On the Concept of Limit Inferior and Limit Superior

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Abstract

This paper is concerned with the giving a generalization of statistically limit inferior and statistically limit superior defined in [15]. Properties of $\Delta\text{-}\limsup_{t\to\infty} f(t)$ and $\Delta\text{-}\liminf_{t\to\infty} f(t)$ is given for a function defined on time scale $\mathbb{T}$.

Keywords— Statistical limit, Cluster point, Limit point, Δ-convergence.

1 Introduction

The theory of statistical convergence has been introduced in [1]. This concept become useful tool for some fundamental subjects of mathematics the last half of the century such as number theory [4], [5], trigono-metric series [6], summability theory [7], measure theory [8], optimization theory [9] and approximation theory [10]. Fridy progressed with the concept of statistically Cauchy sequence in [2] and proved that it is equivalent to statistical convergence. Besides in [3], the notion of the statistical limit point is defined by him.

The theory of time scales was first constructed by Hilger in his Ph. D. thesis in [11]. The concept of time scale is based on the aspect of unite discrete analysis and continuous analysis. The time scale $\mathbb{T}$ is an arbit-rary nonempty closed subset of the real numbers $\mathbb{R}$. In fact, $\mathbb{T}$ is a complete metric space with the usual met-ric. Throughout this paper we consider a time scale $\mathbb{T}$ with the topology that inherits from the real numbers with the standart topology. For detailed information about time scale theory, one can see [12] and [13]. Measure theory on time scales has been introduced in [16], then further studies were made by in [17] and [18]. Deniz-Ufuktepe defined Lebesgue-Stieltjes Δ and ∇-measures and by using these measures they defined an integral which is adaptable to the time scale, specifically Lebesgue-Stieltjes Δ-integral, in [19]. In the light of these studies, let us introduce some time scale and measure theoretic notations. The *forward jump operator* $\sigma\colon \mathbb{T} \to \mathbb{T}$ for each $t \in \mathbb{T}$ by via formula,

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

For $a, b \in \mathbb{T}$ with $a \le b$ we define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] = \{t \in \mathbb{T} : a \le t \le b\}.$$

Open intervals and half-open intervals are defined similarly. Let $\mathcal{S}$ be semiring of left-closed and right-open intervals and m^* be Caratheodory extension of the Lebesgue set function m which is defined by $m([a, b)) = b - a$, associated with the family $\mathcal{S}$ in the time scale $\mathbb{T}$ as in the real case. Also let $\mathfrak{M}(m^*)$ be the σ-algebra of all m^* measurable sets. Recall that $\mathfrak{M}(m^*)$ consists of such a subset E has the property that $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$ for all $A \subset \mathbb{T}$. It is well known that the restriction of m^* to $\mathfrak{M}(m^*)$ which we denote by μ_Δ is a countably additive measu-re on $\mathfrak{M}(m^*)$. This measure called Lebesgue Δ-measure. The measurable subsets of $\mathbb{T}$ is called Δ-measurable and a function $f: \mathbb{T} \to \mathbb{R}$ is called Δ-measu-rable function, if $f^{-1}(\mathcal{O}) \in \mathfrak{M}(m^*)$ for every open subsets $\mathcal{O}$ of $\mathbb{R}$. From [16] we know that if $a, b \in \mathbb{T}$ and $a \leq b$ then

$$\mu_\Delta([a,b)) = b - a \ , \ \mu_\Delta((a,b)) = b - \sigma(a)$$

If $a, b \in \mathbb{T} - \{\max \mathbb{T}\}$ and $a \leq b$ then

$$\mu_\Delta((a,b]) = \sigma(b) - \sigma(a) \ , \ \mu_\Delta([a,b]) = \sigma(b) - a.$$

