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Ideal Limit Superior-Inferior

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Abstract

In this paper the notation of ideal supremum and ideal infimum of real valued sequences is defined. Besides the main properties, it is shown that equality of ideal sup and ideal inf of the sequence is necessary but not sufficient for to existence of usual limit of it. On the other hand, the equality of them is necessary and sufficient for to existence of ideal limit.

1. INTRODUCTION

In 1951, statistical convergence of real valued sequences was introduced by Fast and Steinhaus [8, 19]. The idea of statistical convergence is based on asymptotic density of the subset of natural numbers (see [3]). Let $K \subseteq \mathbb{N}$ and $K(n) = \{k \leq n : k \in K\}$. Asymptotic density of the subset K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}|K(n)|,$$

if the limit exists. The symbol $|K(n)|$ is denote the cardinality of the set $K(n)$. For many years, by using asymptotic density some concepts in Mathematical analysis are generalized [7], [11], [12], etc.

A real valued sequence $x = (x_k)$ is statistical convergent to the element L, if for every $\varepsilon > 0$, the set

$$K(n, \varepsilon) \coloneqq \{k \leq n : |x_k - L| \geq \varepsilon\}$$

has zero asymptotic density, in this case we write

$$st - \lim_{n\to\infty} x_k = L.$$

The concept of statistical convergence has been studied by many authors such as [3], [4], [5], [7], [9], [10], [11], [12], [18], etc.

Let X be a non-empty set and $\mathcal{I}$ be a family of subsets of X. The family $\mathcal{I}$ is called an ideal if it has the properties

(i) $A \cup B \in \mathcal{I}$ for all $A, B \in \mathcal{I}$,

(ii) $A \in \mathcal{I}$ and each $B \subset A$ imply $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset 2^X$ is called admissible if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

Let $\mathcal{I}_\delta$ be the class of all $A \subset \mathbb{N}$ with $\delta(A) = 0$. Then, $\mathcal{I}_\delta$ is a non-trivial admissible ideal.

A real valued sequence $x = (x_k)$ is said to be ideal convergent to L, if for every $\varepsilon > 0$, the set

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$$K(\varepsilon) := \{k \leq n : |x_k - L| \geq \varepsilon\}$$

belongs to $\mathcal{I}$ (see [13], [14]). It is denoted by $\mathcal{I} - \lim_{n\to\infty} x_k = L$.

A non-empty family of sets $\mathcal{F} \subset 2^X$ is a filter on X if $\mathcal{F}$ has the properties

(i) $\emptyset \notin \mathcal{F}$,

(ii) $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$,

(iii) For each $A \in \mathcal{F}$ and each $B \supset A$ imply $B \in \mathcal{F}$,

(see [15] and [17]).

If $\mathcal{I} \subset 2^X$ is a non-trivial ideal then, $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$ is a filter on X.

Ideal limit superior and inferior of a sequence $x = (x_k)$ was given in [6] by using any admissible ideal in the definition of statistical limit superior and inferior which was defined [11]. Some further results about ideal limit-superior and inferior was given by Lahiri and Das in [16].

The idea in the paper [6] and [11] based on to find biggest and smallest statistical limit points of given sequence for calculating st-limit supremum st-limit infimum, respectively. Fridy and Orhan showed that there exists a real valued sequence such that its st-limit superior is not the biggest statistical limit point and st-limit inferior is not the smallest statistical limit point of the sequence ([4], [11]). This phenomena is also true for ideal limit superior and ideal limit inferior [6].

One of the aim of this paper is to give an alternative definition such that the claim of Fridy and Orhan satisfied for any real valued sequences. For this purpose, at first ideal lower and upper bound will be defined and their some basic properties will be investigated. By using this definition of ideal limit supremum and ideal limit infimum will be defined.

2. IDEAL UPPER AND IDEAL LOWER BOUND

In this section $\mathcal{I}$-analogue of statistical upper and statistical lower bound, introduced and studied in [1], [2], will be given.

Definition 2.1. (Ideal Lower Bound) Let $x = (x_k)$ be a real valued sequence. A point $l \in \mathbb{R}$ is an ideal lower bound of the sequence $x = (x_k)$, if the following

(2.1) $$\{k : x_k < l\} \in \mathcal{I} \text{ (or } \{k : x_k \geq l\} \in \mathcal{F}(\mathcal{I}))$$

holds.

