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Araştırma Makalesi - Research Article

Robust Versions of the Lower and Upper Possibilistic Mean - Variance Models for the One Period or Two Periods Cases

Bir ya da İki Periyotlu Durumlar için Alt ve Üst Olabilirlik Ortalama - Varyans Modellerinin Dayanıklı Versiyonları

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ABSTRACT

It is easy to use possibility theory in modeling incomplete information. Robust optimization is an important tool when there is parameter uncertainty. Thus, in this study, we propose robust versions of the lower and upper possibilistic mean - variance (MV) models when there are multiple possibility distribution scenarios. Here, we use entropy as a diversification constraint. In addition, we reduce these robust versions to concave maximization problems. Furthermore, we generalize them for two periods portfolio selection problem by using fuzzy addition and multiplication. On the other hand, these generalizations are not concave maximization problems. Finally, we give an illustrative example by using different solvers in Gams modeling system.

Keywords- Entropy, Fuzzy Arithmetic, Portfolio Selection, Possibility Theory, Robust Optimization

ÖZ

Tam olmayan bilgiyi modellemede olabilirlik teorisini kullanmak kolaydır. Parametre belirsizliği olduğunda dayanıklı optimizasyon önemli bir araçtır. Bu nedenle bu çalışmada, birden çok olabilirlik dağılımı senaryosu olduğunda alt ve üst olabilirlik ortalama - varyans (OV) modellerinin dayanıklı versiyonları önerilmiştir. Burada entropi çeşitlendirme kısıdı olarak kullanılmıştır. Bununla birlikte bu dayanıklı versiyonlar konkav maksimizasyon problemlerine indirgenmiştir. Üstelik bunlar, iki periyotlu portföy seçimi problemine bulanık toplama ve çarpma kullanılarak genelleştirilmiştir. Öte yandan bu genelleştirmeler, konkav maksimizasyon problemleri değildir. Son olarak, Gams modelleme sisteminde farklı çözücüler kullanılarak açıklayıcı bir örnek verilmiştir.

Anahtar Kelimeler- Entropi, Bulanık Aritmetik, Portföy Seçimi, Olabilirlik Teorisi, Dayanıklı Optimizasyon

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I. INTRODUCTION

Fuzzy set theory, which has a wide range of uses, is introduced by Zadeh in [1]. Possibility theory, which is one of them, is also proposed by Zadeh in [2] and enhanced by Dubois and Prade in [3]. Possibility theory is simpler than other uncertainty theories to deal with incomplete information [4]. Thus, it is widely used in many areas [5]. The possibilistic MV model is proposed in [6] for the one period case. Its variants, which are called as the lower and upper possibilistic MV models, are examined in [7,8]. There are also its different variants for the one period case such as two moments models proposed in [9] and a three moments model proposed in [10]. For the multi-period case, we list some of its variants in Table 1. Here, we also mention about the used fuzzy numbers for possibility distributions and whether entropy is used as a diversification constraint or not.

Table 1. The variants of the possibilistic MV model for the multi-period case.

Models	**Fuzzy Number**	**Entropy**
The model in [11]	Trapezoidal	N/A
The models in [12]	Coherent trapezoidal	N/A
The models in [13]	Trapezoidal	N/A
The models in [14]	Triangular	N/A
The model in [15]	LR type	N/A
The model in [16]	Trapezoidal	N/A
The model in [17]	Trapezoidal	Shannon entropy
The model in [18]	Trapezoidal	Shannon entropy
The model in [19]	Trapezoidal	Possibilistic entropy
The models in [20]	Trapezoidal	N/A
The proposed robust versions	Trapezoidal	Shannon entropy

The possibilistic mean - semi variance model is solved with the multiple particle swarm optimization [11]. The models solved with the genetic algorithm capture the heterogeneity of investor attitudes towards the stock market [12]. The models solved with the max-min approach consider several realistic constraints [13]. The models solved with the self adaptive differential evolution algorithm consider higher possibilistic moments [14]. The model solved with the hybrid differential evolution algorithm considers some real investment features [15]. The model solved with the multi-objective evolutionary algorithm considers the liquidity of stocks [16]. The model solved with the fuzzy goal programming considers investor's different investment preferences [17]. The possibilistic mean - semi variance model is solved with the genetic algorithm [18]. The possibilistic mean - semi variance - entropy model is solved with the hybrid intelligent algorithm [19]. The models solved with genetic algorithm considers the possibilistic skewness [20].