In [14], the concept of Δ-density which is generalization of the of concept natural density by using measu-re theoretic approach is given. If A is a Δ-measurable subset of $\mathbb{T}$ and $a = \min \mathbb{T}$, the Δ-density of A in $\mathbb{T}$ is defined by

$$\lim_{s\to\infty} \frac{\mu_\Delta(A(s))}{\sigma(s) - a}$$

(if this limit exists) where $A(s) = \{t \in A: t \leq s\}$. The Δ-density function can be considered as a probabilistic finite additive measure on the algebra of subset of $\mathbb{T}$ which have a Δ-density. By using the Δ-density we obtained a new type of convergence which is generalization of the natural statistical convergence and statistical Cauchy sequences definitions. In [20],
the concepts of the Δ-limit and the Δ-cluster point are given. These concepts are generalization of the concept of the statistical limit and statistical cluster point defining in [3]. Let us remember some of these notions. A Δ-measurable function f is called Δ-convergent to the number L if

$$\delta_\Delta\left(f^{-1}((L - s, L + s))\right) = 1$$

for all $s > 0$. A measurable set K is called Δ-non thin subset of $\mathbb{T}$ if it may have a positive Δ-density or may not have even a Δ-density and a measurable set K is called a Δ-null subset of $\mathbb{T}$ if $\delta_\Delta(K) = 0$. A measurable function $f: \mathbb{T} \to \mathbb{R}$ is called Δ-bounded if there exists a real number r such that $\delta_\Delta(\{t \in \mathbb{T}: |f(t)| \leq r\}) = 1$. The number L is called Δ-cluster point of a measurable function f if $\delta_\Delta(f^{-1}((L - s, L + s)))$ is a Δ-non thin subset of $\mathbb{T}$. We will use the symbol Γ_f to denote all Δ-cluster points of a Δ-measurable function f. The set Γ_f is closed subset of $\mathbb{T}$.

The main purpose of the present paper is to extend the notions of statistical limit inferior and statistical limit superior point defined in [15] by using real valued functions defined on time scale.

2 Δ-limit superior and Δ-limit inferior

In this section, we introduce the notion of Δ-limit superior a Δ-limit inferior for a Δ-measurable function defined on $\mathbb{T}$. We will further with properties of these concepts and we will give the some relations with the Δ-cluster points defined in [20] and classical limit inferior and limit superior points concepts.

Definition 2.1 Let $f: \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. If we consider the following subsets of :

$$A(f) := \{y \in \mathbb{R}: f^{-1}((-\infty, y)) \text{ is a Δ-non thin set}\}$$

$$B(f) := \{y \in \mathbb{R}: f^{-1}((y, \infty)) \text{ is a Δ-non thin set}\}$$

Then the following extended real number

$$\Delta\text{-}\limsup_{t\to\infty} f(t) := \sup B(f)$$

is called Δ-limit supreior of the function f whenever $t \to \infty$. Similarly follwing extended number

$$\Delta\text{-}\liminf_{t\to\infty} f(t) := \inf A(f)$$

is called Δ-limit inferior of the function f whenever $t \to \infty$. Let us start with the expecting property of $\Delta\text{-}\limsup f(t)$ and $\Delta\text{-}\liminf f(t)$.

Proposition 2.2 Let $f: \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then we have

$$\Delta\text{-}\limsup_{t\to\infty} f(t) = -\Delta\text{-}\liminf_{t\to\infty}(-f(t)).$$

Proof From definition of $A(f)$ and $B(f)$ we have,

$$A(-f) = \{y \in \mathbb{R}: \bar{\delta}_\Delta(\{t \in \mathbb{T}: f(t) > -y\}) > 0\}$$

and

$$-A(-f) = \{-y \in \mathbb{R}: \bar{\delta}_\Delta(\{t \in \mathbb{T}: f(t) > -y\}) > 0\}$$
$$= \{y \in \mathbb{R}: \bar{\delta}_\Delta(\{t \in \mathbb{T}: f(t) > y\}) > 0\}$$
$$= B(f).$$

So that we have,

$$-\sup B(f) = -\sup(-A(-f)) = \inf A(-f).$$

Desired equality is easily obtained from above equality.