The set of all ideal lower bounds of the sequence $x = (x_k)$ is denoted by $L_{\mathcal{I}}(x)$:

$$L_{\mathcal{I}}(x) := \left\{l \in \mathbb{R} : \{k : x_k < l\} \in \mathcal{I} \left(\text{or } \{k : x_k \geq l\} \in \mathcal{F}(\mathcal{I})\right)\right\}.$$

Let us also denote the set of all usual lower bounds of the sequence $x = (x_k)$ by $L(x)$:

$$L(x) := \{l \in \mathbb{R} : l \leq x_k \text{ for all } k \in \mathbb{N}\}.$$

Theorem 2.1. If $l \in \mathbb{R}$ is a lower bound of the sequence $x = (x_k)$, then l is an ideal lower bound of the sequence (x_k).

Proof From the definition of usual lower bound we have $l \leq x_k$ for all $k \in \mathbb{N}$. So,

$$\{k : x_k < l\} = \emptyset.$$

Therefore,

$$\{k : x_k < l\} \in \mathcal{I}$$

holds. That is, $L(x) \subset L_{\mathcal{I}}(x)$.

Remark 2.1. The converse of Theorem 2.1 is not true, in general.

Let us consider the sequence $(x_k) = \left(-\frac{1}{k}\right)$ and take $l = -\frac{1}{2} \in \mathbb{R}$. It is clear that $l = -\frac{1}{2}$ is an ideal lower bound of the sequence $x = (x_k)$ because $\left\{k : x_k < -\frac{1}{2}\right\} = \{1\} \in \mathcal{I}$ but it is not usual lower bound.

Definition 2.2. (Ideal Upper Bound) Let $x = (x_k)$ be a real valued sequence. A point $m \in \mathbb{R}$ is an ideal upper bound of the sequence $x = (x_k)$, if the following

(2.2) $$\{k : x_k > m\} \in \mathcal{I} \text{ (or } \{k : x_k \leq m\} \in \mathcal{F}(\mathcal{I}))$$

holds.

The set of all ideal upper bounds of the sequence $x = (x_k)$ is denoted by $U_{\mathcal{I}}(x)$:

$$U_{\mathcal{I}}(x) := \left\{m \in \mathbb{R} : \{k : x_k > m\} \in \mathcal{I} \left(\text{or } \{k : x_k \leq m\} \in \mathcal{F}(\mathcal{I})\right)\right\}.$$

Let us denote the set of usual upper bound of the sequence $x = (x_k)$ by $U(x)$:

$$U(x) := \{m \in \mathbb{R} : x_k \leq m \text{ for all } k \in \mathbb{N}\}.$$

Theorem 2.2. If $m \in \mathbb{R}$ is an usual upper bound of the sequence $x = (x_k)$, then $m \in \mathbb{R}$ is an ideal upper bound.

Proof Since $m \in \mathbb{R}$ is an usual upper bound of the sequence $x = (x_k)$, then we have $x_k \leq m$ for all $k \in \mathbb{N}$. So,

$$\{k : x_k > m\} = \emptyset.$$

Therefore,

$$\{k : x_k > m\} \in \mathcal{I}$$

holds. That is, $U(x) \subset U_{\mathcal{I}}(x)$.

Remark 2.2. The converse of the Theorem 2.2 is not true, in general.

Let us consider the sequence $(x_k) = \left(\frac{1}{k}\right)$ and $m = \frac{1}{2} \in \mathbb{R}$. It is clear that $m = \frac{1}{2}$ is an ideal upper bound of (x_k) because $\left\{k : x_k > \frac{1}{2}\right\} = \{1\} \in \mathcal{I}$, but it is not usual upper bound for the sequence.

Theorem 2.3. a) If $l \in \mathbb{R}$ is an ideal lower bound and $l' < l$, then $l' \in \mathbb{R}$ is also an ideal lower bound of $x = (x_k)$.

b) If $m \in \mathbb{R}$ is an ideal upper bound and $m < m'$, then $m' \in \mathbb{R}$ is also ideal upper bound of the sequence $x = (x_k)$.

Proof a) Assume that $l \in \mathbb{R}$ is an ideal lower bound of the sequence $x = (x_k)$ such that $\{k : x_k < l\} \in \mathcal{I}$. Since $l' < l$, then the following inclusion

$$\{k : x_k < l\} \supset \{k : x_k < l'\}$$

holds. From the hereditary properties of ideal we have $\{k : x_k < l'\} \in \mathcal{I}$. This gives the desired result.

b) Since $m \in \mathbb{R}$ is an ideal upper bound of the sequence $x = (x_k)$, then the set $\{k : x_k > m\} \in \mathcal{I}$. Since $m < m'$, then the inclusion

$$\{k : x_k > m\} \supset \{k : x_k > m'\}$$

holds. From the definition of ideal we have $\{k : x_k > m'\} \in \mathcal{I}$. This gives the desired result.

