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Existence and Decay of Solutions for a Parabolic-Type Kirchhoff Equation with Variable Exponents

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Abstract

This paper deals with a parabolic-type Kirchhoff equation with variable exponents. Firstly, we obtain the global existence of solutions by Faedo-Galerkin method. Later, we prove the decay of solutions by Komornik's inequality.

1. Introduction

In this work, we study the following parabolic-type Kirchhoff equation with variable exponents

$$\begin{cases} \left(1+|u|^{p(x)-2}\right)u_t+\Delta^2 u-M\left(\|\nabla u\|^2\right)\Delta u=|u|^{q(x)-2}u, & \text{in } (x,t)\in\Omega\times(0,T), \\ u(x,t)=\frac{\partial u}{\partial v}(x,t)=0, & \text{on } x\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x), & \text{in } x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n $(n\geq 1)$ with smooth boundary $\partial\Omega$ and

$$M(s)=1+s^\gamma,\ \ \gamma\geq 1.$$

The variable exponents $p(.)$ and $q(.)$ are given as measurable functions on Ω satisfying

$$\begin{cases} 2\leq p^-\leq p(x)\leq p^+\leq p^*, \\ 2\leq q^-\leq q(x)\leq q^+\leq q^*, \end{cases}$$

where

$$\begin{cases} p^-=ess\inf_{x\in\Omega}p(x), & p^+=ess\sup_{x\in\Omega}p(x), \\ q^-=ess\inf_{x\in\Omega}q(x), & q^+=ess\sup_{x\in\Omega}q(x), \end{cases}$$

and

$$p^*,q^*=\begin{cases} \infty, & \text{if } n\leq 4, \\ \frac{2n}{n-4} & \text{if } n>4. \end{cases} \tag{1.2}$$

We also suppose that $p(.)$ and $q(.)$ satisfy the log-Hölder continuity condition:

$$|p(x)-p(y)|\leq-\frac{A}{\log|x-y|},$$

for a.e. $x,y\in\Omega$, $|x-y|<\delta$ with $A>0$, $0<\delta<1$.

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- Parabolic type equation: Many phenomena in physics lead up to problems that deal with parabolic type equations, such as; mathematical description of the reaction-diffusion or diffusion, population dynamic processes and heat transfer [1].
- Kirchhoff equation: The Kirchhoff equation is among the famous wave equation's model which describe small vibration amplitude of elastic strings. This equation has been introduced in 1876 by Kirchhoff [2].
- Variable exponent: The problems with variable exponents arises in many branches in sciences such as electrorheological fluids, nonlinear elasticity theory and image processing [3]-[5].

In [6], Wu et al. established the blow up of solutions with positive initial energy for the following equation

$$u_t - \Delta u = u^{p(x)}.$$

Later, some authors get new results for the same equation to blow up result (see [7]-[10]).
In [11], Qu et al. studied the fourth order parabolic equation as follows

$$u_t + \Delta^2 u = u^{p(x)}.$$

The authors studied the asymptotic behavior of solutions.
When there is no fourth-order term $\Delta^2 u$, (1.1) is reduced to the following equation

$$u_t - M\left(\|\nabla u\|^2\right)\Delta u + |u|^{m(x)-2} u_t = |u|^{r(x)-2} u.$$

Khaldi et al. [12] studied the global existence and stability of solutions.
Recently, problems with variable exponents have been handled carefully in several papers, some results relating the local existence, global existence, blow up and stability have been found ([13]-[17]).
In this work, we considered the existence and decay of solutions of the parabolic type Kirchhoff equation with variable exponents, motivated by above works. To our best knowledge, there is no research, related to the parabolic type Kirchhoff equation (1.1) with fourth-order term ($\Delta^2 u$) and variable exponent source term ($|u|^{q(x)-2}u$), hence, our work is the generalization of the above studies.
This work consists of four parts: Firstly, in part 2, we give some needed theories about Lebesgue and Sobolev space with variable-exponents. Then, in Section 3, we get the existence result by the Faedo-Galerkin method. Moreover, in Section 4, we obtain the decay of solutions by the Komornik's inequality.