On the other hand, to the best of our knowledge, there is not a multi-period model where the upper (lower) possibilistic mean and variance definitions given in [7,8] are used exactly. To fill this gap in the literature, we propose robust versions of the lower (upper) possibilistic MV model for the one period or two periods' cases where we use Shannon entropy as a diversification constraint. Here, we assume that there are multiple possibility distribution scenarios unlike the multi-period models in Table 1. In the one period case, we see that portfolio selection problem is reduced to concave maximization problems. Thus, the proposed robust versions can be solved with the known algorithms in the literature. In the two periods case, we see that portfolio selection problem is given with general nonlinear maximization problems. Here, we use Gams/Octeract, which finds global optima [21].

Due to the linearity of the lower (upper) possibilistic mean - variance model, its solution can be derived analytically. On the other hand, the diversified optimal portfolios can not be uniquely derived with these models when there are not extra constraints [22]. In this study, by using multiple possibility distributions scenarios, we propose their robust versions to overcome this drawback. The main two motivations of this study is to get the diversified optimal portfolios with the proposed robust versions and to generalize these robust versions for the two periods case. The originality and main contribution of this study is that this is the first study considering multiple possibility distributions scenarios for two periods portfolio selection problem. The main limitation of the proposed robust versions is that they can not be effectively used when the asset weights are allowed to be negative. This drawback is also valid for the lower (upper) possibilistic mean - variance model. That is, the proposed robust versions may be preferable for real-world portfolio selection only when the short positions are not allowed in portfolios.

We organize the remainder of paper as follows. Firstly, we formulate the robust versions of the upper and lower possibilistic MV models for the one period case by using only fuzzy addition. Then, we generalize them for the two periods case by using fuzzy addition and multiplication. Secondly, we give an explanatory example to illustrate and compare the proposed robust versions. Then, we conclude the paper.

II. METHODS

A. The Proposed Robust Versions for the One Period Case

In this study, we use trapezoidal fuzzy numbers for possibility distributions as in [7]. The membership function of trapezoidal fuzzy number (a, b, α, β) is as below.

$$A(t)=\begin{cases}1-\dfrac{a-t}{\alpha}, a-\alpha\le t\le a\\ 1, a\le t\le b\\ 1-\dfrac{t-b}{\beta}, b\le t\le b+\beta\\ 0, else\end{cases} \tag{1}$$

In Figure 1, its membership function is shown graphically.

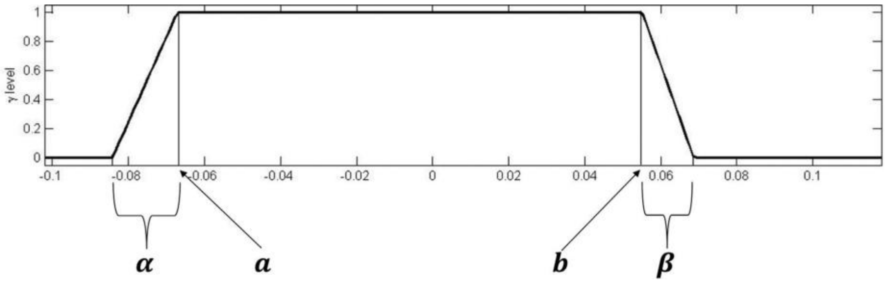

Figure 1. The membership function of trapezoidal fuzzy number [23].