Corollary 2.1. Let $x = (x_k)$ be a real valued sequence. Then, $L_{\mathcal{I}}(x) \cap U_{\mathcal{I}}(x) = \emptyset$.

3. IDEAL SUPREMUM AND IDEAL INFIMUM

In this section, we are going to define ideal supremum and ideal infimum by using ideal upper and ideal lower bound of given sequence.

Definition 3.1. (Ideal Infimum ($\mathcal{I}-\inf$)) A number $s \in \mathbb{R}$ is an ideal infimum of a sequence $x = (x_k)$ if s is supremum of $L_{\mathcal{I}}(x)$. That is, $\mathcal{I}-\inf x_k \coloneqq \sup L_{\mathcal{I}}(x)$.

Definition 3.2. (Ideal Supremum ($\mathcal{I}-\sup$)) A number $S \in \mathbb{R}$ is an ideal supremum of a sequence $x = (x_k)$ if S is infimum of $U_{\mathcal{I}}(x)$. That is, $\mathcal{I}-\sup x_k \coloneqq \inf U_{\mathcal{I}}(x)$.

Let us consider following sequence

$$x_k \coloneqq \begin{cases} k, & \text{if } k \text{ is an odd square,} \\ 2, & \text{if } k \text{ is an even square,} \\ 1, & \text{if } k \text{ is an odd nonsquare,} \\ 0, & \text{if } k \text{ is an odd nonsquare.} \end{cases}$$

and ideal $\mathcal{I}_\delta$. This sequence will be help us to illustrate the concept just defined. Thus, $U_{\mathcal{I}_\delta}(x) = (1, \infty)$ and $L_{\mathcal{I}_\delta}(x) = (-\infty, 0)$. So, $\mathcal{I}_\delta - \sup(x) = 1$ and $\mathcal{I}_\delta - \inf(x) = 0$.

Also, it is known that the set of all $\mathcal{I}_\delta$-limit points is $\{0,1\}$ (see in [11]). This example shows that $\mathcal{I}_\delta - \sup(x)$ equals the biggest $\mathcal{I}_\delta$ limit point and $\mathcal{I}_\delta - \inf(x)$ equals the smallest $\mathcal{I}_\delta$ limit points.

Theorem 3.1. Let $x = (x_k)$ be a real valued sequence. Then,

$$\inf x_k \leq \mathcal{I} - \inf x_k \leq \mathcal{I} - \sup x_k \leq \sup x_k \tag{3.1}$$

holds.

Proof From the definition of usual infimum we have $\{k : \inf x_k > x_k\} = \emptyset \in \mathcal{I}$. So, $\inf x_k \in L_{\mathcal{I}}(x)$. Since $\mathcal{I} - \inf x_k = \sup L_{\mathcal{I}}(x)$, then we have $\mathcal{I} - \inf x_k \geq \inf x_k$.

Analoguously,

$$\mathcal{I} - \sup x_k \leq \sup x_k$$

hold.

To complete the proof it is enough to show that the inequality

$$l \leq m \tag{3.2}$$

holds for an arbitrary $l \in L_{\mathcal{I}}(x)$ and $m \in U_{\mathcal{I}}(x)$.

Let us assume that (3.2) is not true. So, there exist $l' \in L_{\mathcal{I}}(x)$ and $m' \in U_{\mathcal{I}}(x)$ such that $m' < l'$ holds. Since m' is an ideal upper bound, then from Theorem 2.3 (b), l' is also ideal upper bound of the sequence. This is a contradiction.

In the following we give some examples such that the inequality (3.1) is hold.

Example 3.1. i) If $x = (x_k)$ is a constant sequence, then

$$\inf x_k = \mathcal{I} - \inf x_k = \mathcal{I} - \sup x_k = \sup x_k.$$

ii) If we consider the sequence $x = (x_k)$ as

$$x_k = \begin{cases} x_k, & k \leq k_0, k_0 \in \mathbb{N} \\ a, & k > k_0, \end{cases}$$

such that $x_k \leq a$ for all $k \in \{1,2,3, \dots, k_0\}$. Then,

$$\inf x_k \leq \mathcal{I} - \inf x_k \leq \mathcal{I} - \sup x_k = \sup x_k.$$

iii) If we consider the sequence $x = (x_k)$ as

$$x_k = \begin{cases} x_k, & k \leq k_0, k_0 \in \mathbb{N} \\ a, & k > k_0, \end{cases}$$

such that $x_k \geq a$ for all $k \in \{1,2,3,\dots,k_0\}$. Then,

$$\inf x_k = \mathcal{J} - \inf x_k \leq \mathcal{J} - \sup x_k \leq \sup x_k.$$

Theorem 3.2. Let $x = (x_k)$ be a real valued sequence. The following statements are true:

(i) If $x = (x_k)$ is a monotone increasing sequence, then $\mathcal{J} - \inf x_k = \sup x_k$.