2. Preliminaries

Throughout this work, we denote by $\|.\|_p$ the $L^p(\Omega)$ norm. Also, we give some needed theories about Lebesgue space and Sobolev space with variable-exponents (for detailed, see [4, 18, 19]).
Let $p : \Omega \to [1,\infty]$ be a measurable function. We introduce the Lebesgue space with variable exponent $p(.)$

$$L^{p(.)}(\Omega) = \left\{u : \Omega \to R \text{ measurable in } \Omega,\ \ \rho_{p(.)}(\lambda u) \ < \infty,\ \text{ for some } \lambda > 0\right\},$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx.$$

The norm, called Luxemburg's norm, is defined by

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(.)}(\Omega)$ is a Banach space.
Next we define the variable-exponent Sobolev space $W^{m,p(.)}(\Omega)$ as

$$W^{m,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) \text{ such that } D^\alpha u \text{ exists and } D^\alpha u \in L^{p(.)}(\Omega),\ |\alpha| \le m\right\}.$$

Lemma 2.1. *[4]. If*

$$1 \le p_1 := ess\inf_{x\in\Omega} p(x) \le p(x) \le p_2 := ess\sup_{x\in\Omega} p(x) < \infty,$$

then we have

$$\min\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \le \rho_{p(.)}(u) \le \max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\},$$

for any $u \in L^{p(.)}$.

Lemma 2.2. *(Hölder's inequality)[4]. Assume that $p,q,s \ge 1$ are measurable functions defined on Ω such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \text{ for a.e. } y \in \Omega.$$

If $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, then $uv \in L^{s(.)}(\Omega)$ with

$$\|uv\|_{s(.)} \le c\,\|u\|_{p(.)}\,\|v\|_{q(.)}.$$

Lemma 2.3. *[4]. If $p:\Omega\to[1,\infty)$ is a measurable function satisfying (1.2) then the embedding $H_0^2(\Omega)\hookrightarrow H_0^1(\Omega)\hookrightarrow L^{p(.)}$ is continuous and compact.*

Lemma 2.4. *[20]. Let $\varphi:R^+\to R^+$ is a nonincreasing function and suppose that there are two constants $\alpha>0$ and $c>0$ such that*

$$\int_0^\infty \varphi^{\alpha+1}(s)\,ds\le c\varphi^{\alpha}(0)\,\varphi(s)\ \ \forall t\in R^+.$$

Then we have

$$\varphi(t)\le\varphi(0)\left(\frac{c+\alpha t}{c+\alpha c}\right)^{-1/\alpha}\quad\forall t\ge c.$$

3. Existence

In this part, we state and prove the global existence result. Now, let us introduce some functionals as follows:

$$E(t)=\frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}-\int_\Omega\frac{1}{q(x)}|u|^{q(x)}\,dx,$$

$$I(t)=\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}-\int_\Omega|u|^{q(x)}\,dx.$$

Lemma 3.1. *Suppose that (1.2) holds. Then*

$$E'(t)=-\|u_t\|_2^2-\int_\Omega|u|^{p(x)-2}|u_t|^2\,dx\le0,\tag{3.1}$$

and

$$E(t)\le E(0).$$

Proof. We multiply the eq. (1.1) by u_t and integrate over Ω, we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}-\int_\Omega\frac{1}{q(x)}|u|^{q(x)}\,dx\right)\\ =\;&-\|u_t\|_2^2-\int_\Omega|u|^{p(x)-2}|u_t|^2\,dx,\end{aligned}$$

thus

$$E'(t)=-\|u_t\|_2^2-\int_\Omega|u|^{p(x)-2}|u_t|^2\,dx\le0.$$

A simple integration of (3.1) over $(0,T)$, yields

$$E(t)\le E(0).$$

□

Lemma 3.2. *Let assumption (1.2) holds. Further assume that $q_1>2(\gamma+1)$, $I(0)>0$ and*

$$\beta_1+\beta_2<1,$$

where

$$\beta_1=\max\left\{\alpha c_*^{q_1}\left(\frac{2q_1}{q_1-2}E(0)\right)^{(q_1-2)/2},\alpha c_*^{q_2}\left(\frac{mq_1}{q_1-m}E(0)\right)^{(q_2-2)/2}\right\},$$