Let r_i be defined as 1 plus simple return of i^{th} asset. Let the possibility distribution of r_i be (a_i, b_i, α_i, β_i). Then, the lower possibilistic mean and standard deviation of the portfolio are found as below where $E_p^-(\ )$ and $SD_p^-(\ )$ are the lower possibilistic mean and standard deviation operators respectively [7].

$$E_p^-\left(\sum_{i=1}^{n} w_i r_i\right)=\sum_{i=1}^{n} w_i E_p^-(r_i)=\sum_{i=1}^{n} w_i\left(a_i-\frac{\alpha_i}{3}\right)$$
$$SD_p^-\left(\sum_{i=1}^{n} w_i r_i\right)=\sum_{i=1}^{n} w_i SD_p^-(r_i)=\sum_{i=1}^{n} w_i\frac{\alpha_i}{3\sqrt{2}} \tag{2a}$$

The upper possibilistic mean and standard deviation of the portfolio are found as below where $E_p^+(\ )$ and $SD_p^+(\ )$ are the upper possibilistic mean and standard deviation operators respectively [7].

$$E_p^+\left(\sum_{i=1}^{n} w_i r_i\right)=\sum_{i=1}^{n} w_i E_p^+(r_i)=\sum_{i=1}^{n} w_i\left(b_i+\frac{\beta_i}{3}\right)$$
$$SD_p^+\left(\sum_{i=1}^{n} w_i r_i\right)=\sum_{i=1}^{n} w_i SD_p^+(r_i)=\sum_{i=1}^{n} w_i\frac{\beta_i}{3\sqrt{2}} \tag{2b}$$

Let c vary on [0,1]. Based on (2a), the lower possibilistic MV model can be given with the following linear maximization problem [22].

$$\max c\sum_{i=1}^{n} w_i\left(a_i-\frac{\alpha_i}{3}\right)+(1-c)\left(-\sum_{i=1}^{n} w_i\frac{\alpha_i}{3\sqrt{2}}\right)$$
$$s.t. \sum_{i=1}^{n} w_i=1 \tag{3a}$$
$$w_i\ge 0, \forall i$$

Based on (2b), the upper possibilistic MV model can be given with the following linear maximization problem [22].

$$\max c\sum_{i=1}^{n} w_i\left(b_i+\frac{\beta_i}{3}\right)+(1-c)\left(-\sum_{i=1}^{n} w_i\frac{\beta_i}{3\sqrt{2}}\right)$$

$$s.t. \sum_{i=1}^{n} w_i = 1 \tag{3b}$$

$$w_i \geq 0, \forall i$$

Shannon entropy, which is an uncertainty measure is defined with the following concave function. Its main advantage is to form well-diversified portfolios. Its unique minimum is achieved with zero value when the weight of an asset is equal to 1. Its unique maximum is achieved with ln(n) value when the weights of all assets are equal [24].

$$SE(w) = -\sum_{i=1}^{n} w_i \ln w_i \tag{4}$$

We use (4) as a diversification constraint in the proposed robust versions. Then, the feasible set is as below in the one period case. Here, w is the weight vector of assets and w_i is the weight of i^{th} asset.

$$S=\left\{w:\sum_{i=1}^{n} w_i = 1\ and\ w_i \geq 0\ and\ SE(w)\geq \frac{1}{2}\ln n\right\} \tag{5}$$

Let the possibility distribution of r_i be ($a_{i,k}$, $b_{i,k}$, $\alpha_{i,k}$, $\beta_{i,k}$) according to the k^{th} expert. We define the robust version of (3a) as below.

$$\max_{w\in S}\min_{k} c\sum_{i=1}^{n} w_i\left(a_{i,k}-\frac{\alpha_{i,k}}{3}\right)-(1-c)\sum_{i=1}^{n} w_i\frac{\alpha_{i,k}}{3\sqrt{2}} \tag{6a}$$

We define the robust version of (3b) as below.