The following two theorems tell us a necessary and sufficient condition for being a finite valued Δ-limit supreior point and Δ-limit inferior point of a function defined on time scale $\mathbb{T}$.

Theorem 2.3 Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. The real number L is Δ-limit supreior point of the function f if and only if for all $s>0$,

i) $f^{-1}\big((L-s,\infty)\big)$ is a Δ-non thin subset of $\mathbb{T}$,

ii) $f^{-1}\big((L+s,\infty)\big)$ is a Δ-null subset of $\mathbb{T}$.

Proof We will show that (i) and (ii) hold for all $s>0$. Since $\Delta\text{-}\limsup_{t\to\infty} f(t)=L\in\mathbb{R}$ then $B(f)\neq\emptyset$. From sup properties of real numbers, for all $s>0$ there exists $y\in B(f)$ such that $L-s<y$. Since $f^{-1}\big((y,\infty)\big)\subset f^{-1}\big((L-s,\infty)\big)$ and $f^{-1}\big((y,\infty)\big)$ is a Δ-non thin subset of $\mathbb{T}$ then $f^{-1}\big((L-s,\infty)\big)$ is a Δ-non thin subset of $\mathbb{T}$. Now assume that (ii) does not hold. Then there exists $s>0$ such that $f^{-1}\big((L+s,\infty)\big)$ is a Δ-non thin subset of $\mathbb{T}$. That means $L+s\in B(f)$. This contrdicts with $L=\sup B(f)$. Therefore (i) and (ii) hold for all $s>0$. Now we will show that $\Delta\text{-}\limsup_{t\to\infty} f(t)=L\in\mathbb{R}$. From (ii) the real number L is an upper bound of $B(f)$. If M is another upper bound of $B(f)$ then from (i) it should be greater than or equal to L. So that $\sup B(f)=L$.

Theorem 2.4 Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. The real number L is Δ-limit inferior point of the function f if and only if for all $s>0$,

i) $f^{-1}\big((-\infty,L+s)\big)$ is a Δ-non thin subset of $\mathbb{T}$,

ii) $f^{-1}\big((-\infty,L-s)\big)$ is a Δ-null subset of $\mathbb{T}$.

Proof It is easily obtained from Proposition 2.2 and Theorem 2.3.

Theorem 2.5 Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. The real number L is Δ-limit supreior of the function f if and only if $=\sup\Gamma_f$.

Proof We will show that $\sup B(f)=\sup\Gamma_f$. If Γ_f is an unbounded subset of $\mathbb{R}$ then the set $B(f)$ is also unbounded then equality holds. Now let define $\sup\Gamma_f=L_1$ and $\sup B(f)=L_2$ then since the real number L_2 is the Δ-limit supreior of the function f, from Theorem 2.3-(i), $f^{-1}\big((L_2-s,\infty)\big)$ is a Δ-non thin and $f^{-1}\big((L_2+s,\infty)\big)$ is a Δ-null subset of $\mathbb{T}$. If we subtract a Δ-null set from a Δ-non thin set then we obtain a Δ-non thin set. Therefore

$$f^{-1}\big((L_2-s,L_2+s)\big)=f^{-1}\big((L_2-s,\infty)\big)-f^{-1}([L_2+s,\infty))$$

is a Δ-non thin subset of $\mathbb{T}$. So that L_2 is a Δ-cluster point of f and we have $L_1\geq L_2$. Now we assume that $L_1>L_2$ and $s_1:=L_1-L_2>0$. Since

$$f^{-1}\left(\left(L_1-\frac{s_1}{2},L_1+\frac{s_1}{2}\right)\right)\subset f^{-1}\left(\left(L_2+\frac{s_1}{2},\infty\right)\right)$$

and Theorem 2.3 (ii), $f^{-1}\left(\left(L_1-\frac{s_1}{2},L_1+\frac{s_1}{2}\right)\right)$ is a Δ-null subset of $\mathbb{T}$. This contradicts with closedness of Γ_f. That means $L_1\leq L_2$ and so $L_1=L_2$.

