Communications in Advanced Mathematical Sciences
Vol. 6, No. 1, 44-59, 2023
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.1236095

Almost $\eta-$Ricci Solitons on Pseudosymmetric Lorentz Sasakian Space Forms

Tuğba Mert[1]*, Mehmet Atçeken[2]

Abstract
In this paper, we consider pseudosymmetric Lorentz Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz Sasakian space forms admits $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, W_1 and W_2. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz Sasakian space form admits $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made under the some conditions.

Keywords: $\eta-$Ricci Soliton, Lorentz Sasakian Space Form, Ricci-pseudosymmetric Manifold.
2010 AMS: 53C15, 53C25, 53D25

[1] *Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, Sivas, Turkey, ORCID: 0000-0001-8258-8298*
[2] *Department of Mathematics, Faculty of Science and Arts, Aksaray University, Aksaray, Turkey, ORCID: 0000-0002-1242-4359*
***Corresponding author**: tmert@cumhuriyet.edu.tr

Received: 16 January 2023, **Accepted:** 20 March 2023, **Available online:** 31 March 2023
How to cite this article: T. Mert, M. Atçeken, *Almost $\eta-$Ricci Solitons on Pseudosymmetric Lorentz Sasakian Space Forms*, Commun. Adv. Math. Sci., (6)1 (2023) 44-59.

1. Introduction

The notion of Ricci flow was introduced by Hamilton in 1982. With the help of this concept, Hamilton found the canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman used Ricci flow and it surgery to prove Poincare conjecture in [1, 2]. The Ricci flow is an flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g(t) = -2S(g(t)).$$

A Ricci soliton emerges as the limit of the solitons of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [3], Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Ashoka et al. in [4, 5], Bagewadi et al. in [6], Ingalahalli in [7], Bejan and Crasmareanu in [8], Blaga in [9], Chandra et al. in [10], Chen and Deshmukh in [11], Deshmukh et al. in [12], He and Zhu [13], Atçeken et al. in [14], Nagaraja and Premalatta in [15], Tripathi in [16] and many others.

$\phi-$sectional curvature plays an important role for Sasakian manifold. If the $\phi-$sectional curvature of a Sasakian manifold is constant, then the manifold is a Sasakian-space-form [17]. P. Alegre and D. Blair described generalized Sasakian space forms [18]. P. Alegre and D. Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. P. Alegre and A. Carriazo later discussed generalized indefinite Sasakian space forms [19]. Generalized indefinite Sasakian space forms are also called Lorentz-Sasakian space forms, and Lorentz manifolds are of great importance for Einstein's theory of Relativity.

In this paper, we consider Lorentz Sasakian space form admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz Sasakian space form admits $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemannian, concircular, projective, $\mathscr{M}-$projective, W_1 and W_2. Then, again according to the choice of the curvature tensor, necessary conditions for Lorentz Sasakian space form admits $\eta-$Ricci soliton to be Ricci semisymmetric are given. Then some characterizations are obtained and some classifications have been made

2. Preliminaries

Let $\tilde{N}$ be a $(2m+1)-$dimensional Lorentz manifold. If the $\tilde{N}$ Lorentz manifold with (ϕ,ξ,η,g) structure tensors satisfies the following conditions, it is called a Lorentz-Sasakian manifold

$$\phi^2 Y_1 = -Y_1+\eta\left(Y_1\right)\xi, \eta\left(\xi\right)=1, \eta\left(\phi Y_1\right)=0,$$

$$g\left(\phi Y_1,\phi Y_2\right)=g\left(Y_1,Y_2\right)+\eta\left(Y_1\right)\eta\left(Y_2\right), \eta\left(Y_1\right)=-g\left(Y_1,\xi\right),$$

$$\left(\tilde{\nabla}_{Y_1}\phi\right)Y_2=-g\left(Y_1,Y_2\right)\xi-\eta\left(Y_2\right)Y_1, \tilde{\nabla}_{Y_1}\xi=-\phi Y_1,$$

where, $\tilde{\nabla}$ is the Levi-Civita connection according to the Riemannian metric g.

The plane section Π in $T_{Y_1}\tilde{N}$. If the Π plane is spanned by Y_1 and ϕY_1, this plane is called the ϕ-section. The curvature of the ϕ-section is called the ϕ-sectional curvature. If the Lorentz-Sasakian manifold has a constant ϕ-sectional curvature, this manifold is called the Lorentz-Sasakian space form and is denoted by $\tilde{N}(c)$. The curvature tensor of the Lorentz-Sasakian space form $\tilde{N}(c)$ is defined as

$$\begin{aligned}\tilde{R}\left(Y_1,Y_2\right)Y_3 &= \left(\tfrac{c-3}{4}\right)\left\{g\left(Y_2,Y_3\right)Y_1-g\left(Y_1,Y_3\right)Y_2\right\}\\ &+\left(\tfrac{c+1}{4}\right)\left\{g\left(Y_1,\phi Y_3\right)\phi Y_2-g\left(Y_2,\phi Y_3\right)\phi Y_1\right.\\ &+2g\left(Y_1,\phi Y_2\right)\phi Y_3+\eta\left(Y_2\right)\eta\left(Y_3\right)Y_1-\eta\left(Y_1\right)\eta\left(Y_3\right)Y_2\\ &\left.+g\left(Y_1,Y_3\right)\eta\left(Y_2\right)\xi-g\left(Y_2,Y_3\right)\eta\left(Y_1\right)\xi\right\},\end{aligned} \tag{2.1}$$

for all $Y_1,Y_2,Y_3\in\chi\left(\tilde{N}\right)$.

Lemma 2.1. *Let $\tilde{N}(c)$ be the $(2m+1)-$dimensional Lorentz-Sasakian space form. The following relations are hold for the Lorentz-Sasakian space forms.*

$$\tilde{\nabla}_{Y_1}\xi=-\phi Y_1, \tag{2.2}$$

$$\left(\tilde{\nabla}_{Y_1}\phi\right)Y_2=-g\left(Y_1,Y_2\right)\xi-\eta\left(Y_2\right)Y_1,$$

$$\left(\tilde{\nabla}_{Y_1}\eta\right)Y_2=g\left(\phi Y_1,Y_2\right),$$

$$\tilde{R}\left(Y_1,Y_2\right)\xi=\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2, \tag{2.3}$$

$$\eta\left(\tilde{R}\left(Y_1,Y_2\right)Y_3\right)=g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_3\right), \tag{2.4}$$

$$S(Y_1,Y_2)=\left[\frac{(m+2)c-(3m-2)}{2}\right]g(Y_1,Y_2)$$

$$+\frac{(c+1)(m+1)}{2}\eta(Y_1)\eta(Y_2),$$

$$S(Y_1,\xi)=-\left[\frac{(c+1)-4m}{2}\right]\eta(Y_1), \tag{2.5}$$

$$QY_1=\left[\frac{(m+2)c-(3m-2)}{2}\right]Y_1-\frac{(c+1)(m+1)}{2}\eta(Y_1)\xi$$

$$Q\xi=\frac{(c+1)-4m}{2}\xi$$

where $\tilde{R}, S$ are the Riemannian curvature tensor, Ricci curvature tensor of $\tilde{N}(c)$, respectively.