$$\max_{w\in S}\min_{k} c\sum_{i=1}^{n} w_i\left(b_{i,k}+\frac{\beta_{i,k}}{3}\right)-(1-c)\sum_{i=1}^{n} w_i\frac{\beta_{i,k}}{3\sqrt{2}} \tag{6b}$$

We reduce (6a) to the following concave maximization problem.

$$\max_{w\in S} z$$

$$s.t.\ z \leq c\sum_{i=1}^{n} w_i\left(a_{i,k}-\frac{\alpha_{i,k}}{3}\right)-(1-c)\sum_{i=1}^{n} w_i\frac{\alpha_{i,k}}{3\sqrt{2}}, \forall k \tag{7a}$$

We reduce (6b) to the following concave maximization problem.

$$\max_{w\in S} z$$

$$s.t.\ z \leq c\sum_{i=1}^{n} w_i\left(b_{i,k}+\frac{\beta_{i,k}}{3}\right)-(1-c)\sum_{i=1}^{n} w_i\frac{\beta_{i,k}}{3\sqrt{2}}, \forall k \tag{7b}$$

In the one period case, the local maximums of (7a) and (7b) are also the global maximums of them since (7a) and (7b) are concave maximization problems. In this study, we use Gams/Conopt4 to find the local (global) maximums.

B. The Proposed Robust Versions for the Two Periods Case

In the two periods case, the feasible set is as below. Here, w (ω) is the weight vector in the first (second) period.

$$S = \begin{Bmatrix} w,\omega : \sum_{i=1}^{n} w_i = 1 \; and \; w_i \geq 0 \; and \; SE(w) \geq \frac{1}{2}\ln n \\ \sum_{i=1}^{n} \omega_i = 1 \; and \; \omega_i \geq 0 \; and \; SE(\omega) \geq \frac{1}{2}\ln n \end{Bmatrix} \tag{8}$$

Let $r_{1,i}$ be 1 plus simple return of i[th] asset in the first period and $r_{2,i}$ be 1 plus simple return of i[th] asset in the second period with the possibility distribution ($a_{1,i,k}$, $b_{1,i,k}$, $\alpha_{1,i,k}$, $\beta_{1,i,k}$) and ($a_{2,j,k}$, $b_{2,j,k}$, $\alpha_{2,j,k}$, $\beta_{2,j,k}$) respectively. Then, we find the lower possibilistic mean and standard deviation of portfolio as below respectively due to the linearity in (2a).

$$\begin{aligned} E_p^-\left(\left(\sum_{i=1}^{n} w_i r_{1,i}\right)\left(\sum_{j=1}^{n} \omega_j r_{2,j}\right)\right) &= E_p^-\left(\sum_{i=1}^{n}\sum_{j=1}^{n} w_i \omega_j r_{1,i} r_{2,j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \omega_j E_p^-\left(\sum_{i=1}^{n}\sum_{j=1}^{n} r_{1,i} r_{2,j}\right) \\ SD_p^-\left(\left(\sum_{i=1}^{n} w_i r_{1,i}\right)\left(\sum_{j=1}^{n} \omega_j r_{2,j}\right)\right) &= SD_p^-\left(\sum_{i=1}^{n}\sum_{j=1}^{n} w_i \omega_j r_{1,i} r_{2,j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \omega_j SD_p^-\left(\sum_{i=1}^{n}\sum_{j=1}^{n} r_{1,i} r_{2,j}\right) \end{aligned} \tag{9}$$

For positive two trapezoidal fuzzy numbers, we have the following results [25].

