Advances in the Theory of Nonlinear Analysis and its Applications 7 (2023) No. 1, 195–204.
https://doi.org/10.31197/atnaa.1236577
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Triangular functions in solving weakly singular Volterra integral equations

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Abstract

In this paper, the orthogonal triangular functions are employed as a basis functions for solving the weakly singular Volterra integral equations of the second kind. Powerful properties of these functions and some operational matrices are utilized in a direct method to reduce the main singular integral equation to some algebraic system. The presented method does not need any integration for obtaining the constant coefficients. The method is computationally attractive, and applications are demonstrated through illustrative examples.

1. Introduction

Weakly singular integral equations, or integral equations with singular kernels, are important class in the integral equations. In 1896, Mach applied this class of equations in the study of compressible flows around axially symmetric bodies [1]. They have been used for mathematical modeling of many phenomena in physics and engineering, such as; airfoils, contact radiations, fracture mechanics, molecular conductions, and elastodynamics [2]-[4].

Historically, the study of integral equations was started by Abel's problem. The generalized Abel's integral equations on a finite segment appeared in the paper of Zeilon [5] for the first time. Due to increasing the attention to fractional calculus, the study of singular integral equations is attracted more significance. The Riemann-Liouville integral and derivative fractional and Caputo fractional derivative operators have been used in many studies [6]-[7], where defined by singular kernels.

The aim of this study is to present a high order computational method for solving a special case of singular Volterra integral equations of the second kind, defined as follows:

$$y(x) = f(x) - \int_a^x K(x,t)|x-t|^{-\alpha} y(t)dt, \tag{1}$$

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Received January 16, 2023; Accepted: February 9, 2023; Online: February 11, 2023.

$$0 < \alpha < 1, \quad a \le x \le b,$$

where $f(x)$ and $K(x,t)$ are known functions and $y(x)$ is the unknown function that to be determined.
The construction of high order methods for solving equation (1) is, however, not an easy task because of the singularity in the weakly singular kernel. In fact, in this case the solution y is generally not differentiable at the endpoints of the interval [8]-[11], and due to this, to the best of the authors' knowledge the best convergence rate ever achieved remains only at polynomial order. For example, if we set uniform meshes with $n+1$ grid points and apply the spline method so for order m, then the convergence rate is only $O(n^{-2P})$ at most [12]-[13], and it can not be improved by increasing m. One way of remedying this is to introduce graded meshes [12]-[14]. Then the rate is improved to $O(n^{-m})$ [14] which now depends on m, but still at polynomial order.
Some numerical methods for solving weakly singular equations are; Bessel polynomials via collocation method [15], Clenshaw-Curtis-Filon quadrature [16], Laguerre functions [17], B-spline Wavelet Galerkin method [18], Block-pulse functions [19], Lagrange interpolation with Gauss Legendre quadrature nodes [20]. In this work we assume that the $K(x,t) \in [a,b] \times [a,b]$, satisfies in Lipschitz condition, that is:

$$|K(x_1,t) - K(x_2,t)| \le L_s|x_1 - x_2|, \tag{2}$$

and L_s is the Lipschitz constant. The techniques based on polynomials and wavelets are effective to obtain the solution of integral equations. But calculating constant coefficients requires the use of integration formula. Deb et al. introduced a new complementary pair of orthogonal triangular functions (TFs) and their applications to analysis of dynamic systems [21]-[22].
In this work,TFs are applied for solving weakly singular Volterra integral equations of the second kind.
The organization of this paper is arranged as follows. In section 2, we review these functions and their properties, also some operational matrices for these functions are introduced. In section 3, we apply these set of orthogonal functions for approximating the solution of Volterra singular integral equation of the second kind. Using the properties of TFs via Galerkin method, we reduce the singular integral equations to a algebraic system, which can be solved easily using some iterative methods. Powerful search technique can be used to find the optimal coefficients for desirable approximation. In section 4, we illustrate some numerical examples to show the efficiency and accuracy of this method and section 5 contains our conclusion.

2. Brief review of orthogonal triangular functions

In this section, the definitions of TFs and their properties are reviewed [22]. We start with the definition of Block-pulse functions (BPFs).