Precisely, Ricci soliton on a Riemannian manifold $\left(\tilde{N},g\right)$ is defined as a triple (g,ξ,κ_1) on $\tilde{N}$ satisfying

$$L_\xi g+2S+2\kappa_1 g=0,$$

where L_ξ is the Lie derivative operator along the vector field ξ and κ_1 is a real constant. We note that if ξ is a Killing vector field, then the Ricci soliton reduces to an Einstein metric (g,κ_1). Futhermore, in [20], generalization is the notion of $\eta-$Ricci soliton defined by J.T. Cho and M. Kimura as a quadruple $(g,\xi,\kappa_1,\kappa_2)$ satisfying

$$L_\xi g+2S+2\kappa_1 g+2\kappa_2\mu\eta\oplus\eta=0, \tag{2.6}$$

where κ_1 and κ_2 are real constants and η is the dual of ξ and S denotes the Ricci tensor of $\tilde{N}$. Furthermore if κ_1 and κ_2 are smooth functions on $\tilde{N}$, then it called almost $\eta-$Ricci soliton on $\tilde{N}$ [20].

Suppose the quartet $(g,\xi,\kappa_1,\kappa_2)$ is almost $\eta-$Ricci soliton on manifold $\tilde{N}$. Then,

· If $\kappa_1<0$, then $\tilde{N}$ is shrinking.
· If $\kappa_1=0$, then $\tilde{N}$ is steady.
· If $\kappa_1>0$, then $\tilde{N}$ is expanding.

3. Almost $\eta-$Ricci Solitons on Ricci Pseudosymmetric and Ricci Semisymmetric Lorentz Sasakian Space Form

Now let $(g,\xi,\kappa_1,\kappa_2)$ be an almost $\eta-$Ricci soliton on Lorentz Sasakian space form. Then we have

$$\left(L_\xi g\right)(Y_1,Y_2)=L_\xi g(Y_1,Y_2)-g\left(L_\xi Y_1,Y_2\right)-g\left(Y_1,L_\xi Y_2\right)$$

$$=\xi g(Y_1,Y_2)-g([\xi,Y_1],Y_2)-g(Y_1,[\xi,Y_2])$$

$$=g\left(\nabla_\xi Y_1,Y_2\right)+g\left(Y_1,\nabla_\xi Y_2\right)-g\left(\nabla_\xi Y_1,Y_2\right)$$

$$+g(\nabla_{Y_1}\xi,Y_2)-g\left(\nabla_\xi Y_2,Y_1\right)+g(Y_1,\nabla_{Y_2}\xi),$$

for all $Y_1,Y_2\in\Gamma(TM)$. By using ϕ is anti-symmetric and taking into account (2.2) we have

$$\left(L_\xi g\right)(Y_1,Y_2)=0. \tag{3.1}$$

Thus, in a Lorentz Sasakian space form, from (2.6) and (3.1) we have

$$S(Y_1,Y_2)+\kappa_1 g(Y_1,Y_2)+\kappa_2\eta(Y_1)\eta(Y_2)=0. \tag{3.2}$$

It is clear from (3.2) that the $(2m+1)-$dimensional Lorentz Sasakian $\eta-$Ricci soliton $\left(\tilde{N}^{2m+1}, g, \xi, \kappa_1, \kappa_2\right)$ is an $\eta-$Einstein manifold.

For $Y_2 = \xi$ in (3.2) this implies that

$$S(\xi, Y_1) = (\kappa_1 - \kappa_2)\,\eta(Y_1). \tag{3.3}$$

Taking into account of (3.3) we conclude that

$$\kappa_1 - \kappa_2 = \frac{4m-(c+1)}{2}.$$

Definition 3.1. *Let $\tilde{N}(c)$ be an $(2m+1)-$dimensional Lorentz Sasakian space form. If $\tilde{R}\cdot S$ and $Q(g,S)$ are linearly dependent, then the $\tilde{N}(c)$ is said to be **Ricci pseudosymmetric**.*

In this case, there exists a function L_1 on $\tilde{N}(c)$ such that

$$\tilde{R}\cdot S = L_1 Q(g,S).$$

In particular, if $L_1 = 0$, the manifold $\tilde{N}(c)$ is said to be **Ricci semisymmetric**.

Let us now investigate the Ricci pseudosymmetry case of the $(2m+1)-$dimensional Lorentz Sasakian space form.

Theorem 3.2. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g, \xi, \kappa_1, \kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a Ricci pseudosymmetric, then*

$$L_1 = \frac{2\kappa_1-(c+1)+4m}{4m-2\kappa_1-(c+1)},$$

provided $2\kappa_1 \neq 4m-(c+1)$.

Proof. Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be Ricci pseudosymmetric and $(g, \xi, \kappa_1, \kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. Then we have

$$\left(\tilde{R}(Y_1,Y_2)\cdot S\right)(Y_4,Y_5) = L_1 Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1, Y_2, Y_4, Y_5 \in \Gamma\left(T\tilde{N}\right)$. From the last equation, we can easily write

$$\begin{aligned} & S\left(\tilde{R}(Y_1,Y_2)Y_4, Y_5\right) + S\left(Y_4, \tilde{R}(Y_1,Y_2)Y_5\right) \\ & = L_1\left\{S((Y_1\wedge_g Y_2)Y_4, Y_5) + S(Y_4,(Y_1\wedge_g Y_2)Y_5)\right\}. \end{aligned} \tag{3.4}$$

If we choose $Y_5 = \xi$ in (3.4) we get

$$\begin{aligned} & S\left(\tilde{R}(Y_1,Y_2)Y_4, \xi\right) + S\left(Y_4, \tilde{R}(Y_1,Y_2)\xi\right) \\ & = L_1\{S(g(Y_2,Y_4)Y_1 - g(Y_1,Y_4)Y_2, \xi) \\ & + S(Y_4, \eta(Y_1)Y_2 - \eta(Y_2)Y_1)\}. \end{aligned} \tag{3.5}$$