$$\beta_2=\max\left\{\begin{array}{l}(1-\alpha)\,c_*^{q_1}\left(\frac{2(\gamma+1)q_1}{q_1-2(\gamma+1)}E(0)\right)^{(q_1-2(\gamma+1))/(2(\gamma+1))}\\(1-\alpha)\,c_*^{q_2}\left(\frac{2(\gamma+1)q_1}{q_1-2(\gamma+1)}E(0)\right)^{(q_2-2(\gamma+1))/(2(\gamma+1))}\end{array}\right\},$$

with $0<\alpha<1$ and c_* is the best embedding constant of $H_0^2(\Omega)\hookrightarrow L^{q(.)}(\Omega)$. Then $I(t)>0$ for all $t\in[0,T]$.

Proof. Since $I(0)>0$, then by continuity there exists T_* such that

$$I(t)\geq 0,\ \forall t\in[0,T_*]. \tag{3.2}$$

Now, we have for all $t\in[0,T]$ that

$$\begin{aligned}
E(t) &= \frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2 \\
&\quad +\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}-\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\
&\geq \frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} \\
&\quad -\frac{1}{q_1}\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}-I(t)\right) \\
&\geq \frac{q_1-2}{2q_1}\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2\right) \\
&\quad +\frac{q_1-2(\gamma+1)}{2(\gamma+1)q_1}\|\nabla u\|_2^{2(\gamma+1)}+\frac{1}{q_1}I(t).
\end{aligned}$$

Using (3.2), we have

$$\frac{q_1-2}{2q_1}\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2\right)+\frac{q_1-2(\gamma+1)}{2(\gamma+1)q_1}\|\nabla u\|_2^{2(\gamma+1)}\leq E(t).$$

By the definition of E, we obtain

$$\begin{aligned}
\|\Delta u\|_2^2+\|\nabla u\|_2^2 &\leq \frac{2q_1}{q_1-2}E(t) \\
&\leq \frac{2q_1}{q_1-2}E(0),
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\|\nabla u\|_2^{2(\gamma+1)} &\leq \frac{2(\gamma+1)q_1}{q_1-2(\gamma+1)}E(t) \\
&\leq \frac{2(\gamma+1)q_1}{q_1-2(\gamma+1)}E(0).
\end{aligned} \tag{3.4}$$

On the other hand, by Lemma 2.1, we get

$$\begin{aligned}
\int_\Omega |u|^{q(x)}dx &\leq \max\left\{\|u\|_{q(.)}^{q_1},\|u\|_{q(.)}^{q_2}\right\} \\
&= \alpha\max\left\{\|u\|_{q(.)}^{q_1},\|u\|_{q(.)}^{q_2}\right\}+(1-\alpha)\max\left\{\|u\|_{q(.)}^{q_1},\|u\|_{q(.)}^{q_2}\right\}.
\end{aligned}$$

By the embedding of $H_0^2(\Omega)\hookrightarrow H_0^1(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega |u|^{q(x)}dx &\leq \alpha\max\left\{c_*^{q_1}\|\Delta u\|_2^{q_1},c_*^{q_2}\|\Delta u\|_2^{q_2}\right\} \\
&\quad +(1-\alpha)\max\left\{c_*^{q_1}\|\nabla u\|_2^{q_1},c_*^{q_2}\|\nabla u\|_2^{q_2}\right\} \\
&\leq \alpha\max\left\{c_*^{q_1}\|\Delta u\|_2^{q_1-2},c_*^{q_2}\|\Delta u\|_2^{q_2-2}\right\}\|\Delta u\|_2^2 \\
&\quad +(1-\alpha)\max\left\{c_*^{q_1}\|\nabla u\|_2^{q_1-2(\gamma+1)},c_*^{q_2}\|\nabla u\|_2^{q_2-2(\gamma+1)}\right\}\|\nabla u\|_2^{2(\gamma+1)} \\
&\leq \alpha\max\left\{c_*^{q_1}\|\Delta u\|_2^{q_1-2},c_*^{q_2}\|\Delta u\|_2^{q_2-2}\right\}\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2\right) \\
&\quad +(1-\alpha)\max\left\{c_*^{q_1}\|\nabla u\|_2^{q_1-2(\gamma+1)},c_*^{q_2}\|\nabla u\|_2^{q_2-2(\gamma+1)}\right\}\|\nabla u\|_2^{2(\gamma+1)}.
\end{aligned}$$