$$\begin{aligned} &\left(a_{1,i,k}, b_{1,i,k}, \alpha_{1,i,k}, \beta_{1,i,k}\right) \oplus \left(a_{2,j,k} b_{2,j,k}, \alpha_{2,j,k}, \beta_{2,j,k}\right) \\ &\quad = \left(a_{1,i,k} + a_{2,j,k}, b_{1,i,k} + b_{2,j,k}, \alpha_{1,i,k} + \alpha_{2,j,k}, \beta_{1,i,k} + \beta_{2,j,k}\right) \\ &\left(a_{1,i,k}, b_{1,i,k}, \alpha_{1,i,k}, \beta_{1,i,k}\right) \otimes \left(a_{2,j,k} b_{2,j,k}, \alpha_{2,j,k}, \beta_{2,j,k}\right) \\ &\quad \approx \left(a_{1,i,k} a_{2,j,k}, b_{1,i,k} b_{2,j,k}, a_{1,i,k}\alpha_{2,j,k} + \alpha_{1,i,k} a_{2,j,k} - \alpha_{1,i,k}\alpha_{2,j,k}, b_{1,i,k}\beta_{2,j,k} + \beta_{1,i,k} b_{2,j,k} + \beta_{1,i,k}\beta_{2,j,k}\right) \end{aligned} \tag{10}$$

Example: The fuzzy addition of (5, 5, 1, 2) and (6, 6, 3, 4) is equal to (11, 11, 4, 6). The fuzzy multiplication of them is approximately equal to (a, b, α, β) where a=5*6=30, b=5*6=30, α=5*3+1*6-1*3=18 and β=5*4+2*6+2*4=40. Notice that a-α is equal to (5-1)*(6-3)=12 while b+β is equal to (5+2)*(6+4)=70.

We have the following results according to the k[th] expert based on (2a) and (10).

$$\begin{aligned} \Gamma_{i,j,k} &:= E_p^-\left(r_{1,i} r_{2,j}\right) = \left(a_{1,i,k} a_{2,j,k} - \frac{a_{1,i,k}\alpha_{2,j,k} + \alpha_{1,i,k} a_{2,j,k} - \alpha_{1,i,k}\alpha_{2,j,k}}{3}\right) \\ \Pi_{i,j,k} &:= SD_p^-\left(r_{1,i} r_{2,j}\right) = \left(\frac{a_{1,i,k}\alpha_{2,j,k} + \alpha_{1,i,k} a_{2,j,k} - \alpha_{1,i,k}\alpha_{2,j,k}}{3\sqrt{2}}\right) \end{aligned} \tag{11a}$$

We also have the following results according to the k[th] expert based on (2b) and (10).

$$\begin{aligned} \Phi_{i,j,k} &:= E_p^+\left(r_{1,i} r_{2,j}\right) = b_{1,i,k} b_{2,j,k} + \frac{b_{1,i,k}\beta_{2,j,k} + \beta_{1,i,k} b_{2,j,k} + \beta_{1,i,k}\beta_{2,j,k}}{3} \\ \Omega_{i,j,k} &:= SD_p^+\left(r_{1,i} r_{2,j}\right) = \frac{b_{1,i,k}\beta_{2,j,k} + \beta_{1,i,k} b_{2,j,k} + \beta_{1,i,k}\beta_{2,j,k}}{3\sqrt{2}} \end{aligned} \tag{11b}$$

We derive the lower possibilistic mean and standard deviation of portfolio as below respectively according to the k[th] expert based on (9) and (11a) where Γ_k and Π_k are the square matrices.

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \omega_j \Gamma_{i,j,k} &= w^T \Gamma_k \omega \\ \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \omega_j \Pi_{i,j,k} &= w^T \Pi_k \omega \end{aligned} \tag{12}$$

We determine transaction costs function as below similar to [14].

$$TC(w,\omega) = 0.001\sum_{i=1}^{n}|w_i - \omega_i| \tag{13}$$

Based on (12) and (13), we generalize (7a) for the two periods case as below.

$$\begin{aligned} &\max_{w,\omega \in S} z - 0.001\sum_{i=1}^{n}|w_i - \omega_i| \\ &s.t.\ z \le cw^T\Gamma_k\omega - (1-c)w^T\Pi_k\omega,\ \forall k \end{aligned} \tag{14a}$$

Similarly, we generalize (7b) for the two periods case as below where Φ_k and Ω_k are the square matrices, of which elements are as in (11b).

$$\begin{aligned} &\max_{w,\omega \in S} z - 0.001\sum_{i=1}^{n}|w_i - \omega_i| \\ &s.t.\ z \le cw^T\Phi_k\omega - (1-c)w^T\,\Omega_k\omega,\ \forall k \end{aligned} \tag{14b}$$

(14a) and (14b) are general nonlinear maximization problems. Hence, we find the global maximums of them by using Gams/Octeract.