If we make use of (2.3) and (2.5) in (3.5) we have

$$\begin{aligned} & -\left[\tfrac{(c+1)-4m}{2}\right]\eta\left(\tilde{R}(Y_1,Y_2)Y_4\right) + S(Y_4, \eta(Y_2)Y_1 - \eta(Y_1)Y_2) \\ & = L_1\left\{-\left[\tfrac{(c+1)-4m}{2}\right] g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1, Y_4)\right. \\ & \left. + S(Y_4, \eta(Y_1)Y_2 - \eta(Y_2)Y_1)\right\}. \end{aligned} \tag{3.6}$$

If we use (2.4) in the (3.6), we get

$$\begin{aligned} & -\left[\tfrac{(c+1)-4m}{2}\right] g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1, Y_4) \\ & + S(\eta(Y_2)Y_1 - \eta(Y_1)Y_2, Y_4) \\ & = L_1\left\{-\left[\tfrac{(c+1)-4m}{2}\right] g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1, Y_4)\right. \\ & \left. + S(Y_4, \eta(Y_1)Y_2 - \eta(Y_2)Y_1)\right\}. \end{aligned} \tag{3.7}$$

If we use (3.2) in the (3.7), we can write

$$\left[\left(\kappa_1-\tfrac{(c+1)-4m}{2}\right)+\left(\kappa_1+\tfrac{(c+1)-4m}{2}\right)L_1\right]\times \\ g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)=0. \tag{3.8}$$

It is clear from (3.8)

$$L_1=\frac{2\kappa_1-(c+1)+4m}{4m-2\kappa_1-(c+1)}.$$

This completes the proof. □

Thus we have the following corollaries.

Corollary 3.3. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a Ricci semisymmetric, then $\tilde{N}(c)$ is an $\eta-$Einstein manifold with $\kappa_1=\frac{(c+1)-4m}{2}$ and $\kappa_2=(c+1)-4m$.*

Corollary 3.4. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a Ricci semisymmetric, then we observe that:*
i) $\tilde{N}(c)$ is expanding, if $(c+1)>4m$.
ii) $\tilde{N}(c)$ is shrinking, if $(c+1)<4m$.

For a $(2m+1)-$dimensional semi-Riemannian manifold N, the concircular curvature tensor is defined as

$$C\left(Y_1,Y_2\right)Y_3=R\left(Y_1,Y_2\right)Y_3-\frac{r}{2m\left(2m+1\right)}\left[g\left(Y_2,Y_3\right)Y_1-g\left(Y_1,Y_3\right)Y_2\right]. \tag{3.9}$$

For a $(2m+1)-$dimensional Lorentz Sasakian space form, if we choose $Y_3=\xi$ in (3.9) we can write

$$C\left(Y_1,Y_2\right)\xi=\left[1+\frac{r}{2m\left(2m+1\right)}\right]\left[\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2\right], \tag{3.10}$$

and similarly if we take the inner product of both sides of (3.9) by ξ, we get

$$\eta\left(C\left(Y_1,Y_2\right)Y_3\right)=\left[1+\frac{r}{2m\left(2m+1\right)}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_3\right). \tag{3.11}$$

Definition 3.5. *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $C\cdot S$ and $Q(g,S)$ are linearly dependent, then it is said to be* ***concircular Ricci pseudosymmetric.***

In this case, there exists a function L_2 on $\tilde{N}(c)$ such that

$$C\cdot S=L_2Q\left(g,S\right).$$

In particular, if $L_2=0$, the manifold $\tilde{N}(c)$ is said to be **concircular Ricci semisymmetric.**

Let us now investigate the concircular Ricci pseudosymmetry case of the Lorentz Sasakian space form.

Theorem 3.6. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a concircular Ricci pseudosymmetric, then*

$$L_2=\frac{\left[2\kappa_1-(c+1)+4m\right]\left[2m\left(2m+1\right)+r\right]}{2m\left(2m+1\right)\left[4m-(c+1)-2\kappa_1\right]},$$

provided $4m\neq 2\kappa_1+(c+1)$.

Proof. Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be concircular Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. That is mean

$$\left(C\left(Y_1,Y_2\right)\cdot S\right)\left(Y_4,Y_5\right)=L_2Q\left(g,S\right)\left(Y_4,Y_5;Y_1,Y_2\right),$$

for all $Y_1, Y_2, Y_4, Y_5 \in \Gamma\left(T\tilde{N}\right)$. From the last equation, we can easily write

$$\begin{aligned} & S\left(C\left(Y_1,Y_2\right)Y_4,Y_5\right)+S\left(Y_4,C\left(Y_1,Y_2\right)Y_5\right) \\ & = L_2\left\{S\left(\left(Y_1\wedge_g Y_2\right)Y_4,Y_5\right)+S\left(Y_4,\left(Y_1\wedge_g Y_2\right)Y_5\right)\right\}. \end{aligned} \tag{3.12}$$

If we choose $Y_5=\xi$ in (3.12) we get

$$\begin{aligned} & S\left(C\left(Y_1,Y_2\right)Y_4,\xi\right)+S\left(Y_4,C\left(Y_1,Y_2\right)\xi\right) \\ & = L_2\left\{S\left(g\left(Y_2,Y_4\right)Y_1-g\left(Y_1,Y_4\right)Y_2,\xi\right)\right. \\ & \left.+S\left(Y_4,\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1\right)\right\}. \end{aligned} \tag{3.13}$$

If by using (2.5) and (3.10) in (3.13) we have

$$\begin{aligned} & S\left(Y_4,\left[1+\tfrac{r}{2m(2m+1)}\right]\left[\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2\right]\right) \\ & -\left[\tfrac{(c+1)-4m}{2}\right]\eta\left(C\left(Y_1,Y_2\right)Y_4\right) \\ & = L_2\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right. \\ & \left.+S\left(Y_4,\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1\right)\right\}. \end{aligned} \tag{3.14}$$

Substituting (3.11) in (3.14), we get

$$\begin{aligned} & -\left[\tfrac{(c+1)-4m}{2}\right]\left[1+\tfrac{r}{2m(2m+1)}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right) \\ & +\left[1+\tfrac{r}{2m(2m+1)}\right]S\left(\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2,Y_4\right) \\ & = L_2\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right. \\ & \left.+S\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)\right\}. \end{aligned} \tag{3.15}$$

If we use (3.2) in the (3.15), we can write

$$\begin{aligned} & \left[\left(\kappa_1-\tfrac{(c+1)-4m}{2}\right)\left(1+\tfrac{r}{2m(2m+1)}\right)+\left(\kappa_1+\tfrac{(c+1)-4m}{2}\right)L_2\right]\times \\ & g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)=0. \end{aligned}$$

This implies that

$$L_2=\frac{\left[2\kappa_1-(c+1)+4m\right]\left[2m\left(2m+1\right)+r\right]}{2m\left(2m+1\right)\left[4m-(c+1)-2\kappa_1\right]}.$$

This completes the proof. □

We can give the following corollaries.