(ii) If $x = (x_k)$ is a monotone decreasing sequence, then $\mathcal{J} - \sup x_k = \inf x_k$.

Proof We shall give only the proof of (i). Other case can be proved by follows (i). Assume that $x = (x_k)$ is a monotone increasing sequence and $\sup x_k < \infty$ holds. From the definition of supremum, the inequality

$$x_k \leq \sup x_k \tag{3.3}$$

holds for all $k \in \mathbb{N}$ and also for every$\varepsilon > 0$there exists a $k_0 \in \mathbb{N}$ such that

$$\sup x_k - \varepsilon < x_{k_0} \tag{3.4}$$

holds. From (3.3) and (3.4) we have, $\sup x_k \notin L_{\mathcal{J}}(x)$ and the inclusion

$$\{k : x_k < \sup x_k - \varepsilon\} \subset \{1,2,3,\dots,k_0\}$$

holds for any fixed $k_0 \in \mathbb{N}$. Since $\{1,2,3,\dots,k_0\} \in \mathcal{J}$, then $\sup x_k - \varepsilon \in L_{\mathcal{J}}(x)$.

Therefore, Theorem 2.3 gives that

$$L_{\mathcal{J}}(x) = (-\infty, \sup x_k - \varepsilon]$$

for all $\varepsilon > 0$. So,

$$\mathcal{J} - \inf x_k = \sup L_{\mathcal{J}}(x) = \sup x_k.$$

Now, assume that $\sup x_k = \infty$.

It means that, for all $l \in \mathbb{R}$ there exists a number $k_0 = k_0(x) \in \mathbb{N}$ such that $l \leq x_{k_0}$ and from the monotonicity of (x_k) the inequality $x_{k_0} \leq x_k$ holds for all $k \geq k_0$. So, we have

$$\{k : x_k < l\} \subseteq \{1,2,3,\dots,k_0\}.$$

Since, $\{1,2,3,\dots,k_0\} \in \mathcal{J}$, then $l \in L_{\mathcal{J}}(x)$ for an arbitrary point l. Therefore,

$$L_{\mathcal{J}}(x) = (-\infty, \infty) \text{ and } \sup L_{\mathcal{J}}(x) = \infty.$$

This gives the proof.

Corollary 3.1. Let $x = (x_k)$ be a real valued bounded sequence. If $x = (x_k)$ is a monotone decreasing (or increasing) sequence then

$$\lim_{k\to\infty} x_k = \mathcal{J} - \sup x_k \left(\text{or } \lim_{k\to\infty} x_k = \mathcal{J} - \inf x_k\right).$$

Theorem 3.3. Let $x = (x_k)$ be a real valued sequence and $l \in \mathbb{R}$. Then, $\mathcal{J} - \sup x_k = l$ if and only if for every $\varepsilon > 0$,

$$(i)\ \{k : x_k > l + \varepsilon\} \in \mathcal{J}$$

and

$$(ii)\ \{k : x_k \leq l - \varepsilon\} \notin \mathcal{F}(\mathcal{J})$$

hold.

Proof "⇒" Since $\mathcal{J} - \sup x_k = l$, then $l = \inf U_{\mathcal{J}}(x)$. Therefore, we have

$$(a)\ l \leq s,\ \ \forall s \in U_{\mathcal{J}}(x)$$

and

$$(b)\ \forall \varepsilon > 0,\ \ \exists s' \in U_{\mathcal{J}}(x) \text{ such that } s' < l + \varepsilon$$

holds. Hence, from (b) in Theorem 2.3 we have $l+\varepsilon$ is an ideal upper bound. So, (i) holds. Now assume that (ii) is not true. That is, there exists an $\varepsilon_0>0$ such that $\{k : l-\varepsilon_0 > x_k\} \in \mathcal{F}(\mathcal{I})$. It means that $l-\varepsilon_0 \in U_{\mathcal{I}}(x)$. But this is a contradiction with $l=\inf U_{\mathcal{I}}(x)$.

"⇐" Now assume that, (i) and (ii) hold for every $\varepsilon>0$. Then we have $l+\varepsilon \in U_{\mathcal{I}}(x)$ and $l-\varepsilon \notin U_{\mathcal{I}}(x)$ hold, respectively. Its mean that $U_{\mathcal{I}}(x)=[l+\varepsilon,\infty)$ and $\inf U_{\mathcal{I}}(x)=l$.