III. RESULTS AND DISCUSSION

In this section, we examine the proposed robust versions when there are four risky assets (A1, A2, A3 and A4) and two experts. Possibility distributions for the first period are as below according to the first expert.

$$\begin{aligned} &r_{1,1,1} = (1.0115, 1.0115, 0.0115, 0.0085) \quad && r_{1,2,1} = (1.0125, 1.0125, 0.0125, 0.0095) \\ &r_{1,3,1} = (1.0135, 1.0135, 0.0135, 0.0105) && r_{1,4,1} = (1.013, 1.013, 0.013, 0.019) \end{aligned} \tag{15}$$

Possibility distributions for the first period are as below according to the second expert.

$$\begin{aligned} &r_{1,1,2} = (1.013, 1.013, 0.013, 0.019) \quad && r_{1,2,2} = (1.0135, 1.0135, 0.0135, 0.0105) \\ &r_{1,3,2} = (1.0115, 1.0115, 0.0115, 0.0085) && r_{1,4,2} = (1.0125, 1.0125, 0.0125, 0.0095) \end{aligned} \tag{16}$$

Possibility distributions for the second period are as below according to the first expert.

$$\begin{aligned} &r_{2,1,1} = (1.0125, 1.0125, 0.0125, 0.0095) \quad && r_{2,2,1} = (1.0115, 1.0115, 0.0115, 0.0085) \\ &r_{2,3,1} = (1.013, 1.013, 0.013, 0.019) && r_{2,4,1} = (1.0135, 1.0135, 0.0135, 0.0105) \end{aligned} \tag{17}$$

Possibility distributions for the second period are as below according to the second expert.

$$\begin{aligned} &r_{2,1,2} = (1.0135, 1.0135, 0.0135, 0.0105) \quad && r_{2,2,2} = (1.013, 1.013, 0.013, 0.019) \\ &r_{2,3,2} = (1.0125, 1.0125, 0.0125, 0.0095) && r_{2,4,2} = (1.0115, 1.0115, 0.0115, 0.0085) \end{aligned} \tag{18}$$

In this study, we define O1 (O2) as the optimal solution of (7a) or (7b) for the first (second) period when there is not an entropy constraint, EO1 (EO2) as the optimal solution of (7a) or (7b) for the first (second) period when there is an entropy constraint, TEO1 (TEO2) as the optimal solution of (14a) or (14b) for the first (second) period when there is an entropy constraint.

A. The One Period Case

We find the unique optimal solution of (7a) as in Table 2 for two periods separately if c is equal to 0.

Table 2. Optimal solution of (7a) when c is equal to 0.

Assets	O1	EO1	O2	EO2
A1	0.5714	0.5708	0	0
A2	0	0	0.5714	0.5708
A3	0.4286	0.4277	0	0.0015
A4	0	0.0015	0.4286	0.4277

We find the unique optimal solution of (7a) as in Table 3 for two periods separately if c is equal to 0.5 or 1.

Table 3. Optimal solution of (7a) when c is equal to 0.5 or c=1.

Assets	O1	EO1	O2	EO2
A1	0	0	0.6667	0.6606
A2	0.6667	0.6606	0	0
A3	0.3333	0.3272	0	0.0122
A4	0	0.0122	0.3333	0.3272

We find the unique optimal solution of (7b) as in Table 4 for two periods separately if c is equal to 0.

Table 4. Optimal solution of (7b) when c is equal to 0.