Corollary 3.7. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a concircular Ricci semisymmetric, then $\tilde{N}(c)$ is either manifold with scalar curvature $r=-2m(2m+1)$ or $\kappa_1=\frac{(c+1)-4m}{2}$.*

Corollary 3.8. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a concircular Ricci semisymmetric, then we conclude that:*
i) Let $r<2m(2m+1)$.
a) $\tilde{N}(c)$ is expanding, if $(c+1)>4m$.
b) $\tilde{N}(c)$ is shrinking, if $(c+1)<4m$.
ii) Let $r>2m(2m+1)$.
c) $\tilde{N}(c)$ is shrinking, if $(c+1)>4m$.
d)$\tilde{N}(c)$ is expanding, if $(c+1)<4m$.

For a $(2m+1)-$dimensional semi-Riemannian manifold N, the projective curvature tensor is defined as

$$P(Y_1,Y_2)Y_3 = R(Y_1,Y_2)Y_3 - \frac{1}{2m}\left[S(Y_2,Y_3)Y_1 - S(Y_1,Y_3)Y_2\right]. \tag{3.16}$$

For a $(2m+1)-$dimensional Lorentz Sasakian space form, if we choose $Y_3=\xi$ in (3.16) we can write

$$P(Y_1,Y_2)\xi = \frac{c+1}{4m}\left[\eta(Y_2)Y_1 - \eta(Y_1)Y_2\right], \tag{3.17}$$

and in the same way if we take the inner product of both sides of (3.16) by ξ, we get

$$\eta(P(Y_1,Y_2)Y_3) = \frac{c+1}{4m}g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1, Y_3). \tag{3.18}$$

Definition 3.9. *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $P\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be* ***projective Ricci pseudosymmetric.***

In this case, there exists a function L_3 on $\tilde{N}(c)$ such that

$$P\cdot S = L_3 Q(g,S).$$

In particular, if $L_3=0$, the manifold $\tilde{N}(c)$ is said to be **projective Ricci semisymmetric.**

Let us now investigate the projective Ricci pseudosymmetry case of the Lorentz Sasakian space form.

Theorem 3.10. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a projective Ricci pseudosymmetric, then*

$$L_3 = \frac{(c+1)\left[2\kappa_1 - (c+1) + 4m\right]}{2m\left[4m - (c+1) - 2\kappa_1\right]},$$

provided $2\kappa_1 \neq 4m-(c+1)$.

Proof. Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be projective Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. That is mean

$$(P(Y_1,Y_2)\cdot S)(Y_4,Y_5) = L_3 Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5 \in \Gamma\left(T\tilde{N}\right)$. From the last equation, we can easily see

$$\begin{aligned} & S(P(Y_1,Y_2)Y_4,Y_5) + S(Y_4,P(Y_1,Y_2)Y_5) \\ & = L_3\left\{S((Y_1\wedge_g Y_2)Y_4,Y_5) + S(Y_4,(Y_1\wedge_g Y_2)Y_5)\right\}. \end{aligned} \tag{3.19}$$

If we choose $Y_5=\xi$ in (3.19) we get

$$\begin{aligned} & S(P(Y_1,Y_2)Y_4,\xi) + S(Y_4,P(Y_1,Y_2)\xi) \\ & = L_3\{S(g(Y_2,Y_4)Y_1 - g(Y_1,Y_4)Y_2,\xi) \\ & +S(Y_4,\eta(Y_1)Y_2 - \eta(Y_2)Y_1)\}. \end{aligned} \tag{3.20}$$

If we taking into account (2.5) and (3.17) in (3.20), then we have

$$\begin{aligned} & S\left(Y_4,\tfrac{c+1}{4m}\left[\eta(Y_2)Y_1 - \eta(Y_1)Y_2\right]\right) \\ & -\left[\tfrac{(c+1)-4m}{2}\right]\eta(P(Y_1,Y_2)Y_4) \\ & = L_3\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2 - \eta(Y_2)Y_1,Y_4)\right. \\ & \left. +S(Y_4,\eta(Y_1)Y_2 - \eta(Y_2)Y_1)\right\}. \end{aligned} \tag{3.21}$$

If we use (3.18) in the (3.21), we get

$$
\begin{aligned}
&-\left[\tfrac{(c+1)-4m}{2}\right]\left(\tfrac{c+1}{4m}\right) g\left(\eta\left(Y_1\right) Y_2-\eta\left(Y_2\right) Y_1, Y_4\right) \\
&+\left(\tfrac{c+1}{4m}\right) S\left(\eta\left(Y_2\right) Y_1-\eta\left(Y_1\right) Y_2, Y_4\right) \\
&=L_3\left\{-\left[\tfrac{(c+1)-4m}{2}\right] g\left(\eta\left(Y_1\right) Y_2-\eta\left(Y_2\right) Y_1, Y_4\right)\right. \\
&\left.+S\left(\eta\left(Y_1\right) Y_2-\eta\left(Y_2\right) Y_1, Y_4\right)\right\}.
\end{aligned}
\tag{3.22}
$$

If we use (3.2) in the (3.22), we taking into account

$$
\begin{aligned}
&\left[\left(\kappa_1-\tfrac{(c+1)-4m}{2}\right)\left(\tfrac{c+1}{4m}\right)+\left(\kappa_1+\tfrac{(c+1)-4m}{2}\right) L_3\right] \times \\
&g\left(\eta\left(Y_1\right) Y_2-\eta\left(Y_2\right) Y_1, Y_4\right)=0.
\end{aligned}
\tag{3.23}
$$

It is clear from (3.23)

$$
L_3=\frac{(c+1)\left[2\kappa_1-(c+1)+4m\right]}{2m\left[4m-(c+1)-2\kappa_1\right]}.
$$

This completes the proof. □

We have the following corollaries.