Theorem 3.4. Let $x=(x_k)$ be a real valued sequence and $m\in\mathbb{R}$. Then, $\mathcal{I}-\inf x_k = m$ if and only if for every $\varepsilon>0$,

$$(i)\ \{k:\ x_k < m-\varepsilon\} \in \mathcal{I},$$

and

$$(ii)\ \{k:\ x_k \geq m+\varepsilon\} \notin \mathcal{F}(\mathcal{I})$$

hold.

Proof "⇒" Assume that $\mathcal{I}-\inf x_k = m$. That is, $m=\sup L_{\mathcal{I}}(x)$. So, we have

$$(a)\quad s \leq m,\ \ \forall s \in L_{\mathcal{I}}(x)$$

and

$$(b)\ \ \forall\varepsilon>0, \exists s' \in L_{\mathcal{I}}(x) \text{ such that } m-\varepsilon < s'$$

holds. Then, from (b) in Theorem 2.3 we have $m-\varepsilon$ is an ideal upper bound. So, (i) holds. Now assume that (ii) is not hold for any $\varepsilon>0$. That is, there exists an $\varepsilon_0>0$ such that $\{k : x_k \geq m+\varepsilon_0\} \in \mathcal{F}(\mathcal{I})$.

This means that $m+\varepsilon_0 \in L_{\mathcal{I}}(x)$. Since $m<m+\varepsilon_0$, this is a contradiction to assumption on m.

"⇐" Now assume that, (i) and (ii) hold for every $\varepsilon>0$. It is clear that $m-\varepsilon \in L_{\mathcal{I}}(x)$ and $m+\varepsilon \notin L_{\mathcal{I}}(x)$. Therefore, $L_{\mathcal{I}}(x)=(-\infty, m-\varepsilon]$, for all $\varepsilon>0$. So, we have $\sup L_{\mathcal{I}}(x)=m$.

Corollary 3.2. Let $x=(x_k)$ be a real valued sequence. Then,

$$\{k : x_k \notin [\mathcal{I}-\inf x_k, \mathcal{I}-\sup x_k]\} \in \mathcal{I}, \tag{3.5}$$

and

$$\{k : x_k \in [\mathcal{I}-\inf x_k, \mathcal{I}-\sup x_k]\} \in \mathcal{F}(\mathcal{I}) \tag{3.6}$$

hold.

Theorem 3.5. Let $x=(x_k)$ and $y=(y_k)$ be any real valued sequences. Then,

$$\mathcal{I}-\sup(x_k+y_k) = \mathcal{I}-\sup x_k + \mathcal{I}-\sup y_k,$$

and

$$\mathcal{I}-\inf(x_k+y_k) = \mathcal{I}-\inf x_k + \mathcal{I}-\inf y_k,$$

hold.

Proof Let $\mathcal{I}-\sup x_k = m$ and $\mathcal{I}-\sup y_k = l$. So, from Theorem 3.3 we have

$$\left\{k : x_k > l+\frac{\varepsilon}{2}\right\} \in \mathcal{I} \ \text{ and } \ \left\{k : y_k > m+\frac{\varepsilon}{2}\right\} \in \mathcal{I}$$

for any $\varepsilon>0$. Therefore,

$$\left\{k : x_k > l+\frac{\varepsilon}{2}\right\} \cup \left\{k : y_k > m+\frac{\varepsilon}{2}\right\} = \{k : x_k+y_k > l+m+\varepsilon\} \in \mathcal{I}.$$

Consequently, $\mathcal{I}-\sup(x_k+y_k)=m+l$.

The other one can be proved by using Theorem 3.4.

Theorem 3.6. Let $x=(x_k)$ be a real valued sequence. Then, the equality

$$\mathcal{I}-\inf(-x_k) = -(\mathcal{I}-\sup x_k)$$

holds.

Proof Let $\mathcal{I}-\sup x_k = l$. From Theorem 3.3 we have

$$\{k : -x_k < -l-\varepsilon\} = \{k : x_k > l+\varepsilon\} \in \mathcal{I}.$$

So, we have $\mathcal{I}-\inf(-x_k) = -l$. Therefore, $\mathcal{I}-\sup x_k = -(\mathcal{I}-\inf(-x_k))$ holds.

Definition 3.3. (Peak Point [12]) A point x_l is called upper (or lower) peak point of the sequence $x=(x_k)$ if the inequality $x_l \geq x_k$ (or $x_l \leq x_k$) holds for all $k \geq l$.

Theorem 3.7. Let $x=(x_k)$ be a real valued sequence. If x_{k_0} is an upper (or lower) peak point of (x_k), then x_{k_0} is an ideal upper (or an ideal lower) bound of the sequence.