Assets	O1	EO1	O2	EO2
A1	0	0.0076	0.6667	0.635
A2	0.6667	0.635	0	0.0076
A3	0.3333	0.3574	0	0.3574
A4	0	0	0.3333	0

We find the unique optimal solution of (7b) as in Table 5 for two periods separately if c is equal to 0.5.

Table 5. Optimal solution of (7b) when c is equal to 0.5.

Assets	O1	EO1	O2	EO2
A1	0	0.0012	0.5653	0.5635
A2	0.5653	0.5635	0	0.0012
A3	0	0	0.4347	0.4353
A4	0.4347	0.4353	0	0

We find the unique optimal solution of (7b) as in Table 6 for two periods separately if c is equal to 1.

Table 6. Optimal solution of (7b) when c is equal to 1.

Assets	O1	EO1	O2	EO2
A1	0.4231	0.422	0	0.0018
A2	0	0.0018	0.4231	0.422
A3	0	0	0.5769	0.5762
A4	0.5769	0.5762	0	0

Based on the tables given in this subsection, we can say that O1 (O2) and EO1 (EO2) are nearly the same and the proposed robust versions give sufficiently diversified optimal portfolios even if there is not an entropy constraint.

B. The Two Periods Case

We find the optimal solution of (14a) as in Table 7 when c is equal to 0.

Table 7. Optimal solution of (14a) when c is equal to 0.

Assets	TEO1	EO1	TEO2	EO2
A1	0.2585	0.5708	0.2585	0
A2	0.2415	0	0.2415	0.5708
A3	0.2969	0.4277	0.2969	0.0015
A4	0.2031	0.0015	0.2031	0.4277

We find the optimal solution of (14a) as in Table 8 when c is equal to 0.5.

Table 8. Optimal solution of (14a) when c is equal to 0.5.

Assets	TEO1	EO1	TEO2	EO2
A1	0.3661	0	0.3661	0.6606
A2	0.1339	0.6606	0.1339	0
A3	0.1518	0.3272	0.1518	0.0122
A4	0.3482	0.0122	0.3482	0.3272

We find the optimal solution of (14a) as in Table 9 when c is equal to 1.

Table 9. Optimal solution of (14a) when c is equal to 1.

Assets	TEO1	EO1	TEO2	EO2
A1	0.1190	0	0.1190	0.6606
A2	0.3809	0.6606	0.3809	0
A3	0.25	0.3272	0.25	0.0122
A4	0.25	0.0122	0.25	0.3272

We find the optimal solution of (14b) as in Table 10 when c is equal to 0.

Table 10. Optimal solution of (14b) when c is equal to 0.

Assets	TEO1	EO1	TEO2	EO2
A1	0.5	0.0076	0.5	0.635
A2	0.0002	0.635	0.0002	0.0076
A3	0	0.3574	0	0
A4	0.4998	0	0.4998	0.3574

We find the optimal solution of (14b) as in Table 11 when c is equal to 0.5.

Table 11. Optimal solution of (14b) when c is equal to 0.5.

Assets	TEO1	EO1	TEO2	EO2
A1	0.3889	0.0012	0.3889	0.5635
A2	0.1112	0.5635	0.1112	0.0012
A3	0.2962	0	0.2962	0.4353
A4	0.2037	0.4353	0.2037	0

We find the optimal solution of (14b) as in Table 12 when c is equal to 1.

Table 12. Optimal solution of (14b) when c is equal to 1.

Assets	TEO1	EO1	TEO2	EO2
A1	0.2202	0.422	0.2202	0.0018
A2	0.2798	0.0018	0.2798	0.422
A3	0.1756	0	0.1756	0.5762
A4	0.3244	0.5762	0.3244	0

Based on the tables given in this subsection, we can say that TEO1 (TEO2) and EO1 (EO2) are not close to each other whereas TEO1 and TEO2 are nearly the same. This is because, there are the effects of transaction costs and fuzzy multiplication in the two periods case. We also note that TEO1 (TEO2) is more diversified than EO1 (EO2). For these reasons, the use of (14a) or (14b) is a better choice than the use of (7a) or (7b) for two periods separately especially when the experts have different predictions about two consecutive periods.