By (3.3) and (3.4), we obtain

$$\int_\Omega |u|^{q(x)}dx\leq \beta_1\left(\|\Delta u\|_2^2+\|\nabla u\|_2^2\right)+\beta_2\|\nabla u\|_2^{2(\gamma+1)}. \tag{3.5}$$

Since $\beta_1+\beta_2<1$, then

$$\int_\Omega |u|^{q(x)}dx<\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}. \tag{3.6}$$

This implies that

$$I(t)>0,\ \ \forall t\in[0,T_*].$$

Repeating the above procedure, we can extend T_* to T. □

Theorem 3.3. *(Existence of weak solution). Suppose that (1.2) holds. Let $u_0 \in L^2(\Omega)$ be given. Then the problem (1.1) admits a weak local solution*

$$u \in L^\infty\left((0,T), H_0^2(\Omega)\right), \quad u_t \in L^2\left((0,T), L^2(\Omega)\right).$$

Proof. We shall use the Faedo-Galerkin method of approximation. Let $\{v_l\}_{l=1}^{\infty}$ be a basis of $H_0^2(\Omega)$ which forms a complete orthonormal system in $L^2(\Omega)$. Denote by

$$V_k = span\{v_1, v_2, ..., v_k\},$$

the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^{\infty}$. After normalization, we get $\|v_l\| = 1$ and for any given integer k, we consider the approximate solution

$$u_k(t) = \sum_{l=1}^{k} u_{lk}(t) v_l,$$

where u_k are the solutions to the problem

$$\begin{aligned} & \left(u_k'(t), v_l\right) + \left(\Delta^2 u_k(t), v_l\right) \\ & - \left(M\left(\int_\Omega |\nabla u_k(t)|^2 dx\right) \Delta u_k(t), v_l\right) + \left(|u_k(t)|^{p(x)-2} u_k'(t), v_l\right) \\ = & \left(|u_k(t)|^{q(x)-2} u_k(t), v_l\right), \quad l = 1, 2, ..., k, \end{aligned} \tag{3.7}$$

$$u_k(0) = u_{0k} = \sum_{l=1}^{k} (u_k(0), v_l) v_l \to u_0 \text{ in } L^2(\Omega). \tag{3.8}$$

Note that we can solve the system (3.7) and (3.8) by Picard's iterative method for ordinary differential equations. Therefore, there exists a solution in $[0, T_*)$ for some $T_* > 0$ and we can extend this solution to the whole interval $[0, T]$ for any given $T > 0$ by making use of the a priori estimates below. We multiply the equation (3.7) by $u_{lk}'(t)$ and summing over l from 1 to k, we have

$$\begin{aligned} & \frac{d}{dt}\left(\begin{array}{c} \frac{1}{2}\|\Delta u_k(t)\|_2^2 + \frac{1}{2}\|\nabla u_k(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u_k(t)\|^{2(\gamma+1)} \\ -\int_\Omega \frac{1}{q(x)}|u_k(t)|^{q(x)} dx \end{array}\right) \\ = & -\left\|u_{t,k}(t)\right\|_2^2 - \int_\Omega |u_k(t)|^{p(x)-2}\left|u_{t,k}(t)\right|^2 dx. \end{aligned} \tag{3.9}$$

Then

$$E'(u_k(t)) = -\left\|u_{t,k}(t)\right\|_2^2 - \int_\Omega |u_k(t)|^{p(x)-2}\left|u_{t,k}(t)\right|^2 dx \leq 0.$$

Integrating (3.9) over $(0,T)$, we get

$$\begin{aligned} & \frac{1}{2}\|\Delta u_k(t)\|_2^2 + \frac{1}{2}\|\nabla u_k(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u_k(t)\|^{2(\gamma+1)} - \int_\Omega \frac{1}{q(x)}|u_k(t)|^{q(x)} dx \\ & + \int_0^t \left\|u_{t,k}(s)\right\|_2^2 ds + \int_0^t \int_\Omega |u_k(s)|^{p(x)-2}\left|u_{t,k}(s)\right|^2 dxds \\ \leq & E(0). \end{aligned} \tag{3.10}$$