2.1. BPFs

Definition 2.1. For $M \in \mathbb{N}$, a $M-$set of BPFs in the $[0,T)$ is defined as [19]

$$B_i(x) = \begin{cases} 1, & ih \le x < (i+1)h \\ 0, & o.w. \end{cases}$$

where $i = 0,1,...,M-1$ and $h = \frac{T}{M}$. BPFs have some significant properties, such as; disjointness:

$$B_i(x)B_j(x) = \begin{cases} B_i(x), & i = j \\ 0, & i \ne j \end{cases} \quad i,j = 0,1,...,M-1, \tag{3}$$

orthogonality:

$$\langle B_i(t), B_j(t)\rangle = \int_0^1 B_i(t)B_j(t)dt = \begin{cases} \frac{1}{M}, & i = j \\ 0, & i \ne j \end{cases} \tag{4}$$

$$i,j = 0,1,...,M-1.$$

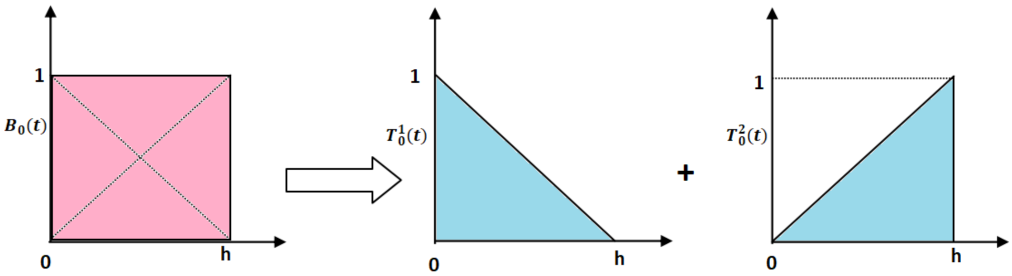

Figure 1: Construction of TFs by BPFs

2.2. Function approximation by BPFs

A square integrable $f(x)$ may be expanded by BPF series in $x \in [0, T)$ as

$$f(x) = \sum_{j=0}^{M-1} c_j B_j(x) = C_M^T \mathbf{B}_M(x), \tag{5}$$

where the constant coefficients c_j are given by

$$c_j = \frac{1}{h} \int_{jh}^{(j+1)h} f(x)dx, \qquad \mathbf{B}_M(x) = [B_0(x), ..., B_{M-1}(x)]^T. \tag{6}$$

Another important property of BPFs is completeness. In the other words, For every $f \in L^2[0, 1)$, when m approaches to the infinity, Parseval's identity holds, that is

$$\int_0^T f^2(x)dx = \sum_{j=0}^{\infty} c_j^2 \|B_j(x)\|^2.$$

2.3. TFs

To present a complementary pair of orthogonal TF sets and compare their basic properties with those of BPF sets, we define the m-set triangular functions. Suppose $\psi_0(x)$ be the first component of a M-set BPFs, we put

$$B_0(x) = T_0^1(x) + T_0^2(x),$$

where T_0^1 and T_0^2 functions are shown in Fig. 1.

Definition 2.2. Two M-sets of triangular functions are defined over the interval $[0, T)$ as

$$T_i^1(x) = \begin{cases} 1 - \frac{x-ih}{h}, & ih \leq x < (i+1)h \\ 0, & o.w. \end{cases} \tag{7}$$

$$T_i^2(x) = \begin{cases} \frac{x-ih}{h}, & ih \leq x < (i+1)h \\ 0, & o.w. \end{cases} \tag{8}$$

where, $i = 0, ..., M-1$ and $h = \frac{T}{M}$. In this paper, it is assumed that $T = 1$, so TFs are defined over $[0, 1)$, and $h = \frac{1}{M}$.
From the definition of TFs, it is clear that they are disjoint and complete [22]. Also Tfs are orthogonal:

$$\int_0^1 T_i^r(x) T_j^s(x)dt = \delta_{ij} \Delta_{r,s}, \tag{9}$$

where δ_{ij} is Kronecker delta and

$$\Delta_{r,s} = \begin{cases} \frac{h}{3} & r = s \in \{1, 2\}, \\ \frac{h}{6} & r \neq s. \end{cases}$$

Now we put

$$\mathbf{T}_M(x) = \left[T_0^1(x), ..., T_{M-1}^1(x), T_0^2(x), ..., T_{M-1}^2(x)\right]^T, \tag{10}$$

$\mathbf{T}_M(x)$ is called the 1D-TF vector. From the above representation and disjointness property, it is clear that

$$\mathbf{T}_M(x).\mathbf{T}_M^T(x) = diag\left\{\mathbf{T}_M(x)\right\}, \tag{11}$$

where $diag\left\{\mathbf{T}_M(x)\right\}$ is $2M \times 2M$ diagonal matrix.