Proof Assume that $x_{k_0}$is an upper peak point of the sequence $x=(x_k)$ such that $x_k \leq x_{k_0}$ holds for all $k \geq k_0$. So, the inclusion $\{k : x_k > x_{k_0}\} \subset \{1,2,3,\dots,k_0\}$ holds. Since $\mathcal{I}$ is an admissible ideal, then $\{k : x_k > x_{k_0}\} \in \mathcal{I}$. This gives that x_{k_0} is an ideal upper bound of $x=(x_k)$.

Theorem 3.8. If $\lim_{k\to\infty} x_k = l$, then $\mathcal{I}-\sup x_k = \mathcal{I}-\inf x_k = l$.

Proof Assume $\lim_{k\to\infty} x_k = l$. That is, for any $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$|x_k - l| < \varepsilon \tag{3.7}$$

holds for every $k \geq k_0$. So, it is clear from (3.7) that following inclusions

$$\{k : x_k < l-\varepsilon\} \subset \{1,2,3,\dots,k_0\},\ \{k : x_k > l+\varepsilon\} \subset \{1,2,3,\dots,k_0\} \tag{3.8}$$

hold. By using (3.8) we obtain $\{k : x_k < l-\varepsilon\} \in \mathcal{I}$ and $\{k : x_k > l+\varepsilon\} \in \mathcal{I}$. So, for any $\varepsilon > 0$,

$$l-\varepsilon \in L_{\mathcal{I}}(x), \qquad l+\varepsilon \in U_{\mathcal{I}}(x)$$

hold. Also, from Theorem 2.3 we have

$$L_{\mathcal{I}}(x) = (-\infty, l) \text{ and } U_{\mathcal{I}}(x) = (l,\infty).$$

Therefore,

$$\mathcal{I}-\inf x_k = \sup(-\infty,l) = l \text{ and } \mathcal{I}-\sup x_k = \inf(l,\infty) = l$$

are obtained.

Remark 3.1. The converse of Theorem 3.8 is not true, in general.

Let us consider a sequence $x=(x_k)$ as follows:

$$x_k := \begin{cases} 1, & k = n^2, n = 1,2,\dots, \\ 0, & \text{otherwise} \end{cases}$$

and the ideal $\mathcal{I}_\delta$. It is clear that $\mathcal{I}_\delta - \inf x_k = \mathcal{I}_\delta - \sup x_k = 0$, but it is not convergent to 0.

Theorem 3.9. $\mathcal{I}-\lim_{k\to\infty} x_k = l$ if and only if $\mathcal{I}-\sup x_k = \mathcal{I}-\inf x_k = l$.

Proof"⇒" Assume that $\mathcal{I}-\lim_{k\to\infty} x_k = l$. So, we have

$$\{k : |x_k - l| \geq \varepsilon\} \in \mathcal{I} \tag{3.9}$$

for any $\varepsilon > 0$. From (3.9) and

$$\{k : |x_k - l| \geq \varepsilon\} = \{k : x_k \geq l+\varepsilon\} \cup \{k : x_k \leq l-\varepsilon\}$$

then we have

$$\{k : x_k \geq l+\varepsilon\} \in \mathcal{I}, \tag{3.10}$$

and

$$\{k : x_k \leq l-\varepsilon\} \in \mathcal{I}. \tag{3.11}$$

Also, from (3.10) and (3.11) we have

$$\{k : x_k > l + \varepsilon\} \in \mathcal{I} \tag{3.12}$$

and

$$\{k : x_k < l - \varepsilon\} \in \mathcal{I}. \tag{3.13}$$

The equation (3.12) gives $l + \varepsilon$ is an ideal upper bound and (3.13) gives $l - \varepsilon$ is an ideal lower bound. Therefore,

$$L_{\mathcal{I}}(x) = (-\infty, l) \text{ and } U_{\mathcal{I}}(x) = (l, \infty)$$

for all $\varepsilon > 0$. So, we have

$$\sup L_{\mathcal{I}}(x) = l, \quad \inf U_{\mathcal{I}}(x) = l.$$

"$\Leftarrow$" Assume that

$$\mathcal{I} - \sup x_k = \mathcal{I} - \inf x_k = l.$$

That is,

$$l = \sup L_{\mathcal{I}}(x) = \inf U_{\mathcal{I}}(x).$$

From the definition of usual supremum and infimum for any $\varepsilon > 0$, there exists $l' \in L_{\mathcal{I}}(x)$ and $l'' \in U_{\mathcal{I}}(x)$ such that the inequalities

$$l - \varepsilon < l' \text{and } l'' < l + \varepsilon$$

hold.