Then, from (3.6), the inequality (3.10) becomes

$$\begin{aligned} & \frac{q_1-2}{2q_1}\sup_{t\in(0,T)}\|\Delta u_k(t)\|_2^2 + \frac{q_1-2}{2q_1}\sup_{t\in(0,T)}\|\nabla u_k(t)\|_2^2 \\ & + \frac{q_1-2(\gamma+1)}{2(\gamma+1)q_1}\sup_{t\in(0,T)}\|\nabla u_k(t)\|_2^{2(\gamma+1)} + \int_0^t \left\|u_{t,k}(s)\right\|_2^2 ds \\ & + \int_0^t \int_\Omega |u_k(s)|^{p(x)-2}\left|u_{t,k}(s)\right|^2 dxds \\ \leq & E(0). \end{aligned} \tag{3.11}$$

From (3.11), we conclude that

$$\begin{cases} \{u_k\} \text{ is uniformly bounded in } L^\infty\left([0,T], H_0^2(\Omega)\right), \\ \left\{u_k'\right\} \text{ is uniformly bounded in } L^2\left([0,T], L^2(\Omega)\right). \end{cases} \tag{3.12}$$

Furthermore, we have from Lemma 2.3 and (3.12) that

$$\begin{cases} \left\{|u_k|^{q(x)-2}u_k\right\} \text{ is uniformly bounded in } L^\infty\left([0,T],L^2(\Omega)\right), \\ \left\{|u_k|^{p(x)-2}u_k'\right\} \text{ is uniformly bounded in } L^\infty\left([0,T],L^2(\Omega)\right). \end{cases} \tag{3.13}$$

By (3.12) and (3.13) we infer that there exist a subsequence of u_k and a function u such that

$$\begin{cases} u_k \rightharpoonup u \text{ weakly star in } L^\infty\left([0,T],H_0^2(\Omega)\right), \\ u_k' \rightharpoonup u' \text{ weakly star in } L^2\left([0,T],L^2(\Omega)\right), \\ |u_k|^{q(x)-2}u_k \rightharpoonup |u|^{q(x)-2}u \text{ weakly star in } L^\infty\left([0,T],L^2(\Omega)\right), \\ |u_k|^{p(x)-2}u_k' \rightharpoonup |u|^{p(x)-2}u' \text{ weakly star in } L^\infty\left([0,T],L^2(\Omega)\right). \end{cases} \tag{3.14}$$

By the Aubin-Lions compactness lemma (see [21]), we conclude from (3.14) that

$$u_k \rightharpoonup u \text{ strongly in } C\left([0,T],H_0^2(\Omega)\right),$$

yields

$$u_k \rightharpoonup u \text{ everywhere in } \Omega\times[0,T]. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\begin{cases} |u_k|^{q(x)-2}u_k \rightharpoonup |u|^{q(x)-2}u \text{ weakly in } L^\infty\left([0,T],L^2(\Omega)\right), \\ |u_k|^{p(x)-2}u_k' \rightharpoonup |u|^{p(x)-2}u' \text{ weakly in } L^\infty\left([0,T],L^2(\Omega)\right). \end{cases}$$

Letting $k\to\infty$ and passing to the limit in (3.7) we have

$$\begin{aligned} &\left(u'(t),v_l\right)+\left(\Delta^2u(t),v_l\right)-\left(M\left(\int_\Omega|\nabla u(t)|^2dx\right)\Delta u(t),v_l\right) \\ &+\left(|u(t)|^{p(x)-2}u_k'(t),v_l\right), \\ =\ &\left(|u(t)|^{q(x)-2}u(t),v_l\right),\ \ l=1,2,...,k. \end{aligned}$$

Since $\{v_l\}_{l=1}^{\infty}$ is a basis of $H_0^2(\Omega)$, we deduce that u satisfies equation (1.1). From (3.14) and Lemma 3.1.7 of [22] with $B=L^2(\Omega)$ we infer that

$$u_k(0)\rightharpoonup u(0) \text{ weakly in } L^2(\Omega). \tag{3.16}$$

We get from (3.8) and (3.16) that $u(0)=u_0$. The proof of the Theorem is now finished. □

Theorem 3.4. *Let the assumptions of Lemma 3.2 hold. Then the local solution of (1.1) is global.*