2.4. Function approximation by TFs

The expansion of a function $f(x) \in L^2[0, 1)$ with 1D-TFs is given by

$$f(x) \simeq \sum_{j=0}^{M-1} \alpha_j T_j^1(x) + \sum_{j=0}^{M-1} \beta_j T_j^2(x) = C_M^T \mathbf{T}_M(x), \tag{12}$$

where

$$C_M = \left[\alpha_0, ..., \alpha_{M-1}, \beta_0, ..., \beta_{M-1}\right]^T, \tag{13}$$

and the constant coefficients α_j and β_j are the samples of function $f(x)$ such that

$$\alpha_j = f(jh), \qquad \beta_j = f((j+1)h).$$

As a result there is no need for integration. The vector C_M is called the 1D-TF coefficient vector.

2.5. TFs Operational matrix of integration

In this section, we construct the operational matrix of integration for TFs. For this purpose, we put

$$\int_0^x T^1(t)dt = \Theta_1 T^1(x) + \Theta_2 T^2(x), \tag{14}$$

$$\int_0^x T^2(t)dt = \Theta_1 T^1(x) + \Theta_2 T^2(x), \tag{15}$$

where Θ_1 and Θ_2 are $m \times m$ square matrices, as:

$$\Theta_1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & ... & 1 \\ 0 & 0 & 1 & ... & 1 \\ 0 & 0 & 0 & ... & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Theta_2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & 1 & 1 & ... & 1 \\ 0 & 0 & 1 & ... & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

therefore we can write

$$\int_0^x \mathbf{T}_M(t)dt = \Pi \mathbf{T}_M(x), \tag{16}$$

where, Π is operational matrix of integration that is given by

$$\Pi = \begin{pmatrix} \Theta_1 & \Theta_2 \\ \Theta_1 & \Theta_2 \end{pmatrix}_{2M \times 2M} \tag{17}$$

now the integral of every function $f(x)$ by using triangular functions can be approximated by

$$\int_0^x f(t)dt = \int_0^x C^T.\mathbf{T}_M(t)dt = C^T \Pi \mathbf{T}_M(x). \tag{18}$$

2.6. Approximation of two variable function by TFs

Now, we apply TFs for approximating the two variable function $\omega(x,t)$. We obtain

$$\omega(x,t) \quad \simeq \mathbf{T}_{M_1}(t)^T \Omega_{M_1,M_2} \mathbf{T}_{M_2}(x)^T, \tag{19}$$

where $\Omega_{M_1\times M_2}$ is $2M_1 \times 2M_2$ coefficients matrix. According to TFs approximation, we can decompose it as follows:

$$\Omega_{M_1\times M_2} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} T(t),$$

where, $W_{ij},\ i,j = 1,2$, are $M_1 \times M_2$ matrices, can be computed by sampling the function $\omega(x,t)$ at the nodes $x_i = ih_1$ and $t_j = jh_2$ as follows:

$$\Big[W_{11}\Big]_{i,j} = \omega\left(x_i, t_j\right)\Big|_{i=0,\dots,M_1-1}^{j=0,\dots,M_2-1},$$

$$\Big[W_{12}\Big]_{i,j} = \omega\left(x_i, t_j\right)\Big|_{i=0,\dots,M_1-1}^{j=1,\dots,M_2},$$

$$\Big[W_{21}\Big]_{i,j} = \omega\left(x_i, t_j\right)\Big|_{i=1,\dots,M_1}^{j=0,\dots,M_2-1},$$

$$\Big[W_{22}\Big]_{i,j} = \omega\left(x_i, t_j\right)\Big|_{i=1,\dots,M_1}^{j=1,\dots,M_2}.$$

Proposition 1. Let H be a $2M \times 2M$ matrix. It can be concluded that

$$\mathbf{T}_M^T(t) H \mathbf{T}_M(t) \simeq \hat{H} \mathbf{T}_M(t), \tag{20}$$

in which, $\hat{H}$ is a $2M$-vector with elements equal to the diagonal entries of matrix H.