Corollary 3.11. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a projective Ricci semisymmetric, then $\tilde{N}(c)$ is either real space form with constant section curvature $c=-1$ or $\kappa_1=\frac{(c+1)-4m}{2}$.*

Corollary 3.12. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a projective Ricci semisymmetric, then we conclude provided that $c+1\neq 0$:*
i) The soliton $\tilde{N}(c)$ is expanding, if $(c+1)>4m$.
ii) The soliton $\tilde{N}(c)$ is shrinking, if $(c+1)<4m$.

For a $(2m+1)-$dimensional semi-Riemannian manifold N, the $\mathscr{M}-$projective curvature tensor is defined as

$$
\begin{aligned}
&\mathscr{M}\left(Y_1, Y_2\right) Y_3=R\left(Y_1, Y_2\right) Y_3-\tfrac{1}{2m}\left[S\left(Y_2, Y_3\right) Y_1-S\left(Y_1, Y_3\right) Y_2\right. \\
&\left.+g\left(Y_2, Y_3\right) Q Y_1-g\left(Y_1, Y_3\right) Q Y_2\right]
\end{aligned}
\tag{3.24}
$$

For a $(2m+1)-$dimensional Lorentz Sasakian space form, if we choose $Y_3=\xi$ in (3.24) we can write

$$
\begin{aligned}
&\mathscr{M}\left(Y_1, Y_2\right) \xi=\tfrac{c+1}{4m}\left[\eta\left(Y_2\right) Y_1-\eta\left(Y_1\right) Y_2\right] \\
&+\tfrac{1}{2m}\left[\eta\left(Y_2\right) Q Y_1-\eta\left(Y_1\right) Q Y_2\right].
\end{aligned}
\tag{3.25}
$$

On the other hand, if we take the inner product of both sides of (3.24) by ξ, we get

$$
\begin{aligned}
&\eta\left(\mathscr{M}\left(Y_1, Y_2\right) Y_3\right)=\tfrac{c+1}{4m} g\left(\eta\left(Y_1\right) Y_2-\eta\left(Y_2\right) Y_1, Y_3\right) \\
&-\tfrac{1}{2m} S\left(\eta\left(Y_2\right) Y_1-\eta\left(Y_1\right) Y_2, Y_3\right).
\end{aligned}
\tag{3.26}
$$

Definition 3.13. *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $\mathscr{M}\cdot S$ and $Q(g,S)$ are linearly dependent, then it is said to be $\mathscr{M}-$**projective Ricci pseudosymmetric.***

In this case, there exists a function L_4 on $\tilde{N}(c)$ such that

$$
\mathscr{M}\cdot S=L_4 Q(g,S).
$$

In particular, if $L_4=0$, the manifold $\tilde{N}(c)$ is said to be $\mathscr{M}-$**projective Ricci semisymmetric.**

Let us now investigate the $\mathscr{M}-$projective Ricci pseudosymmetric case of the Lorentz Sasakian space form admitting almost $\eta-$Ricci soliton.

Theorem 3.14. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $\mathscr{M}-$projective Ricci pseudosymmetric, then*

$$L_4 = \frac{4\kappa_1[(c+1)-2m]-(c+1)[(c+1)-4m]-4\kappa_1^2}{4m[2\kappa_1-(c+1)+4m]},$$

provided $2\kappa_1 \neq (c+1)-4m$.

Proof. Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be $\mathscr{M}-$projective Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. That is mean

$$(\mathscr{M}(Y_1,Y_2)\cdot S)(Y_4,Y_5) = L_4 Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5 \in \Gamma\left(T\tilde{N}\right)$. From the last equation, we have

$$\begin{aligned} &S(\mathscr{M}(Y_1,Y_2)Y_4,Y_5)+S(Y_4,\mathscr{M}(Y_1,Y_2)Y_5) \\ &= L_4\left\{S((Y_1\wedge_g Y_2)Y_4,Y_5)+S(Y_4,(Y_1\wedge_g Y_2)Y_5)\right\}. \end{aligned} \tag{3.27}$$

If we choose $Y_5=\xi$ in (3.27) we get

$$\begin{aligned} &S(\mathscr{M}(Y_1,Y_2)Y_4,\xi)+S(Y_4,\mathscr{M}(Y_1,Y_2)\xi) \\ &= L_4\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi) \\ &+S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \end{aligned} \tag{3.28}$$

If we make use of (2.5) and (3.25) in (3.28), we have

$$\begin{aligned} &-\left[\tfrac{(c+1)-4m}{2}\right]\eta(\mathscr{M}(Y_1,Y_2)Y_4) \\ &+S\left(Y_4,\tfrac{c+1}{4m}[\eta(Y_2)Y_1-\eta(Y_1)Y_2]\right. \\ &\left.+\tfrac{1}{2m}[\eta(Y_2)QY_1-\eta(Y_1)QY_2]\right) \\ &= L_4\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right. \\ &+S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \end{aligned} \tag{3.29}$$

If we by using (3.26) in the (3.29), we get

$$\begin{aligned} &-\tfrac{(c+1)[(c+1)-4m]}{8m}g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4) \\ &+\tfrac{(c+1)-4m}{4m}S(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4) \\ &+S\left(Y_4,\tfrac{c+1}{4m}[\eta(Y_2)Y_1-\eta(Y_1)Y_2]\right. \\ &\left.+\tfrac{1}{2m}[\eta(Y_2)QY_1-\eta(Y_1)QY_2]\right) \\ &= L_4\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right. \\ &+S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\}. \end{aligned} \tag{3.30}$$

If we use (3.2) in the (3.30), we can write

$$\begin{aligned}
&-\tfrac{(c+1)[(c+1)-4m]}{8m} g\left(\eta\left(Y_1\right) Y_2-\eta\left(Y_2\right) Y_1, Y_4\right) \\
&-\tfrac{\kappa_1[(c+1)-4m]}{4m} g\left(\eta\left(Y_2\right) Y_1-\eta\left(Y_1\right) Y_2, Y_4\right) \\
&-\tfrac{\kappa_1(c+1)}{4m} g\left(Y_4, \eta\left(Y_2\right) Y_1-\eta\left(Y_1\right) Y_2\right) \\
&-\tfrac{\kappa_1}{2m} S\left(\eta\left(Y_2\right) Y_1-\eta\left(Y_1\right) Y_2, Y_4\right) \\
&=L_4\left\{-\left[\tfrac{(c+1)-4m}{2}\right] g\left(\eta\left(Y_1\right) Y_2-\eta\left(Y_2\right) Y_1, Y_4\right)\right. \\
&\left.-\kappa_1 g\left(Y_4, \eta\left(Y_1\right) Y_2-\eta\left(Y_2\right) Y_1, Y_4\right)\right\}.
\end{aligned} \tag{3.31}$$

Again, if we use (3.2) in the (3.31), we obtain

$$\begin{aligned}
&\left[\tfrac{\kappa_1[(c+1)-4m]}{4m}+\tfrac{\kappa_1(c+1)}{4m}-\tfrac{(c+1)[(c+1)-4m]}{8m}\right. \\
&\left.-\tfrac{\kappa_1^2}{2m}+L_4\left(\tfrac{(c+1)-4m}{2}-\kappa_1\right)\right] \times \\
&g\left(\eta\left(Y_1\right) Y_2-\eta\left(Y_2\right) Y_1, Y_4\right)=0.
\end{aligned} \tag{3.32}$$

It is clear from (3.32)

$$L_4=\frac{4\kappa_1[(c+1)-2m]-(c+1)[(c+1)-4m]-4\kappa_1^2}{4m[2\kappa_1-(c+1)+4m]},$$

which proves our assertion □

We have the following corollaries.