Proof. We have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}-\int_\Omega\frac{1}{q(x)}|u|^{q(x)}dx, \\ &\geq \frac{q_1-2}{2q_1}\|\Delta u\|_2^2+\frac{q_1-2}{2q_1}\|\nabla u\|_2^2+\frac{q_1-2(\gamma+1)}{2(\gamma+1)q_1}\|\nabla u\|_2^{2(\gamma+1)}, \end{aligned}$$

which implies that

$$\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}\leq CE(t). \tag{3.17}$$

By Lemma 3.1, we get

$$\|\Delta u\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}\leq CE(0).$$

□

4. Decay

In this part, we state and prove the decay of solutions. Firstly, we give the following lemma.

Lemma 4.1. *Let the assumptions of Lemma 3.2 hold. Then*

$$\int_{\Omega} |u|^{p(x)}\, dx \leq cE(t),$$

where $c > 0$.

Proof.

$$\begin{aligned}\int_{\Omega} |u|^{p(x)}\, dx &= \max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\},\\ &\leq \max\left\{c_*^{p_1}\|\Delta u\|_2^{p_1}, c_*^{p_2}\|\Delta u\|_2^{p_2}\right\},\\ &\leq \max\left\{c_*^{p_1}\|\Delta u\|_2^{p_1-2}, c_*^{p_2}\|\Delta u\|_2^{p_2-2}\right\}\|\Delta u\|_2^2.\end{aligned}$$

Using (3.3), we have

$$\int_{\Omega} |u|^{p(x)}\, dx \leq cE(t).$$

□

Theorem 4.2. *Let the assumptions of Lemma 3.2 hold. Then*

$$E(t) \leq E(0)\left(\frac{c+rt}{c+rc}\right)^{-1/r}, \quad \forall t \geq c,$$

where $c > 0$.

Proof. Multiplying the equation (1.1) by $u(t)E^q(t)$ $(q>0)$ and then integrating over $\Omega \times (S,T)$, we get

$$\begin{aligned}&\int_S^T\int_{\Omega} E^q(t)\left[u\Delta^2 u + uu_t - u\left(M\left(\int_{\Omega}|\nabla u|^2 dx\right)\Delta u + uu_t|u|^{p(x)-2}\right)\right] dxdt\\ = &\int_S^T E^q(t)\int_{\Omega}|u|^{q(x)}\, dxdt.\end{aligned}$$

Then

$$\begin{aligned}&\int_S^T\int_{\Omega} E^q(t)\left(|\Delta u|^2 + uu_t + |\nabla u|^2 + \|\nabla u\|_2^{2\gamma}|\nabla u|^2 + uu_t|u|^{p(x)-2}\right) dxdt\\ = &\int_S^T E^q(t)\int_{\Omega}|u|^{q(x)}\, dxdt.\end{aligned}$$

We adding and substracting the term

$$\int_S^T E^q(t)\int_{\Omega}\left(\beta_1\left(|\Delta u|^2 + |\nabla u|^2\right) + \beta_2\|\nabla u\|_2^{2\gamma}|\nabla u|^2\right) dxdt,$$

and use (3.5), we obtain

$$\begin{aligned}
&(1-\beta_1)\int_S^T E^q(t)\int_\Omega \left(|\Delta u|^2\right)dxdt \\
&+(1-\beta_1)\int_S^T E^q(t)\int_\Omega |\nabla u|^2\,dxdt \\
&+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \left(\|\nabla u\|_2^{2\gamma}|\nabla u|^2\right)dxdt \\
&+\int_S^T E^q(t)\int_\Omega (uu_t)\,dxdt \\
&+\int_S^T E^q(t)\int_\Omega \left(uu_t|u|^{p(x)-2}\right)dxdt \\
=\ & -\int_S^T E^q(t)\int_\Omega \left(\beta_1|\Delta u|^2+\beta_1|\nabla u|^2+\beta_2\|\nabla u\|_2^{2\gamma}|\nabla u|^2-|u|^{q(x)}\right)dxdt \\
\le\ & 0. \qquad (4.1)
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\xi\int_S^T E^q(t)\int_\Omega \begin{pmatrix} \frac{1}{2}|\Delta u|^2+\frac{1}{2}|\nabla u|^2 \\ +\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2\gamma}|\nabla u|^2-\frac{|u(t)|^{q(x)}}{q(x)} \end{pmatrix} dxdt \\
\le\ & (1-\beta_1)\int_S^T E^q(t)\int_\Omega |\Delta u|^2\,dxdt \\
&+(1-\beta_1)\int_S^T E^q(t)\int_\Omega |\nabla u|^2\,dxdt \\
&+(1-\beta_2)\int_S^T E^q(t)\int_\Omega \|\nabla u\|_2^{2\gamma}|\nabla u|^2\,dxdt, \qquad (4.2)
\end{aligned}$$