Proposition 2. Let X be a $2M$-dimensional column vector. So, it can be similarly concluded that

$$\mathbf{T}_M^T(t) \mathbf{T}_M(t) X \simeq \tilde{X} \mathbf{T}_M(t), \tag{21}$$

where, $\tilde{X} = diag(X)$ is $2M \times 2M$ diagonal matrix.

Inner product of triangular functions in the interval $[0,1]$ can be written in matrix form as

$$\int_0^1 \mathbf{T}_M(t).\mathbf{T}_M^T(t) dt = R, \tag{22}$$

where R is defined as

$$R = \begin{pmatrix} \frac{h}{3} I_M & \frac{h}{6} I_M \\ \\ \frac{h}{6} I_M & \frac{h}{3} I_M \end{pmatrix}, \tag{23}$$

and I_M is M-dimensional identity matrix.

3. Description of the numerical method

In this section, we solve the singular integral equation (1) by triangular functions. First by using Lipschitz condition of kernel function, we tarnsform the singular integral equation to a nonsingular integral equation. For this purpose, the integral term in equation (1) can be written as:

$$\int_0^x K(x,t)|x-t|^{-\alpha}y(t)dt = \int_0^x \Big(K(x,t)-K(x,x)\Big)|x-t|^{-\alpha}y(t)dt$$
$$+K(x,x)\int_0^x |x-t|^{-\alpha}y(t)dt,$$

and

$$\int_0^x |x-t|^{-\alpha}y(t)dt = \int_0^x |x-t|^{-\alpha}\Big(y(t)-y(x)\Big)dt + y(x)\int_0^x |x-t|^{-\alpha}dt,$$

thus we have

$$\left|\int_0^x \left(K(x,t)-K(x,x)\right)(x-t)^{-\alpha}y(t)dt\right| \leq \int_0^x \Big|K(x,t)-K(x,x)\Big|\,|x-t|^{-\alpha}\,\big|y(t)\big|dt$$
$$\leq \int_0^x L_s|x-t|^{1-\alpha}|y(t)|dt,$$

now as $x \to t$,

$$\left|\int_0^x \left(K(x,t)-K(x,x)\right)(x-t)^{-\alpha}y(t)dt\right| \to 0. \tag{24}$$

On the other hand

$$\left|\int_0^x (x-t)^{-\alpha}\Big(y(t)-y(x)\Big)dt\right| \leq \int_0^x |x-t|^{-\alpha}\big|y(t)-y(x)\big|dt$$
$$\leq y'(\xi)\int_0^x |x-t|^{1-\alpha}dt$$
$$\leq y'(\xi)\frac{|x-t|^{2-\alpha}}{2-\alpha},$$

so, as $x \to t$

$$\left|\int_0^x (x-t)^{-\alpha}\left(y(t)-y(x)\right)dt\right| \to 0. \tag{25}$$

Now we introduce the function $H(x,t)$,

$$H(x,t) = \begin{cases} K(x,t)(x-t)^{-\alpha} & x \neq t \\ 0 & x = t. \end{cases} \tag{26}$$

So the integral term of equation (1) can be written as:

$$\int_0^x K(x,t)(x-t)^{-\alpha}y(t)dt = \int_0^x H(x,t)y(t)dt, \tag{27}$$

and we note that the new kernel function is not singular in $[0,1]$. Thus the integral equation 1 can be rewritten as follows:

$$y(x) = f(x) + \int_0^x H(x,t)y(t)dt. \tag{28}$$

For solving current equation, the unknown and known functions in equation (1) are expanded in terms of the triangular functions as follows:

$$y(x) \simeq y_M(x) = C_M^T\mathbf{T}_M(x) = \mathbf{T}_M^T(x)C_M, \tag{29}$$

$$f(x) \simeq f_M(x) = F_M^T \mathbf{T}_M(x) = \mathbf{T}_M^T(x) F_M, \tag{30}$$

and two variable kernel function $H(x,t)$ can be written as

$$H(x,t) = \mathbf{T}_M^T(x) H_{M,M} \mathbf{T}_M(t),$$

by substituting these relations in equation (1), we get

$$C_M^T \mathbf{T}_M(x) = F_M^T \mathbf{T}_M(x) + \int_0^x \mathbf{T}_M^T(x) H_{M,M} \mathbf{T}_M(t) \mathbf{T}_M^T(t) C_M dt, \tag{31}$$