Corollary 3.15. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $\mathscr{M}-$projective Ricci semisymmetric, then*

$$\kappa_1=\frac{(c+1)-4m}{2},$$

or

$$\kappa_1=\frac{c+1}{2}.$$

Corollary 3.16. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a $\mathscr{M}-$projective Ricci semisymmetric, then we observe that:*
i) $\tilde{N}(c)$ is shrinking, if κ_1 is between $\frac{(c+1)-4m}{2}$ and $\frac{c+1}{2}$,
ii) $\tilde{N}(c)$ is steady for $\kappa_1=\frac{(c+1)-4m}{2}$ and $\kappa_1=\frac{c+1}{2}$,
iii) $\tilde{N}(c)$ is expanding for other cases of κ_1.

For a $(2m+1)-$dimensional semi-Riemannian manifold N, the W_1-curvature tensor is defined as

$$W_1\left(Y_1,Y_2\right)Y_3=R\left(Y_1,Y_2\right)Y_3+\frac{1}{2m}\left[S\left(Y_2,Y_3\right)Y_1-S\left(Y_1,Y_3\right)Y_2\right]. \tag{3.33}$$

For a $(2m+1)-$dimensional Lorentz Sasakian space form, if we choose $Y_3=\xi$ in (3.33), we can write

$$W_1\left(Y_1,Y_2\right)\xi=\frac{8m-(c+1)}{4m}\left[\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2\right], \tag{3.34}$$

and similarly if we take the inner product of both sides of (3.33) by ξ, we get

$$\eta\left(W_1\left(Y_1,Y_2\right)Y_3\right)=\frac{8m-(c+1)}{4m}g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_3\right). \tag{3.35}$$

Definition 3.17. *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $W_1 \cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_1-**Ricci pseudosymmetric.***

In this case, there exists a function L_5 on $\tilde{N}(c)$ such that

$$W_1 \cdot S = L_5 Q(g,S).$$

In particular, if $L_5 = 0$, the manifold $\tilde{N}(c)$ is said to be W_1-**Ricci semisymmetric.**

Let us now investigate the W_1-Ricci pseudosymmetric case of the Lorentz Sasakian space form.

Theorem 3.18. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a W_1-Ricci pseudosymmetric, then*

$$L_5 = \frac{[8m-(c+1)][2\kappa_1-(c+1)+4m]}{4m[4m-(c+1)-2\kappa_1]},$$

provided $2\kappa_1 \neq 4m-(c+1)$.

Proof. Let be assume that Lorentz Sasakian space form $\tilde{N}(c)$ be W_1-Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form $\tilde{N}(c)$. That is mean

$$(W_1(Y_1,Y_2)\cdot S)(Y_4,Y_5) = L_5 Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5 \in \Gamma\left(T\tilde{N}\right)$. From the last equation, we have

$$\begin{aligned} &S(W_1(Y_1,Y_2)Y_4,Y_5)+S(Y_4,W_1(Y_1,Y_2)Y_5) \\ &= L_5\left\{S((Y_1\wedge_g Y_2)Y_4,Y_5)+S(Y_4,(Y_1\wedge_g Y_2)Y_5)\right\}. \end{aligned} \tag{3.36}$$

If we choose $Y_5=\xi$ in (3.36) we get

$$\begin{aligned} &S(W_1(Y_1,Y_2)Y_4,\xi)+S(Y_4,W_1(Y_1,Y_2)\xi) \\ &= L_5\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi) \\ &+S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}. \end{aligned} \tag{3.37}$$

If we make use of (2.5) and (3.34) in (3.37), we have

$$\begin{aligned} &S\left(Y_4,\tfrac{8m-(c+1)}{4m}[\eta(Y_2)Y_1-\eta(Y_1)Y_2]\right) \\ &-\left[\tfrac{(c+1)-4m}{2}\right]\eta(W_1(Y_1,Y_2)Y_4) \\ &= L_5\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right. \\ &\left.+S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\right\}. \end{aligned} \tag{3.38}$$

If we use (3.35) in the (3.38), we get

$$\begin{aligned} &\tfrac{[4m-(c+1)][8m-(c+1)]}{8m}g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4) \\ &+\tfrac{8m-(c+1)}{4m}S(\eta(Y_2)Y_1-\eta(Y_1)Y_2,Y_4) \\ &= L_5\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right. \\ &\left.+S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right\}. \end{aligned} \tag{3.39}$$

If we use (3.2) in the (3.39), we can write

$$\left\{\tfrac{8m-(c+1)}{4m}\left[\kappa_1+\tfrac{4m-(c+1)}{2}\right]+L_5\left[\tfrac{(c+1)-4m}{2}+\kappa_1\right]\right\}\times \\ g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_4\right)=0 \tag{3.40}$$

It is clear from (3.40)

$$L_5=\frac{[8m-(c+1)]\left[2\kappa_1-(c+1)+4m\right]}{4m\left[4m-(c+1)-2\kappa_1\right]}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 3.19. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a W_1-Ricci semisymmetric, then $\tilde{N}(c)$ is either real space form with $c=8m-1$ constant section curvature or $\kappa_1=\frac{(c+1)-4m}{2}$.*

Corollary 3.20. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a W_1-Ricci semisymmetric, then we conclude that:*
i) Let $8m>c+1$.
a) $\tilde{N}(c)$ is expanding, if $(c+1)>4m$.
b) $\tilde{N}(c)$ is shrinking, if $(c+1)<4m$.
ii) Let $8m<c+1$.
c) $\tilde{N}(c)$ is shrinking, if $(c+1)>4m$.
d) $\tilde{N}(c)$ is expanding, if $(c+1)<4m$.