where

$$\xi=\min\{(1-\beta_1),(1-\beta_2)\}.$$

By (4.1), (4.2) and the definition of $E(t)$, we otain

$$\begin{aligned}
\xi\int_S^T E^{q+1}(t)\,dt \quad \le\ & -\int_S^T E^q(t)\int_\Omega uu_t dxdt \qquad (4.3)\\
& -\int_S^T E^q(t)\int_\Omega uu_t|u|^{p(x)-2}\,dxdt.
\end{aligned}$$

We estimate the terms on the right-hand side of (4.3). For the first term, we use the Young's inequality

$$AB\le\frac{\varepsilon}{\eta_1}A^{\eta_1}+\frac{1}{\eta_2\varepsilon^{\eta_2/\eta_1}}B^{\eta_2},\ \ A,B\ge 0,\ \varepsilon>0 \text{ and } \frac{1}{\eta_1}+\frac{1}{\eta_2}=1,$$

and get

$$-\int_S^T E^q(t)\int_\Omega uu_t dxdt\le\int_S^T E^q(t)\int_\Omega\left(\varepsilon c|u|^2+c_\varepsilon|u_t|^2\right)dxdt. \qquad (4.4)$$

We use again the Young's inequality to get

$$\begin{aligned}
&-\int_S^T E^q(t)\int_\Omega uu_t|u|^{p(x)-2}\,dxdt \\
=\ & -\int_S^T E^q(t)\int_\Omega |u|^{(p(x)-2)/2}u_t|u|^{(p(x)-2)/2}u\,dxdt \\
\le\ & \int_S^T E^q(t)\int_\Omega\left(\varepsilon c|u|^{p(x)}+c_\varepsilon|u_t|^{p(x)-2}u_t^2\right)dxdt. \qquad (4.5)
\end{aligned}$$

By (4.4) and (4.5) the inequality (4.3) becomes

$$\begin{aligned}\xi\int_S^T E^{q+1}(t)\,dt &\leq \int_S^T E^q(t)\int_\Omega\left(\varepsilon c|u|^2+c_\varepsilon|u_t|^2\right)dxdt\\&+\int_S^T E^q(t)\int_\Omega\left(\varepsilon c|u|^{p(x)}+c_\varepsilon|u_t|^{p(x)-2}u_t^2\right)dxdt\\&\leq \varepsilon c\int_S^T E^q(t)\int_\Omega\left(|u|^2+|u|^{p(x)}\right)dxdt\\&+c_\varepsilon\int_S^T E^q(t)\int_\Omega\left(|u_t|^2+|u|^{p(x)-2}u_t^2\right)dxdt.\end{aligned}$$

We use (3.17) , Lemma 4.1 and definition of $E^{'}(t)$ to obtain

$$\xi\int_S^T E^{q+1}(t)\,dt\leq\varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon\int_S^T E^q(t)\left(-E^{'}(t)\right)dt.$$

This implies

$$\begin{aligned}\xi\int_S^T E^{q+1}(t)\,dt &\leq \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon\left(E^{q+1}(s)-E^{q+1}(T)\right)\\&\leq \varepsilon c\int_S^T E^{q+1}(t)\,dt+c_\varepsilon E^q(0)E(s).\end{aligned}$$

Choosing ε so small such that $\xi>\varepsilon c$, we arrive at

$$\int_S^T E^{q+1}(t)\,dt\leq cE^q(0)E(s).$$

By taking $T\to\infty$, we obtain

$$\int_S^\infty E^{q+1}(t)\,dt\leq cE^q(0)E(s).$$

Thus, Komornik's Lemma implies the desired result. □

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