considering equation (20), the current equation is written as

$$C_M^T \mathbf{T}_M(x) = F_M^T \mathbf{T}_M(x) + \mathbf{T}_M^T(x) H_{M,M} \tilde{C}_M \int_0^x \mathbf{T}_M(t) dt, \tag{32}$$

also by substituting equation (15) in equation (16), we have

$$C_M^T \mathbf{T}_M(x) = F_M^T \mathbf{T}_M(x) + \mathbf{T}_M^T(x) H_{M,M} \tilde{C}_M P \mathbf{T}_M(x), \tag{33}$$

putting $H_{M,M}\tilde{C}_M P = \Lambda_M$ and applying equation (20), we get

$$C_M^T \mathbf{T}_M(x) = F_M^T \mathbf{T}_M(x) + \hat{\Lambda}_M \mathbf{T}_M(x). \tag{34}$$

On the other hand $\hat{\Lambda}_M$ can be written as

$$\hat{\Lambda}_M = C_M^T \Delta_M, \tag{35}$$

where Δ_M is $2M \times 2M$ matrix, defined as follows

$$[\Delta_M]_{i,j} = P_{ik} \left[H_{M,M}\right]_{jk}, \qquad i,j,k = 1,2,...,2M,$$

therefore in equation (27), we get

$$C_M^T \mathbf{T}_M(x) = F_M^T \mathbf{T}_M(x) + C_M^T \Delta_M \mathbf{T}_M(x), \tag{36}$$

so the Volterra integral equation (1) is reduced into a linear algebraic system as

$$C_M^T \left(I - \Delta_M\right) = F_M^T, \tag{37}$$

by solving this system, the unknown vector C_M is obtained and consequently $y(x)$ is approximated.

4. Illustrative Examples

In this section, for showing the accuracy and efficiency of the described method, we present some examples, then we compare the results of our methods with the results of some other methods.

Example 1.

Consider the following singular integral equation of the second kind

$$y(x) + \int_0^x \frac{y(t)}{(x-t)^{1/2}} dt = x^2 + \frac{16}{15} x^{5/2},$$

with the exact solution $y(x) = x^2$. The numerical solutions for $y(x)$ are obtained for $m = 5, 10, 20$ and results are tabulated in table 1. Figure 2 shows the exact and approximated solutions ($M = 10$) of example 1, where the continuous line is the plot of exact solution and the numerical solution is plotted by discreet

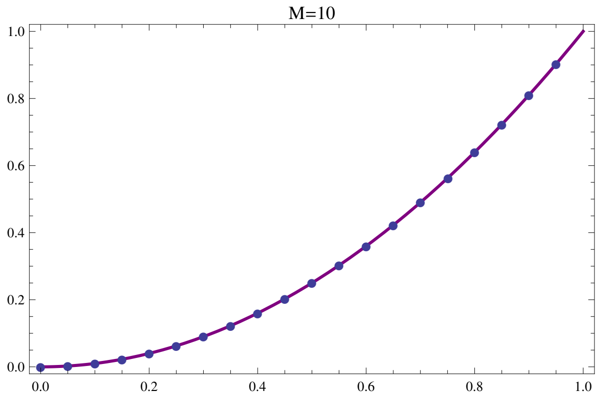

Figure 2: Exact and numerical solutions of example 1 for $M = 10$.