For a $(2m+1)-$dimensional semi-Riemannian manifold N, the W_2-curvature tensor is defined as

$$W_2\left(Y_1,Y_2\right)Y_3=R\left(Y_1,Y_2\right)Y_3-\frac{1}{2m}\left[g\left(Y_2,Y_3\right)QY_1-g\left(Y_1,Y_3\right)QY_2\right]. \tag{3.41}$$

For a $(2m+1)-$dimensional Lorentz Sasakian spacew form $\tilde{N}(c)$, if we choose $Y_3=\xi$ in (3.41), we can write

$$W_2\left(Y_1,Y_2\right)\xi=\left[\eta\left(Y_2\right)Y_1-\eta\left(Y_1\right)Y_2\right] \\ -\tfrac{1}{2m}\left[\eta\left(Y_1\right)QY_2-\eta\left(Y_2\right)QY_1\right]. \tag{3.42}$$

Furthermore, if we take the inner product of both sides of (3.41) by ξ, we get

$$\eta\left(W_2\left(Y_1,Y_2\right)Y_3\right)=g\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_3\right) \\ +\tfrac{1}{2m}S\left(\eta\left(Y_1\right)Y_2-\eta\left(Y_2\right)Y_1,Y_3\right). \tag{3.43}$$

Definition 3.21. *Let $\tilde{N}(c)$ be a $(2m+1)-$dimensional Lorentz Sasakian space form. If $W_2\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_2-**Ricci pseudosymmetric.***

In this case, there exists a function L_6 on $\tilde{N}(c)$ such that

$$W_2\cdot S=L_6Q(g,S).$$

In particular, if $L_6=0$, the manifold $\tilde{N}(c)$ is said to be W_2-**Ricci semisymmetric.**
Let us now investigate the W_2-Ricci pseudosymmetric of the Lorentz Sasakian space form.

Theorem 3.22. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a W_2-Ricci pseudosymmetric, then*

$$L_6=\frac{\kappa_1(1-2m)+m\left[(c+1)-4m\right]+\kappa_1^2}{m\left[2\kappa_1+(c+1)-4m\right]},$$

provided $2\kappa_1\neq 4m-(c+1)$.

Proof. Let be assume that Lorentz Sasakian space form be W_2-Ricci pseudosymmetric and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on Lorentz Sasakian space form. That is mean

$$(W_2(Y_1,Y_2)\cdot S)(Y_4,Y_5)=L_6Q(g,S)(Y_4,Y_5;Y_1,Y_2),$$

for all $Y_1,Y_2,Y_4,Y_5\in\Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned}&S(W_2(Y_1,Y_2)Y_4,Y_5)+S(Y_4,W_2(Y_1,Y_2)Y_5)\\&=L_6\left\{S((Y_1\wedge_g Y_2)Y_4,Y_5)+S(Y_4,(Y_1\wedge_g Y_2)Y_5)\right\}.\end{aligned}\tag{3.44}$$

If putting $Y_5=\xi$ in (3.44), we get

$$\begin{aligned}&S(W_2(Y_1,Y_2)Y_4,\xi)+S(Y_4,W_2(Y_1,Y_2)\xi)\\&=L_6\{S(g(Y_2,Y_4)Y_1-g(Y_1,Y_4)Y_2,\xi)\\&+S(Y_4,\eta(Y_2)Y_1-\eta(Y_1)Y_2)\}.\end{aligned}\tag{3.45}$$

If we make use of (2.5) and (3.42) in (3.45), we have

$$\begin{aligned}&-\left[\tfrac{(c+1)-4m}{2}\right]\eta(W_2(Y_1,Y_2)Y_4)\\&+S(Y_4,[\eta(Y_2)Y_1-\eta(Y_1)Y_2]\\&-\tfrac{1}{2m}[\eta(Y_1)QY_2-\eta(Y_2)QY_1])\\&=L_6\left\{-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right.\\&+S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\}.\end{aligned}\tag{3.46}$$

If we use (3.43) in the (3.46), we get

$$\begin{aligned}&-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&+\tfrac{1}{2m}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&+S(Y_4,[\eta(Y_2)Y_1-\eta(Y_1)Y_2]\\&-\tfrac{1}{2m}[\eta(Y_1)QY_2-\eta(Y_2)QY_1]\\&=L_6\{S(Y_4,\eta(Y_1)Y_2-\eta(Y_2)Y_1)\\&\left.-\left[\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\right\}.\end{aligned}\tag{3.47}$$

If we use (3.2) in the (3.47), we have

$$\begin{aligned}&\left[\kappa_1-\tfrac{\kappa_1}{2m}-\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&+\tfrac{\kappa_1}{2m}S(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\\&=-L_6\left[\kappa_1+\tfrac{(c+1)-4m}{2}\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\end{aligned}\tag{3.48}$$

Again, if we use (3.2) in (3.48), we obtain

$$\begin{aligned}&\left[\kappa_1-\tfrac{\kappa_1}{2m}-\tfrac{(c+1)-4m}{2}-\tfrac{\kappa_1^2}{2m}\right.\\&\left.+L_6\left(\kappa_1+\tfrac{(c+1)-4m}{2}\right)\right]g(\eta(Y_1)Y_2-\eta(Y_2)Y_1,Y_4)\end{aligned}\tag{3.49}$$

It is clear from (3.49)

$$L_6=\frac{\kappa_1\,(1-2m)+m\,[(c+1)-4m]+\kappa_1^2}{m\,[2\kappa_1+(c+1)-4m]}.$$

This completes the proof. □

We can give a result of this theorem as follows.

Corollary 3.23. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a W_2- Ricci semisymmetric, then*

$$\kappa_1=-\frac{1}{2}\left[-(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right],$$

or

$$\kappa_1=\frac{1}{2}\left[(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right].$$

Corollary 3.24. *Let $\tilde{N}(c)$ be Lorentz Sasakian space form and $(g,\xi,\kappa_1,\kappa_2)$ be almost $\eta-$Ricci soliton on $\tilde{N}(c)$. If $\tilde{N}(c)$ is a W_2-Ricci semisymmetric, then we observe that*
i) $\tilde{N}(c)$ is shrinking, if κ_1 is between $-\frac{1}{2}\left[-(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right]$ and $\frac{1}{2}\left[(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right]$,
ii) $\tilde{N}(c)$ is steady for $-\frac{1}{2}\left[-(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right]$
and $\frac{1}{2}\left[(2m-1)+\sqrt{-4(c+2)m+20m^2+1}\right]$,
iii) $\tilde{N}(c)$ is expanding for other cases of κ_1.