Since l'' is an ideal upper bound, then the following inclusion

$$\{k : x_k \geq l + \varepsilon\} \subset \{k : x_k \geq l''\}$$

holds. So, we have

$$\{k : x_k \geq l + \varepsilon\} \in \mathcal{I}. \tag{3.14}$$

Since l' is an ideal lower bound, then the following inclusion

$$\{k : x_k \leq l - \varepsilon\} \subset \{k : x_k \leq l'\}$$

holds. So, we have

$$\{k : x_k \leq l - \varepsilon\} \in \mathcal{I}. \tag{3.15}$$

From the facts (3.14)-(3.15) and

$$\{k : |x_k - l| \geq \varepsilon\} = \{k : x_k \geq l + \varepsilon\} \cup \{k : x_k \leq l - \varepsilon\},$$

we have

$$\{k : |x_k - l| \geq \varepsilon\} \in \mathcal{I}.$$

Consequently, the sequence $x = (x_k)$ is ideal convergent to $l \in \mathbb{R}$.

Definition 3.4. Two real valued sequence $x = (x_k)$ and $y = (y_k)$ are called ideal equivalent if $\{k : x_k \neq y_k\} \in \mathcal{I}$. It is denoted by $x \approx y$.

Theorem 3.10. If $x = (x_k)$ and $y = (y_k)$ are ideal equivalent sequences, then

$$\mathcal{I} - \inf x_k = \mathcal{I} - \inf y_k \text{ and } \mathcal{I} - \sup x_k = \mathcal{I} - \sup y_k$$

are hold.

Proof Since the sequence $x = (x_k)$ and $y = (y_k)$ are equivalent, then the set $A = \{k : x_k \neq y_k\}$ belongs to ideal. Take into consider an arbitrary element $l \in L_{\mathcal{I}}(x)$. Since l is an ideal lower bound of the sequence $x = (x_k)$, then we have

$$\{k : x_k < l\} \in \mathcal{I}.$$

Since

$$\{k : y_k < l\} = \{k : x_k \neq y_k < l\} \cup \{k : x_k = y_k < l\} \subset$$
$$\subset A \cup \{k : x_k = y_k < l\},$$

then we have

$$\{k : y_k < l\} \in \mathcal{I}. \tag{3.16}$$

From (3.16) that, the element $l \in \mathbb{R}$ is an ideal lower bound of the sequence $y = (y_k)$. That is, $L_{\mathcal{I}}(x) \subset L_{\mathcal{I}}(y)$. If we consider arbitrary point $l \in L_{\mathcal{I}}(y)$, it can be obtained easily $l \in L_{\mathcal{I}}(x)$ such that $L_{\mathcal{I}}(y) \subset L_{\mathcal{I}}(x)$. Therefore,

$$L_{\mathcal{I}}(y) = L_{\mathcal{I}}(x) \tag{3.17}$$

holds. Since $\sup L_{\mathcal{I}}(y) = \sup L_{\mathcal{I}}(x)$, then $\mathcal{I} - \inf x_k = \mathcal{I} - \inf y_k$ is obtained. By using the same idea as given for $\mathcal{I} - \inf$ above $\mathcal{I} - \sup x_k = \mathcal{I} - \sup y_k$ can be obtained easily.

Remark 3.2. The converse of Theorem 3.10 is not true, in general.

Let us consider $\mathcal{I}_\delta$ as an ideal and sequences $x = (x_k)$ and $y = (y_k)$ as follows:

$$x_k := 1 - \frac{1}{k}, \quad y_k := 1 + \frac{1}{k}$$

for all $k \in \mathbb{N}$. It is clear from Theorem 3.8 that

$$\mathcal{I} - \inf x_k = \mathcal{I} - \inf y_k = 1 \quad \text{and} \quad \mathcal{I} - \sup x_k = \mathcal{I} - \sup y_k = 1.$$

But, the set

$$A = \{k : x_k \neq y_k\} = \mathbb{N} \notin \mathcal{I}_\delta.$$

So, $x = (x_k)$ and $y = (y_k)$ are not ideal equivalent.

4. IDEAL LIMIT INFIMUM AND IDEAL LIMIT SUPREMUM

In this section, we will define $\mathcal{I} - \limsup x_k$ and $\mathcal{I} - \liminf x_k$ by using $\mathcal{I} - \sup x_k$ and $\mathcal{I} - \inf x_k$.

Definition 4.1. Let $x = (x_k)$ be a sequence of real numbers and $\mathcal{I}$ be an admissible ideal. Then

$$\mathcal{I} - \liminf_{k \to \infty} x_k := \mathcal{I} - \sup_k \gamma_k$$

and

$$\mathcal{I} - \limsup_{k \to \infty} x_k := \mathcal{I} - \inf_k \beta_k$$

where $\gamma_k := \mathcal{I} - \inf_{n \geq k}\{x_n, x_{n+1}, \dots\}$ and $\beta_k := \mathcal{I} - \sup_{n \geq k}\{x_n, x_{n+1}, \dots\}$ for $k \in \mathbb{N}$.