Table 1: Exact and approximate solutions of example 1 for different values of M.

x	$M = 5$	$M = 10$	$M = 20$	Exact
0	0	0	0	0
0.1	0.099156	0.010064	0.010001	0.01
0.2	0.041086	0.040052	0.040000	0.04
0.3	0.0884362	0.090012	0.090003	0.09
0.4	0.167382	0.160038	0.160005	0.16
0.5	0.254899	0.250049	0.250007	0.25
0.6	0.368320	0.360066	0.360002	0.36
0.7	0.491981	0.490015	0.490000	0.49
0.8	0.642579	0.640084	0.640001	0.64
0.9	0.811309	0.810033	0.640005	0.81
1	1.007747	1.000045	1.000000	1

Table 2: Exact and approximate solutions of example 2 for $M = 5, 10, 20$.

x	$M = 5$	$M = 10$	$M = 20$	Exact
0	0	0	0	0
0.1	0.032544	0.031695	0.031622	0.0316228
0.2	0.0896607	0.0894410	0.0894427	0.0894427
0.3	0.161052	0.164373	0.164317	0.164317
0.4	0.253317	0.252986	0.252982	0.252982
0.5	0.353488	0.353502	0.353553	0.353553
0.6	0.469214	0.464760	0.464758	0.464758
0.7	0.581139	0.585631	0.585662	0.585662
0.8	0.717351	0.715596	0.715542	0.715542
0.9	0.856024	0.853849	0.853815	0.853815
1	1.002064	1.000082	1.000004	1

plot markers for $x_j = 0.5j,\ j = 0, 1, ..., 19$.

Example 2. Consider the following singular integral equation

$$u(x) - \int_0^x \frac{u(t)}{(x-t)^{1/2}} dt = x^{3/2} - \frac{3}{8}\pi x^2, \tag{38}$$

with the exact solution $u(x) = x^{\frac{3}{2}}$.

The solution for $y(x)$ is obtained by the method in section 3 for $m = 5, 10$ and 20. In Table 2, we present exact and approximate solutions of example 2 in some arbitrary points.

Example 3. Consider the equation

$$y(x) = \frac{1}{\sqrt{x+1}} + \frac{\pi}{8} - \frac{1}{4}\arcsin(\frac{1-x}{1+x}) - \frac{1}{4}\int_0^x \frac{y(t)}{\sqrt{x-t}}dt,$$

with exact solution $y(x) = \frac{1}{\sqrt{x+1}}$. The solution for $y(x)$ is obtained by the method in section 3 for $m = 5, 10$ and 20. In Table 3, we report the exact and approximate solutions of example 3 and compare with the numerical results of [19] (BPFs method) for $k = 16$ and [23] (Legendre wavelets method) for $k = 1, M = 5$ in some arbitrary points.

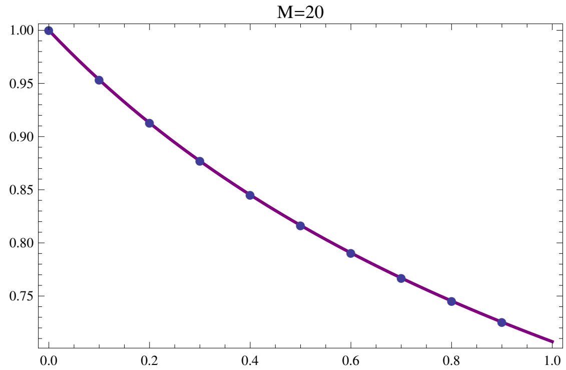

Figure 3: Exact and numerical solutions of example 3 for $M = 20$.

Table 3: Comparison of approximate solutions of example 3 for $M = 10, 20$ and BPFs for $k = 16$ and Legendre wavelets for $k = 1, M = 5$.

x	$M = 10$	$M = 20$	[19] $k = 16$	[23] $k = 1, M = 5$	Exact
0	1.000048	1.000000	0.99734	0.999432	1
0.1	0.953471	0.953463	–	–	0.953463
0.2	0.912860	0.912871	0.911748	0.91232	0.912871
0.3	0.877000	0.877058	–	–	0.877058
0.4	0.845167	0.845159	0.848041	0.8453212	0.845154
0.5	0.816484	0.816497	–	–	0.816497
0.6	0.790522	0.790569	0.788293	0.7905387	0.790569
0.7	0.766904	0.766960	–	–	0.766965
0.8	0.745316	0.745356	0.746027	0.745342	0.745356
0.9	0.725499	0.725476	–	–	0.725476
1	0.707137	0.707106	0.70423	0.707163	0.707107

5. Conclusions

In this paper, we proposed an advanced numerical model in solving Volterra Abel integral equation of the second kind by means of orthogonal triangular functions. The approach can be extended to nonlinear singular integral equations with little additional work. Further research along these lines is under progress and will be reported in due time.

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