Theorem 2.6 Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. The real number L is Δ-limit inferior of the function f if and only if $=\inf\Gamma_f$.

Proof It is easily obtained from Proposition 2.2 and Theorem 2.5.

Theorem 2.7 Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. Then we have $\liminf_{t\to\infty} f(t)\leq\Delta\text{-}\liminf_{t\to\infty} f(t)\leq$ $\Delta\text{-}\limsup_{t\to\infty} f(t)\leq\limsup_{t\to\infty} f(t)$.

Proof The case of $\Delta\text{-}\limsup_{t\to\infty} f(t)=\infty$ or $\Delta\text{-}\liminf_{t\to\infty} f(t)=-\infty$ are obvious. Now, assume that $\Delta\text{-}\limsup_{t\to\infty} f(t)=L_1\in\mathbb{R}$ and $\Delta\text{-}\liminf_{t\to\infty} f(t)=L_2\in\mathbb{R}$. From Theorem 2.3 (ii), for any $s>0$, the set $f^{-1}\big((L_1+s,\infty)\big)$ is a Δ-null subset of $\mathbb{T}$ then $f^{-1}\big((-\infty,L_1+s)\big)$ is a Δ-non thin subset of $\mathbb{T}$. From this, we have $L_2\leq L_1+s$ for all $s>0$. Therefore

$$\Delta\text{-}\liminf_{t\to\infty} f(t)\leq\ \Delta\text{-}\limsup_{t\to\infty} f(t).$$

Now, first inequality is easily obtained from Proposition 2.2.

Example Let $\mathbb{T}=[0,\infty)$ and (M_n) strictly increasing unbounded sequence in $\mathbb{T}$. Take a sequence (s_n) such that $0<s_n<M_{n+1}-M_n$ and $s_n\to 0$. One can easily see that $A:=\cup_{n\in\mathbb{N}}[M_n,M_n+s_n]$ is a Δ-null set. If we define $f:\mathbb{T}\to\mathbb{R}$

$$f(t):=\begin{cases}L', & t\in A\\ L, & t\in\mathbb{T}-A\end{cases}$$

where L and L' fixed real numbers such that $L<L'$ then we have $\Delta\text{-}\limsup_{t\to\infty} f(t)=L$ and $\limsup_{t\to\infty} f(t)=L'$.

Theorem 2.8 Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-bounded function. The function f is Δ-converge to the real number L if and only if

$$\Delta\text{-}\liminf_{t\to\infty} f(t) = \Delta\text{-}\limsup_{t\to\infty} f(t).$$

Proof If f is Δ-converge to the real number L then $\delta_\Delta\left(f^{-1}\big((L-s,L+s)\big)\right)=1$ for all $s>0$. This implies that both of $f^{-1}\big((L+s,\infty)\big)$ and $f^{-1}\big((-\infty,L-s)\big)$ are Δ-null subsets of $\mathbb{T}$ for all $s>0$. By same argument both of $f^{-1}\big((-\infty,L+s)\big)$ and $f^{-1}\big((L-s,\infty)\big)$ are Δ-non thin subset of $\mathbb{T}$ for all $s>0$. Therefore from Theorem 2.3 and Theorem 2.4 we have,

$$\Delta\text{-}\liminf_{t\to\infty} f(t) = \Delta\text{-}\limsup_{t\to\infty} f(t) = L.$$

Now let $\Delta\text{-}\liminf_{t\to\infty} f(t) = \Delta\text{-}\limsup_{t\to\infty} f(t) = L$. From Theorem 2.3 and Theorem 2.4 we have $\delta_\Delta\left(f^{-1}\big((L-s,L+s)\big)\right)=1$.

3 References

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