Lemma 4.1. Let $x = (x_k)$ be a real valued sequence and (n_k) be an arbitrary monotone increasing sequence of positive natural numbers. Then, the following statements are true:

(i) If $\mathcal{I} - \sup x_k = l$, then $\mathcal{I} - \sup x_{n_k} = l$,

(ii) If $\mathcal{I} - \inf x_k = m$, then $\mathcal{I} - \inf x_{n_k} = m$.

Proof We shall prove only (i) here. Assume that $\mathcal{I} - \sup x_k = l$. From Definiton 3.1 and Theorem 3.2 we have

$$\{k : x_k > l + \varepsilon\} \in \mathcal{I} \quad \text{and} \quad \{k : x_k \leq l - \varepsilon\} \notin \mathcal{F}(\mathcal{I})$$

for every $\varepsilon > 0$. Since $\{n_k : x_{n_k} > l + \varepsilon\} \subset \{k : x_k > l + \varepsilon\}$ and $\{n_k : x_{n_k} \leq l - \varepsilon\} \subset \{k : x_k \leq l - \varepsilon\}$ then $\{n_k : x_{n_k} > l + \varepsilon\} \in \mathcal{I}$ and $\{n_k : x_{n_k} \leq l - \varepsilon\} \notin \mathcal{F}(\mathcal{I})$.

Therefore, $\mathcal{I} - \sup x_{n_k} = l$.

Lemma 4.2. Let $x = (x_k)$ be sequence of real numbers.

(i) If $\beta_n \coloneqq \mathcal{I} - \sup_{k\geq n}\{x_k, x_{k+1}, ...\}$ for all $n \in \mathbb{N}$, then $(\beta_n)_{n\in\mathbb{N}}$ is a constant sequence and $\beta_n \coloneqq \mathcal{I} - \sup x_k$ for all $n \in \mathbb{N}$.

(ii) If $\gamma_n \coloneqq \mathcal{I} - \inf_{k\geq n}\{x_k, x_{k+1}, ...\}$ for all $n \in \mathbb{N}$, then $(\gamma_n)_{n\in\mathbb{N}}$ is a constant sequence and $\gamma_n \coloneqq \mathcal{I} - \inf x_k$ for all $n \in \mathbb{N}$.

Theorem 4.1. Let $x = (x_k)$ be a sequence of real numbers. Then, the following statements are true:

(i) $\mathcal{I} - \liminf_{k\to\infty} x_k = \mathcal{I} - \inf x_k = \sup L_{\mathcal{I}}(x)$,

(ii) $\mathcal{I} - \limsup_{k\to\infty} x_k = \mathcal{I} - \sup x_k = \inf U_{\mathcal{I}}(x)$.

Proof (i) Since

$$\mathcal{I} - \liminf_{k\to\infty} x_k = \mathcal{I} - \sup\left(\mathcal{I} - \inf_{n\geq k} x_k\right),$$

then

$$\mathcal{I} - \liminf_{k\to\infty} x_k = \mathcal{I} - \sup(\gamma_n) = \mathcal{I} - \sup_n(\mathcal{I} - \inf x_k) = \mathcal{I} - \inf x_k.$$

Since

$$\mathcal{I} - \limsup_{k\to\infty} x_k = \mathcal{I} - \inf\left(\mathcal{I} - \sup_{k\geq n} x_k\right),$$

then

$$\mathcal{I} - \limsup_{k\to\infty} x_k = \mathcal{I} - \inf_n(\beta_n) = \mathcal{I} - \inf(\mathcal{I} - \sup x_k) = \mathcal{I} - \sup x_k.$$

Corollary 4.1. Let $x = (x_k)$ be a real valued sequence. Then,

(i) $\mathcal{I} - \liminf x_k \leq \mathcal{I} - \limsup x_k$,

(ii) $\liminf x_k \leq \mathcal{I} - \liminf x_k \leq \mathcal{I} - \limsup x_k \leq \limsup x_k$,

(iii) $\mathcal{I} - \liminf(x_k + y_k) = \mathcal{I} - \liminf x_k + \mathcal{I} - \liminf y_k$,

(iv) $\mathcal{I} - \limsup(x_k + y_k) = \mathcal{I} - \limsup x_k + \mathcal{I} - \limsup y_k$,

(v) $\mathcal{I} - \liminf(-x_k) = -(\mathcal{I} - \limsup x_k)$.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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