4. Conclusion

In this paper, we consider pseudosymmetric Lorentz Sasakian space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentz Sasakian space forms admits $\eta-$Ricci soliton have introduced according to the choice of some special curvature tensors such as Riemann, concircular, projective, $\mathscr{M}-$projective, W_1 and W_2. Then, again according to the choice of the curvature tensor, necessary conditions are given for Lorentz Sasakian space form admits $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made under the some conditions.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, http://arXiv.org/abs/math/0211159, (2002), 1–39.

[2] G. Perelman, *Ricci flow with surgery on three manifolds*, http://arXiv.org/abs/math/0303109, (2003), 1–22.

[3] R. Sharma, *Certain results on k-contact and (k,μ)-contact manifolds*, J. Geom., **89** (2008),138–147.

[4] S.R. Ashoka, C.S. Bagewadi, G. Ingalahalli, *Certain results on Ricci Solitons in $\alpha-$Sasakian manifolds*, Hindawi Publ. Corporation, Geometry, Vol.(2013), Article ID 573925,4 Pages.

[5] S.R. Ashoka, C.S. Bagewadi, G. Ingalahalli, *A geometry on Ricci solitons in $(LCS)_n$ manifolds*, Diff. Geom.-Dynamical Systems, **16** (2014), 50–62.

[6] C.S. Bagewadi, G. Ingalahalli, *Ricci solitons in Lorentzian-Sasakian manifolds*, Acta Math. Acad. Paeda. Nyire., **28** (2012), 59-68.

[7] G. Ingalahalli, C. S. Bagewadi, *Ricci solitons in $\alpha-$Sasakian manifolds*, ISRN Geometry, Vol.(2012), Article ID 421384, 13 Pages.

[8] C.L. Bejan, M. Crasmareanu, *Ricci Solitons in manifolds with quasi-contact curvature*, Publ. Math. Debrecen, **78** (2011), 235-243.

[9] A. M. Blaga, *$\eta-$Ricci solitons on para-kenmotsu manifolds*, Balkan J. Geom. Appl., **20** (2015), 1–13.

[10] S. Chandra, S.K. Hui, A. A. Shaikh, *Second order parallel tensors and Ricci solitons on $(LCS)_n$-manifolds*, Commun. Korean Math. Soc., **30** (2015), 123–130.

[11] B.Y. Chen, S. Deshmukh, *Geometry of compact shrinking Ricci solitons*, Balkan J. Geom. Appl., **19** (2014), 13–21.

[12] S. Deshmukh, H. Al-Sodais, H. Alodan, *A note on Ricci solitons*, Balkan J. Geom. Appl.,**16** (2011), 48–55.

[13] C. He, M. Zhu, *Ricci solitons on Sasakian manifolds*, arxiv:1109.4407V2, [Math DG], (2011).

[14] M. Atçeken, T. Mert, P. Uygun, *Ricci-Pseudosymmetric $(LCS)_n$ $-$manifolds admitting almost $\eta-$Ricci solitons*, Asian J. Math. Comput. Research, **29**(2), 23-32,2022.

[15] H. Nagaraja, C. R. Premalatta, *Ricci solitons in Kenmotsu manifolds*, J. Math. Analysis, **3**(2) (2012), 18–24.

[16] M. M. Tripathi, *Ricci solitons in contact metric manifolds*, arxiv:0801,4221 V1, [Math DG], (2008).

[17] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Volume 203 of Progress in Mathematics, Birkhauser Boston, Inc., Boston, MA, USA, 2nd edition, 2010.

[18] P. Alegre, D. E. Blair, A. Carriazo, *Generalized Sasakian space form*, Israel J. Math., **141** (2004), 157-183.

[19] P. Alegre, A. Carriazo, *Semi-Riemannian generalized Sasakian space forms*, Bulletin of the Malaysian Math. Sci. Soc., **41**(1) (2018), 1–14.

[20] J.T. Cho, M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J, **61**(2) (2009), 205-212.

[21] G. Ayar, M. Yıldırım, *$\eta-$Ricci solitons on nearly Kenmotsu manifolds*, Asian-European J. Math., **12**(6), 2040002 (2019).

[22] G. Ayar, M. Yıldırım, *Ricci solitons and gradient Ricci solitons on nearly Kenmotsu manifolds*, Facta Universitatis, Series: Mathematics and Informatics, (2019), 503-510.

[23] M.Yıldırım, G. Ayar, *Ricci solitons and gradient Ricci solitons on nearly Cosymplectic manifolds*, J. Univers. Math., **4**(2) (2021), 201-208.

[24] G. Ayar, D. Dilek, *Ricci Solitons on Nearly Kenmotsu Manifolds with Semi-symmetric Metric Connection*, Journal of Engineering Technology and Applied Sciences, **4**(3) (2019), 131-140.

[25] G. Ayar, *Kenmotsu manifoldlarda konformal ricci solitonlar*, Afyon Kocatepe Üniversitesi Fen Ve Mühendislik Bilimleri Dergisi **19**(3) (2019), 635-642.

[26] M. Turan, C. Yetim, S.K. Chaubey, *On quasi-Sasakian 3-manifolds admitting $\eta-$Ricci solitons*, Filomat, **33**(15) (2019), 4923–4930.

[27] G. Ayar, S. K. Chaubey, *M-projective curvature tensor over cosymplectic manifolds*, Differ. Geom. Dyn. Syst., **21** (2019), 23-33.

[28] G. Ayar, H.R Cavusoglu, *Conharmonic curvature tensor on nearly cosymplectic manifolds with generalized tanaka-webster connection*, Sigma J. Eng. Nat. Sci, **39**(5) (2021), 9-13.

[29] G. Ayar, *Pseudo-projective and quasi-conformal curvature tensors on Riemannian submersions*, Math. Meth. App. Sci., **44**(17), 13791-13798.

[30] G. Ayar, *Some curvature tensor relations on nearly cosymplectic manifolds with Tanaka-Webster Connection*, Univers. J. Math. Appl., **5**(1) (2022), 24-31.

[31] S.K. Chaubey, R. H. Ojha,*On the m-projective curvature tensor of a Kenmotsu manifold*, Differ. Geom. Dyn. Syst., **12**(2010), 52-60.

[32] S. K Chaubey ,S. Prakash, R Nivas, *Some Properties of M-projective curvature tensor m- in Kenmotsu manifolds*, Bulletin of Mathematical Analysis and Applications, **4**(3) (2012), 48-56.