C. Comparisons of the Existing Models and Their Proposed Robust Versions

In this subsection, we compare the given results with the results of the lower (upper) possibilistic MV model where L1 is the optimal solution of (3a) or (3b) for the first period according to the first expert. That is, the possibility distributions are as in (15) for the existing models. For the other cases, we have the similar results.

We find the unique optimal solution of (3a) as in Table 13 when c is equal to 0.

Table 13. Optimal solution of (3a) when c is equal to 0.

Assets	L1	O1	EO1	TEO1
A1	1	0.5714	0.5708	0.2585
A2	0	0	0	0.2415
A3	0	0.	0.4277	0.2969
A4	0	0.4286	0.0015	0.2031

We find the unique optimal solution of (3a) as in Table 14 when c is equal to 0.5.

Table 14. Optimal solution of (3a) when c is equal to 0.5.

Assets	L1	O1	EO1	TEO1
A1	0	0	0	0.3661
A2	0	0.6667	0.6606	0.1339
A3	1	0.3333	0.3272	0.1518
A4	0	0	0.0122	0.3482

We find the unique optimal solution of (3a) as in Table 15 when c is equal to 1.

Table 15. Optimal solution of (3a) when c is equal to 1.

Assets	L1	O1	EO1	TEO1
A1	0	0.	0	0.1190
A2	0	0.6667	0.6606	0.3809
A3	1	0.3333	0.3272	0.25
A4	0	0	0.0122	0.25

We find the unique optimal solution of (3b) as in Table 16 when c is equal to 0.

Table 16. Optimal solution of (3b) when c is equal to 0.

Assets	L1	O1	EO1	TEO1
A1	1	0	0.0076	0.5
A2	0	0.6667	0.635	0.0002
A3	0	0.3333	0.3574	0
A4	0	0	0	0.4998

We find the unique optimal solution of (3b) as in Table 17 when c is equal to 0.5.

Table 17. Optimal solution of (3b) when c is equal to 0.5.

Assets	L1	O1	EO1	TEO1
A1	0	0	0.0012	0.3889
A2	0	0.5653	0.5635	0.1112
A3	0	0	0	0.2962
A4	1	0.4347	0.4353	0.2037

We find the unique optimal solution of (3b) as in Table 18 when c is equal to 1.

Table 18. Optimal solution of (3b) when c is equal to 1.

Assets	L1	O1	EO1	TEO1
A1	0	0.4231	0.422	0.2202
A2	0	0	0.0018	0.2798
A3	0	0	0	0.1756
A4	1	0.5769	0.5762	0.3244

Based on the tables given in this subsection, we can say that L1 is not diversified unlike O1, EO1 and TEO1. That is, by using the proposed robust versions, we get the diversified optimal portfolios, which are robust to the worst-case scenario by definition. Thus, we believe that the proposed robust versions are superior to the existing models especially for conservative investors.

IV. CONCLUSIONS

In this study, we propose the robust versions of the lower (upper) possibilistic MV model for the one period or two periods' cases when there are multiple possibility distribution scenarios based on the different expert opinions. The main limitation of these models is that they can not be effectively used when the short positions are allowed in portfolios. It is sufficient to make local optimization in the one period case whereas it is necessary to make global optimization in the two periods case. Because we use only fuzzy addition in the one period case whereas we use fuzzy addition and multiplication in the two periods case. That is, two periods case should be preferred when applicable due to conveying higher information. In our illustrative example, we get diversified optimal portfolios even if there is not an entropy constraint. Furthermore, the diversified optimal portfolios are robust to the worst-case scenario by definition. For these reasons, we conclude that the proposed robust versions are more preferable alternatives especially for conservative investors.

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