arc lengths of the spherical indicatries of the evolute curve in* E^3, Int. Math. Forum, **9**(18) (2014), 857–869.
[27] I. S. Fischer, *Dual-Number Methods in Kinematics*, Statics and Dynamics CRC Press, Boca Raton, London, New York, Washington DC, 1999.